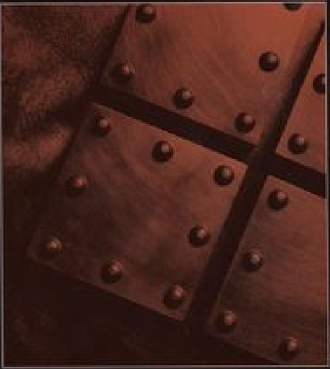
ARTHUR P. BORESI
RICHARD J. SCHMIDT

ADVANCED MECHANICS OF MATERIALS



Sixth Edition

SOLUTION MANUAL

SOLUTIONS MANUAL

to accompany

Sixth Edition

ADVANCED MECHANICS OF MATERIALS

ARTHUR P. BORESI
Emeritus Professor
In Civil and Architectural Engineering
The University of Wyoming and Laramie
And
Emeritus Professor
In Theoretical and Applied Mechanics
University of Illinois, Urbana-Champaign

RICHARD J. SCHMIDT
Professor
Civil and Architectural Engineering
The University of Wyoming

To order books or for customer service call 1-800-CALL-WILEY (225-5945).

Copyright © 2003 by John Wiley & Sons, Inc.

Excerpts from this work may be reproduced by instructors for distribution on a not-for-profit basis for testing or instructional purposes only to students enrolled in courses for which the textbook has been adopted. *Any other reproduction or translation of this work beyond that permitted by Sections 107 or 108 of the 1976 United States Copyright Act without the permission of the copyright owner is unlawful. Requests for permission or further information should be addressed to the Permissions Department, John Wiley & Sons, Inc., 605 Third Avenue, New York, NY 10158-0012.*

ISBN 0-471-27179-9

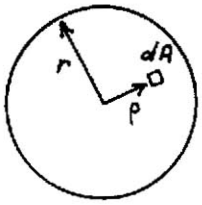
United States of America

John Wiley & Sons, Inc.

1.22	ALLOY STEEL FIG. 1.3 & 1.4	STRUCTURAL STEEL FIG. 1.5
YIELD POINT	N/A	265 MPa
YIELD STRENGTH	450 MPa	N/A
UPPER YIELD POINT	N/A	280 MPa
LOWER YIELD POINT	N/A	265 MPa
MODULUS OF RESILIENCE	N/A	0.1855 MPa
ULTIMATE TENSILE STRENGTH	715 MPa	470 MPa
STRAIN AT FRACTURE	0.23	0.26
PER CENT ELONGATION	23%	26%

1.23 ASSUME:

1. PLANE SECTIONS NORMAL TO THE AXIS OF THE ROD REMAIN PLANE UNDER APPLICATION OF THE LOAD.
2. SHEAR STRAINS VARY LINEARLY FROM THE LONGITUDINAL AXIS.
3. HOOKE'S LAW APPLIES



EQUILIBRIUM: $\Sigma M_x = 0$

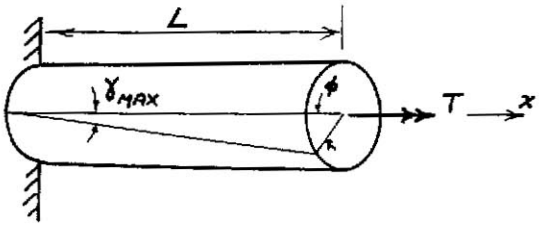
$$T = \int_A \rho \tau \, dA \quad (a)$$

COMPATIBILITY:

$$\gamma = \frac{\gamma_{max}}{r} \rho \quad (b)$$

HOOKE'S LAW:

$$\tau = G\gamma \quad (c)$$



SUB (c) INTO (b) & THEN (b) INTO (a)

$$T = \frac{\tau_{max}}{r} \int_A \rho^2 dA$$

$$T = \frac{\tau_{max}}{r} J \; ; \; J = \int_A \rho^2 dA$$

$$\underline{\underline{\tau_{max} = \frac{Tr}{J} \quad \text{ON SURFACE}}}$$

$$\underline{\underline{\tau = \frac{T\rho}{J} \quad \text{AT ANY } \rho}} \quad (d)$$

FROM GEOMETRY OF DEFORMATION:

$$\gamma_{max} = \frac{r\phi}{L} \; ; \; \gamma = \frac{\rho\phi}{L} \quad (e)$$

SUB (d) & (e) INTO (c)

$$\frac{T\rho}{J} = G\left(\frac{\rho\phi}{L}\right)$$

$$\underline{\underline{\phi = \frac{TL}{GJ}}}$$

1.24 ASSUME: 1. PLANE SECTIONS NORMAL TO THE AXIS OF THE BAR REMAIN PLANE UNDER APPLICATION OF THE LOAD.
2. HOOKE'S LAW APPLIES.

EQUILIBRIUM:

$$P = \sigma A \qquad (a)$$

HOOKE'S LAW:

$$\sigma = E\varepsilon \qquad (c)$$

CONTINUITY:

$$\Delta L = \int_0^L \varepsilon \, dx \qquad (b)$$

GEOMETRY

$$A(x) = b\left(d_0 - \tfrac{x}{L}(d_0 - d_1)\right) \qquad (d)$$

SUB (d) INTO (a)

$$\underline{\underline{\sigma = P/\left[b\left(d_0 - \tfrac{x}{L}(d_0 - d_1)\right)\right]}} \qquad (e)$$

SUB. (c) & (e) INTO (b):

$$\Delta L = \int_0^L \frac{P}{Eb}\left[d_0 - \tfrac{x}{L}(d_0 - d_1)\right]^{-1} dx$$

$$\underline{\underline{\Delta L = \frac{PL}{Eb}\int_0^L \frac{dx}{d_0 L - x(d_0 - d_1)} = \frac{PL}{Eb}\left(\frac{1}{d_0 - d_1}\right)\ln\frac{d_0}{d_1}}}$$

1.25

RODS: $A_R = 4\left(\frac{\pi}{4}15^2\right) = 706.9\ mm^2$

PIPE: $A_p = \frac{\pi}{4}\left(100^2 - 90^2\right) = 1492\ mm^2$

AFTER ASSEMBLY: $T_R = C_p = 4(65) = 260\ kN$

$$\sigma_R = \frac{4(65000)}{706.9} = 367.8\ MPa$$

$$\sigma_p = \frac{-4(65000)}{1492} = -174.3\ MPa$$

AFTER PRESSURE IS APPLIED:

EQUILIBRIUM:

$$p\left(\frac{\pi}{4}90^2\right) = \Delta T_R + \Delta C_p \qquad (a)$$

COMPATIBILITY:

$$\Delta L_R = \Delta L_p; \quad \frac{\Delta T_R L}{A_R E} = \frac{\Delta C_p L}{A_p E} \qquad (b)$$

LEAKAGE REQUIREMENT:

$$\Delta C_p = 260\ kN \qquad (c)$$

SUB (c) INTO (b):

$$\Delta T_R = 123.2\ kN \qquad (d)$$

SUB (c) & (d) INTO (a)

$$\underline{\underline{p = 60.23\ MPa}}$$

$$\underline{\underline{\Delta\sigma_R = \frac{123200}{706.9} = 174.3\ MPa}}$$

$$\underline{\underline{\sigma_{R(FINAL)} = 542.1\ MPa}}$$

1.26 (a) Figure a shows the bars subjected to force P. Figure b shows the free-body diagrams of the steel bar, the aluminum bar, and point A. By the free-body diagram of point A,

$$P = P_s + P_a \qquad (a)$$

By Eq. (1.2) and Figs. a and b,

$$\delta_a = \frac{P_s L_s}{E_s A_s} = \frac{P_a L_a}{E_a A_a} \qquad (b)$$

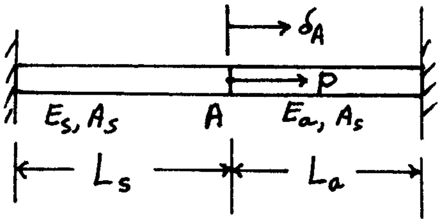


Figure a

By Eqs. (a) and (b),

$$P_s = \frac{P_a E_s A_s L_a}{E_a A_a L_s} = \frac{(P-P_s)(E_s A_s L_a)}{E_a A_a L_s}$$

Solving this equation for P_s, we find

$$P_s = \frac{P E_s A_s L_a}{E_s A_s L_a + E_a A_a L_s}$$. Hence, by Eq. (1.1),

the stress in the steel bar is

$$\sigma_s = \frac{P_s}{A_s} = \frac{P}{A_s}\left(\frac{E_s A_s L_a}{E_s A_s L_a + E_a A_a L_s}\right)$$

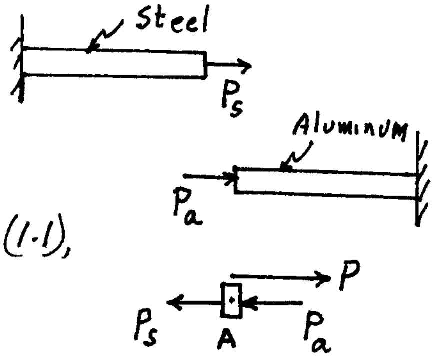


Figure b

Similarly, by Eqs. (a) and (b),

$$P_a = \frac{P_s E_a A_a L_s}{E_s A_s L_a} = \frac{(P-P_a)(E_a A_a L_s)}{E_s A_s L_a}$$

or

$$P_a = \frac{P E_a A_a L_s}{E_s A_s L_a + E_a A_a L_s}$$

Hence, the stress in the aluminum bar is

$$\sigma_a = \frac{P}{A_a}\left(\frac{E_a A_a L_s}{E_s A_s L_a + E_a A_a L_s}\right)$$

(cont.)

1.26 cont.

(b) When the left wall is displaced to the right by an amount δ, the point A is displaced to the right by an amount δ_A (Fig. c).

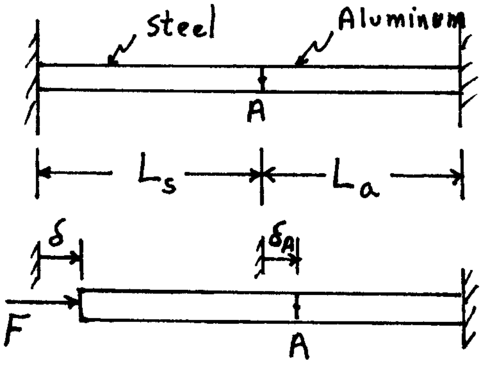


Figure C

By Eq. (1.2) and Fig. c,

$$\delta_A = \frac{FL_a}{E_aA_a} \quad (a)$$

and

$$\delta - \delta_A = \frac{FL_s}{E_sA_s} \quad (b)$$

By Eqs. (a) and (b),

$$\delta = \delta_A + \frac{FL_s}{E_sA_s} = F\left(\frac{L_a}{E_aA_a} + \frac{L_s}{E_sA_s}\right)$$

or

$$F = \frac{\delta E_aA_aE_sA_s}{E_sA_sL_a + E_aA_aL_s}$$

Hence, by Eq. (1.1), the stress in the steel bar is

$$\sigma_s = \frac{F}{A_s} = \frac{\delta E_aA_aE_s}{E_sA_sL_a + E_aA_aL_s}$$

and the stress in the aluminum bar is

$$\sigma_a = \frac{F}{A_a} = \frac{\delta E_aE_sA_s}{E_sA_sL_a + E_aA_aL_s}$$

1.27

(a) Consider the free-body diagram of a cable (Fig. a). By equilibrium,

$$\Sigma F_y = T - W = 0$$

$W = AL\rho g$, g = acceleration of gravity

$$A = \pi D^2/4$$

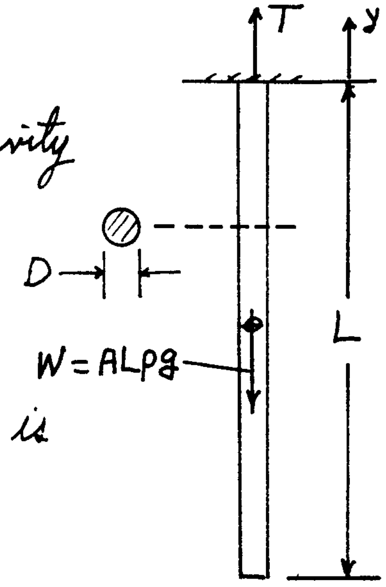


Figure a

Therefore,

$$T = AL\rho g \qquad (a)$$

So the maximum stress in the cable is

$$\sigma_{max} = \frac{T}{A} = L\rho g \qquad (b)$$

For the steel cable, by Eq. (b),

$$\sigma_{max} = \frac{1}{10}\sigma_u = \frac{1030}{10}\,\text{MPa} = L(7920)(9.81)$$

or

$$L = 1325.7 \text{ m} \qquad (c)$$

Similarly for the aluminum cable

$$\sigma_{max} = \frac{1}{10}\sigma_u = \frac{570}{10}\,\text{MPa} = L(2770)(9.81)$$

or

$$L = 2097.6 \text{ m} \qquad (d)$$

Consequently, the aluminum cable can be longer before exceeding a stress greater than $\frac{1}{10}\sigma_u$

(b) For a cable of length L subjected to a load P at its end, the elongation δ is [see Eq. (1.3)]

$$\delta = \frac{\sigma L}{E} \qquad (e)$$

(Cont.)

1.27 Cont. Now consider the elongation of an element of length dx cut from the cable (Fig. b). For an element of length dx, the elongation $d\delta$ is, by Eq. (e),

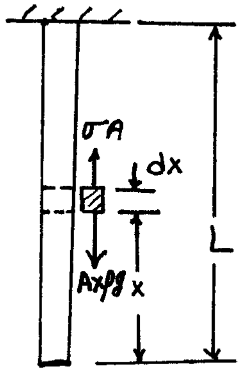


Figure b

$$d\delta = \frac{\sigma}{E}dx = \frac{1}{E}\left(\frac{Ax\rho g}{A}\right)dx \qquad (f)$$

where A is the cross-sectional area of the cable and $Ax\rho g$ is the weight of the part of the cable below the cross section at distance x from the lower end.

Integration of Eq. (f) from $x = 0$ to $x = L$ yields the total elongation Δ of the cable as

$$\Delta = \frac{1}{2}\,\frac{AL^2\rho g}{EA} = \frac{1}{2}\,\frac{WL}{EA} \qquad (g)$$

where $W = AL\rho g$ is the weight of the cable. Equations (g) and (1.2) show that the elongation of the cable due to its weight is equal to the elongation due to a load equal to half its weight applied at its end.

For a steel cable with $\sigma = \frac{1}{10}\sigma_u$, $L = 1325.7$ m, and Eq. (g) yields

$$\Delta = \tfrac{1}{2}(1325.7)^2(7920)(9.81)/(193\times10^9) = 0.354\ \text{m}$$

For an aluminum cable with $\sigma = \frac{1}{10}\sigma_u$, $L = 2097.6$ m, and

$$\Delta = \tfrac{1}{2}(2097.6)^2(2770)(9.81)/(72\times10^9) = 0.830\ \text{m}$$

(c) By Eq. (a), the tension at the top of the cable (Fig. a) due to the weight is $AL\rho g$. The tension at the top of the cable due to a load P applied at

(cont.)

1.27 cont. is P. So, the total tension is $P+AL\rho g$. Therefore, the maximum stress in the cable is

$$\sigma_{max} = \frac{1}{A}(P+AL\rho g) = \frac{P}{A} + L\rho g \qquad (h)$$

For $\sigma_{max} = \frac{1}{5}\sigma_u$, $L = 1000$ m and $D = 0.075$ m, Eq. (h) yields

For a steel cable: $P = 566.8$ kN

For an aluminum cable: $P = 383.6$ kN

The steel cable can lower a cage 1.478 times as heavy as an aluminum cable.

1.28 By Eq. (1.4), with $J = \pi d^4/32$ and $\rho = d/2$, we obtain at $\tau = 30$ MPa,

$$30\text{ MPa} = \frac{16T}{\pi d^3} \qquad (a)$$

and by Eq. (1.5), with $\psi/L = 0.005$ rad/m,

$$0.005\text{ rad/m} = \frac{32T}{G\pi d^4} \qquad (b)$$

Assume that the maximum shear stress and the maximum angle of twist occur simultaneously. Then, by Eqs. (a) and (b)

$$T = \frac{(30\times10^6)\pi d^3}{16} = \frac{(0.005)(77\times10^9)\pi d^4}{32}$$

or $d = 0.1558$ m. For $d < 0.1558$ m, Eq. (b) yields the smallest twisting moment T; that is, the maximum angle of twist occurs before the maximum shear stress.

1.29 (a) By Fig. a, the area A of the cross section of the beam is

$$A=(6.5)(200+300)=3250\ \text{mm}^2$$

and

$$A\bar{y}=(6.5)(300)(150)+(6.5)(200)(303.25)=686725\ \text{mm}^3$$

or

$$\bar{y}=211.3\ \text{mm} \qquad \text{(a)}$$

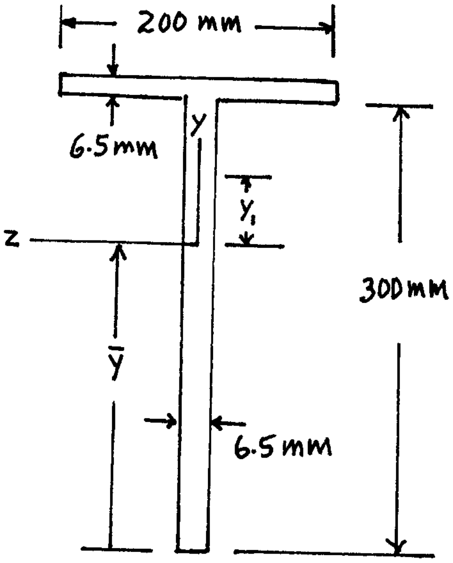


Figure a

(b) By Fig. b, the support reactions at A and B are determined by the equations

$$\sum F_y = A+B-21-(10)(3)=0$$

$$\overset{+}{\circlearrowleft}\ \sum M_A = 3B+(21)(1)-(10)(3)(1.5)=0 \qquad \text{(b)}$$

The solution of Eqs. (b) is

$$A=43\ \text{kN},\quad B=8\ \text{kN} \qquad \text{(c)}$$

With Fig. (b) and Eqs. (c), the shear and moment diagrams may be drawn (Fig. c).

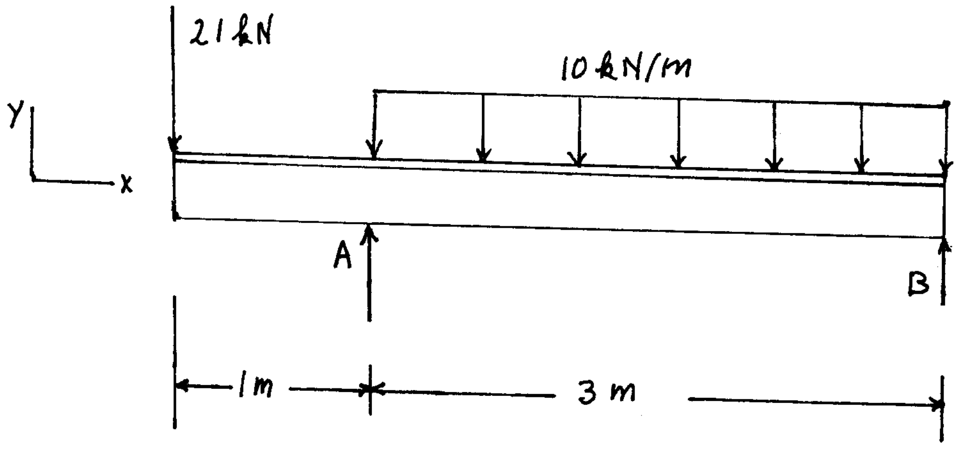


Figure b

(cont.)

1.29 cont.

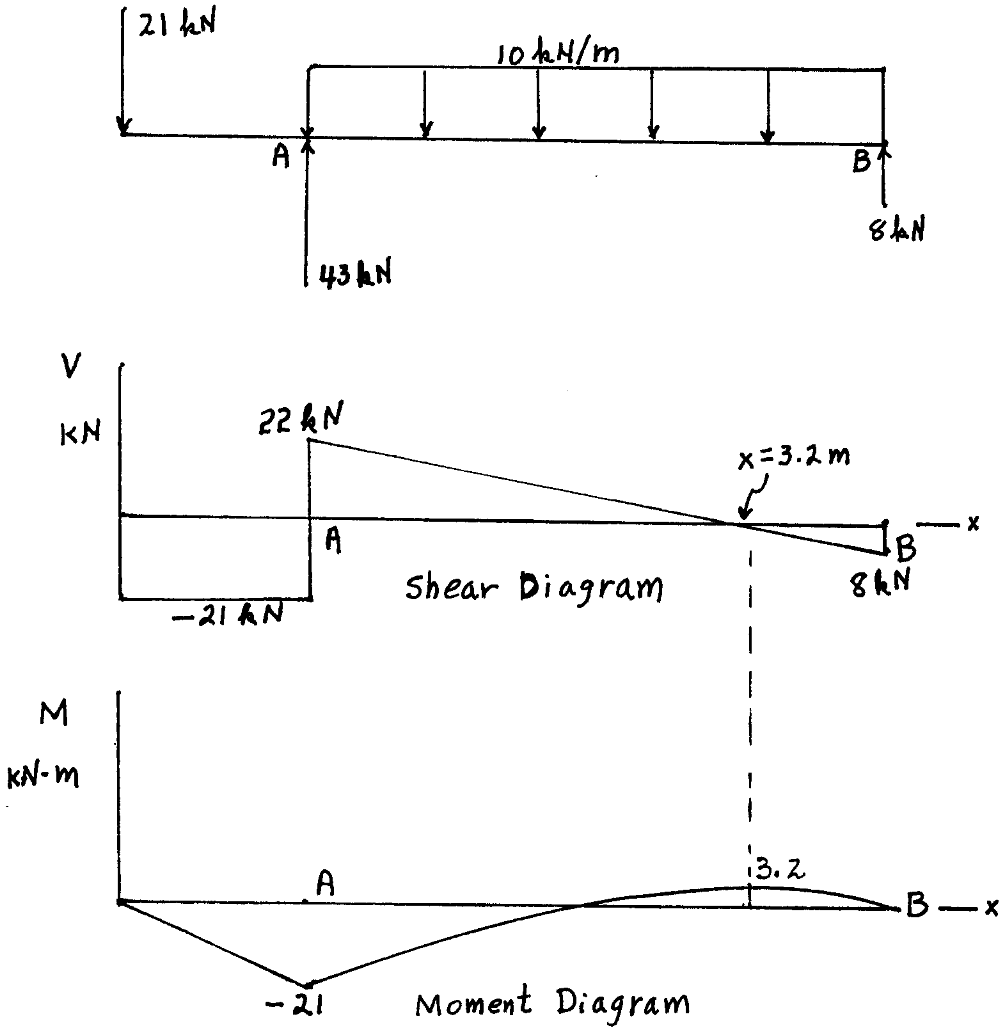


Figure c

1.30

By Fig. a, the moment of the cross-sectional area above y_1, with respect to the z axis is

$$Q = \int_{211.3}^{95.2} y\,dA = (6.5)\left[(88.7-y_1)\left(\frac{88.7+y_1}{2}\right)\right. + (200)(6.5)(91.95)$$

or

$$Q = 3.25(7867.7 - y_1^2) + 119535 \ [\mathrm{mm}^3] \quad (a)$$

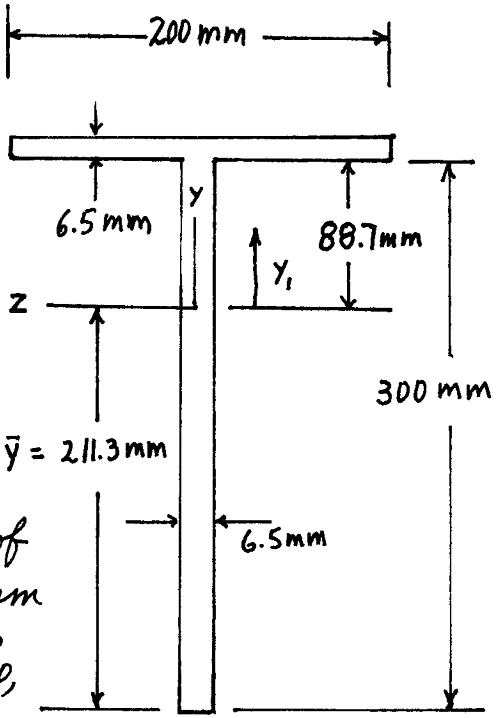


Figure a

By the shear diagram of the beam (Fig. b), the maximum shear occurs at section A and is $V = 22$ kN.

The area moment of inertia of the cross section of the T-beam is $I_z = 32.948 \times 10^6$ mm⁴. So, above the horizontal centroidal axis z, by Eqs. (1.9) and (a), the stress in the web is

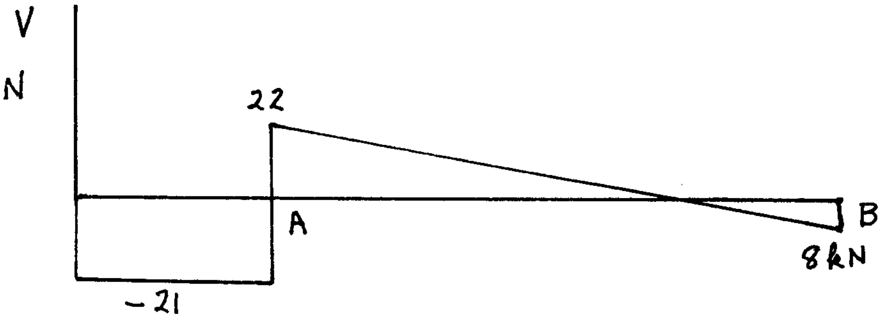


Figure b

$$\tau = \frac{VQ}{Ib} = \frac{(22)[3.25(7867.7 - y_1^2) + 119535]}{(32.948 \times 10^6)(6.5)} \quad [\mathrm{N \cdot mm^{-2}}] \ (b)$$

(cont.)

1.30 cont. By Eq. (b),

$\tau = 14.906$ MPa for $y_1 = 0$ (c)

$\tau = 12.279$ MPa for $y_1 = 88.7$ mm (d)

Below the horizontal centroidal axis z (Fig. c),

$$Q = (6.5)(211.3 - y_1)\left(\frac{211.3 + y_1}{2}\right) = 3.25(211.3^2 - y_1^2) \quad (e)$$

For $y_1 = 0$, Eq. (e) yields $Q = 145\,105$ mm^3, and therefore,

$$\tau = \frac{VQ}{Ib} = \frac{(22)(145\,105)}{(32.948\times10^6)(6.5)} = 14.906 \text{ MPa}$$

as given previously [Eq. (c)].

For $y_1 = 211.3$, $Q = 0$ and $\tau = 0$. Hence, the maximum shear stress is $\tau_{max} = 14.906$ MPa at section A on the centroidal axis z. The minimum shear stress is $\tau_{min} = 0$ at the bottom of the web.

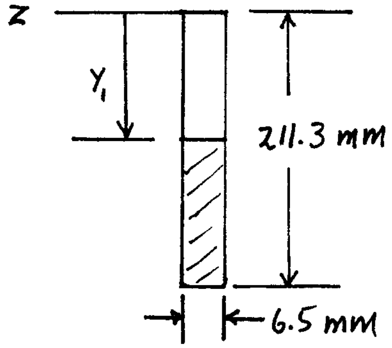


Figure c

1.31

The stress-strain curve is shown in Fig. a. By Fig. a, the ultimate strength is $\sigma_u \approx 665$ MPa. Since each rectangular box under the stress-strain curve represents $(100)(0.05) = 5$ MN·m/m^3 of energy and we estimate that there are approximately 29.5 boxes under the stress-strain curve, we find

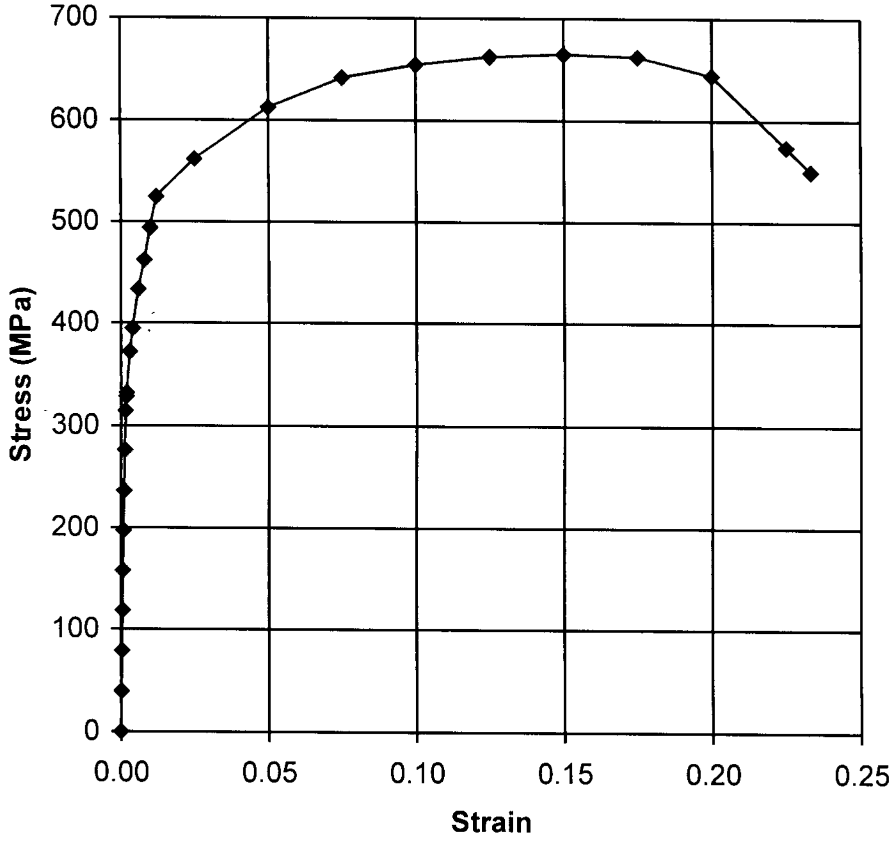


Figure a

that the modulus of toughness is

$$U_F = (5)(29.5) = 147.5 \text{ MN}\cdot\text{m/m}^3.$$

By numerical integration,

$$U_F = 144.2 \text{ MN}\cdot\text{m/m}^3$$

1.32

Extending the straight line portion of the stress-strain curve (Fig. a), we see that the slope of the line is

$$E = \frac{400}{.002} = 200 \text{ GPa}.$$

also, by Fig. a,

$$\sigma_{YS} = 395 \text{ MPa} \quad \text{and} \quad \sigma_{PL} = 315 \text{ MPa}$$

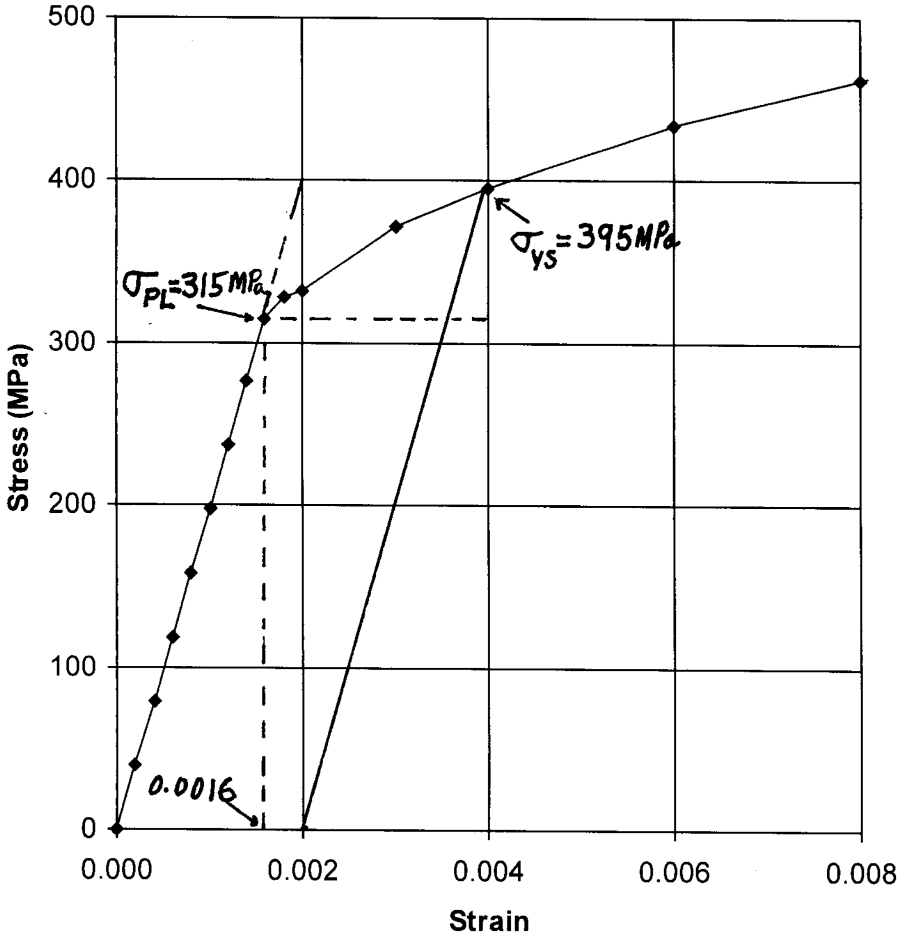


Figure a

The area under the stress-strain curve to the yield strength is approximately (see Fig. a)

$$\text{Area} = \tfrac{1}{2}(315)(0.0016) + (315)(0.004-0.016) + \tfrac{1}{2}(395-315)(0.004-0.0016)$$

or,

$$\text{Modulus of Resilience} = 1.104 \ \text{MN}\cdot\text{m}/\text{m}^3$$

1.33 The area of the test specimen is

$$A = \pi D^2/4 = \pi (20)^2/4 = 314.16 \text{ mm}^2$$

the stress at a load $P = 75.4$ kN is

$$\sigma = \frac{P}{A} = 240 \text{ MPa} \qquad (a)$$

The strain of the specimen is

$$\epsilon = \frac{\text{elongation}}{\text{gage length}} = \frac{0.330}{100} = 0.0033 \quad (b)$$

By Eqs. (a) and (b), the modulus of elasticity is

$$E = \frac{\sigma}{\epsilon} = 72.73 \text{ GPa}$$

The radial strain in the bar at a load of 75.4 kN is

$$\epsilon_r = \frac{19.978 - 20.0}{20} = -0.0011 \qquad (c)$$

By Eqs. (b) and (c), the Poisson ratio is

$$\nu = -\frac{\epsilon_r}{\epsilon} = 0.33$$

Since the load and elongation are proportional up to $P = 75.4$ kN, the proportional limit is

$$\sigma_{PL} = \frac{P}{A} = \frac{75.4}{314.16} = 240 \text{ MPa}$$

1.34

The initial area of the test specimen was

$$A_0 = \pi D^2/4 = \pi (10)^2/4 = 78.54 \text{ mm}^2 \quad (a)$$

Since the reduction of area of the specimen is 55%, the area at fracture is, with Eq. (a),

$$A_F = (0.45) A_0 = 35.34 \text{ mm}^2 \quad (b)$$

Therefore, the true fracture stress is, with $P = 43.2$ kN at fracture (see Table P1.31),

$$\sigma_t = \frac{43.2}{35.34} = 1220 \text{ MPa}$$

The engineering fracture stress is

$$\sigma_E = \frac{43.2}{78.54} = 550 \text{ MPa}$$

The ratio $\sigma_E/\sigma_t = \left(\frac{550}{1220}\right)(100\%) = 45\%$.

This result was to be expected, since the area at fracture is 45% of the original area

2.1 With $\sigma_{xx} = 50$ MPa, $\sigma_{yy} = -30$ MPa, $\sigma_{zz} = 20$ MPa, $\sigma_{xy} = 5$ MPa, $\sigma_{xz} = -30$ MPa, $\sigma_{yz} = 0$, and $l = m = n = 1/\sqrt{}$, Eq. (2.10) yields

$$\sigma_{Px} = l\sigma_{xx} + m\sigma_{yx} + n\sigma_{zx} = \frac{1}{\sqrt{3}}(50 + 5 - 30) = 14.434 \text{ MPa}$$

$$\sigma_{Py} = l\sigma_{xy} + m\sigma_{yy} + n\sigma_{zy} = \frac{1}{\sqrt{3}}(5 - 30 + 0) = -14.434 \text{ MPa}$$

$$\sigma_{Pz} = l\sigma_{xz} + m\sigma_{yz} + n\sigma_{zz} = \frac{1}{\sqrt{3}}(-30 + 0 + 20) = -5.774 \text{ MPa}$$

then, by Eq. (2.11),

$$\sigma_{PN} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2mn\sigma_{yz} + 2ln\sigma_{xz} + 2lm\sigma_{xy}$$
$$= \frac{1}{3}[50 - 30 + 20 + 2(0) + 2(-30) + 2(5)] = -3.333 \text{ MPa}$$

Therefore, by Eq. (2.12),

$$\sigma_{PS} = \sqrt{\sigma_{Px}^2 + \sigma_{Py}^2 + \sigma_{Pz}^2 - \sigma_{PN}^2} = \sqrt{(14.434)^2 + (-14.434)^2 + (-5.774)^2 - (-3.333)^2}$$

or

$$\sigma_{PS} = 20.950 \text{ MPa}$$

2.2 By Figure P2.2, the section cut off by plane A-A is shown in Fig. b. By equilibrium of forces,

$$\Sigma F_x = -\sigma(S\sin\theta) - \tau_0(S\cos\theta) - \tau_0 S(\cos\theta) = 0$$

$$\Sigma F_y = -\tau(S\sin\theta) + \tau_0 S(\sin\theta) = 0$$

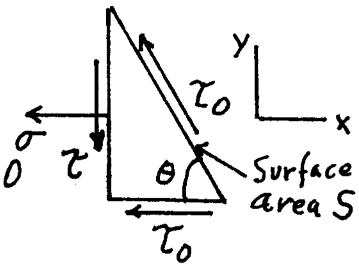


Figure a

Therefore,

$$\sigma = -2\tau_0\cot\theta \text{ (compression)}$$

$$\tau = \tau_0$$

2.3

(a) By Eqs. (2.21), with the given stress components,

$$I_1 = \sigma_{xx} = 20, \quad I_2 = -\sigma_{xz}^2 = -300, \quad I_3 = 0 \qquad (a)$$

By Eqs. (a) and (2.20), $\sigma^3 - 20\sigma^2 - 300\sigma = 0$, or $\sigma(\sigma - 30)(\sigma + 10) = 0$. Hence, $\sigma = 0$, $\sigma = 30$, $\sigma = -10$.

Ordering the stresses, we have

$$\sigma_1 = 30 \text{ MPa}, \quad \sigma_2 = 0, \quad \sigma_3 = -10 \text{ MPa}$$

(b) With $\sigma = \sigma_1$, Eqs. (2.18) yield

$$\begin{aligned}
(20-30)l + (0)m + 10\sqrt{3}\,n &= 0 \Rightarrow l = \sqrt{3}\,n && (b)\\
-30m &= 0 \Rightarrow m = 0 && (c)\\
10\sqrt{3}\,l + (0)m - 30n &= 0 && (d)
\end{aligned}$$

Where

$$l^2 + m^2 + n^2 = 1 \qquad (e)$$

The solution of Eqs. (b), (c), and (e) is

$$l = \pm\sqrt{3}/2, \quad m = 0, \quad n = \pm 1/2 \qquad (f)$$

Note that Eqs. (f) satisfy Eq. (d) identically.

2.4

(a) By Eq. (2.19) and the given stress components, the principal stresses are the roots of

$$\begin{vmatrix} (-100-\sigma) & 0 & -80 \\ 0 & (20-\sigma) & 0 \\ -80 & 0 & (20-\sigma) \end{vmatrix} = 0$$

or

$$(20-\sigma)(\sigma-60)(\sigma+140) = 0$$

Hence, $\sigma_1 = 60 \text{ MPa}, \quad \sigma_2 = 20 \text{ MPa}, \quad \sigma_3 = -140 \text{ MPa} \qquad (a)$

(Cont.)

2.4 cont.

(b) By Eqs.(2.22) and (a),

$$9\tau_{oct}^2 = (\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2 = 67200$$

or

$$\tau_{oct} = 86.41\ \text{MPa} \qquad \text{(b)}$$

(c) By Eqs. (2.39) and (a),

$$\tau_{max} = \frac{1}{2}(\sigma_1-\sigma_3) = 100\ \text{MPa} \qquad \text{(c)}$$

By Eqs. (b) and (c),

$$\tau_{max} = 1.16\,\tau_{oct}$$

(d) By Mohr's circle in the x-z plane (Fig. a), the maximum shear stress occurs on planes for which $2\theta_1 = 90^\circ$ (points P and P'); that is, on planes for which the direction cosines are $(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}})$, relative to principal stress axes, and planes perpendicular to these planes (Fig. b).

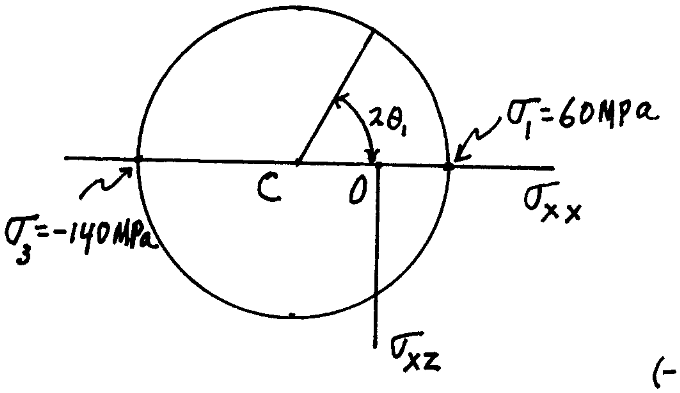
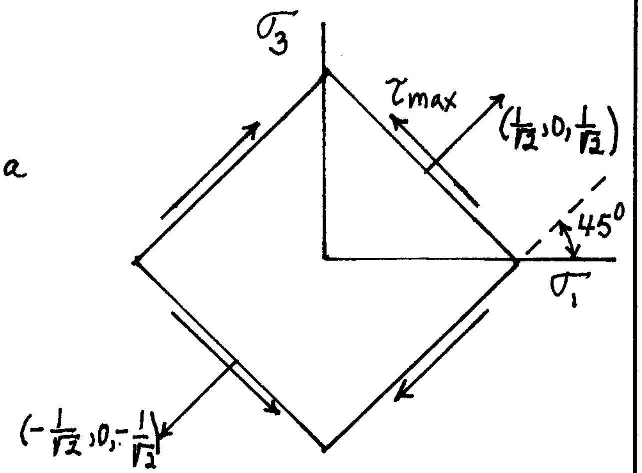


Figure a



σ_3
τ_{max}
$(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$
45^0
σ_1
$(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

Figure b

2.5

(a) a unit vector in the direction of the vector $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ has direction cosines

$$l = \frac{1}{\sqrt{6}}, \quad m = \frac{2}{\sqrt{6}}, \quad n = \frac{1}{\sqrt{6}} \qquad (a)$$

Hence, by Eqs. (a) and (2.10) and the given stress components, the stress vector on the plane normal to the vector $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ has components

$$\begin{aligned} \sigma_{Px} &= l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz} = \frac{1}{\sqrt{6}}[80 + 2(20) + 40] = \frac{160}{\sqrt{6}} \\ \sigma_{Py} &= l\sigma_{xy} + m\sigma_{yy} + n\sigma_{zy} = \frac{1}{\sqrt{6}}[20 + 2(60) + 10] = \frac{150}{\sqrt{6}} \qquad (b) \\ \sigma_{Pz} &= l\sigma_{xz} + m\sigma_{yz} + n\sigma_{zz} = \frac{1}{\sqrt{6}}[40 + 2(10) + 20] = \frac{80}{\sqrt{6}} \end{aligned}$$

By Eqs. (b) and (2.9), the stress vector is

$$\underset{\sim}{\sigma}_P = 65.32\hat{\imath} + 61.24\hat{\jmath} + 32.66\hat{k}$$

(b) With the given stress components and Eqs. (2.21), the stress invariants are

$$I_1 = 80 + 60 + 20 = 160, \quad I_2 = 5500, \quad I_3 = 0 \qquad (c)$$

By Eqs. (c) and (2.20),

$$\sigma^3 - 160\sigma^2 + 5500\sigma + 0 = 0$$

or

$$\sigma(\sigma^2 - 160\sigma + 5500) = 0 \qquad (d)$$

The roots of Eq. (d) are

$$\sigma_1 = 110 \text{ MPa}, \quad \sigma_2 = 50 \text{ MPa}, \quad \sigma_3 = 0$$

(c) The maximum shear stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 55 \text{ MPa}$$

(cont.)

2.5 cont. (d) The octahedral shear stress is given by

$$\tau_{oct}^2 = \frac{1}{9}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right] = \frac{18200}{9}$$

or

$$\tau_{oct} = 44.97 \text{ MPa}$$

2.6 (a) Given

$$T = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix} \qquad (a)$$

By Eqs. (a) and (2.21), the stress invariants of T are

$$I_1 = 4+6+8 = 18$$

$$I_2 = \begin{vmatrix} 4 & 1 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix} + \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} = 99 \qquad (b)$$

$$I_3 = \begin{vmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{vmatrix} = 160$$

(b) The 45° rotation of axes (X, Y) is shown in Fig. a. The direction cosines are given in Table a. Then, by Eqs. (a) and (2.15), with Table a,

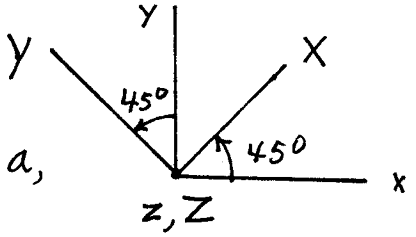


Figure a

$$\sigma_{XX} = \left(\tfrac{1}{\sqrt{2}}\right)^2(4) + \left(\tfrac{1}{\sqrt{2}}\right)^2(6) + (0)(8) + 2\left(\tfrac{1}{\sqrt{2}}\right)(0)(0) + 2(0)\left(\tfrac{1}{\sqrt{2}}\right)(2) + 2\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right)(1) = 6 \text{ MPa}$$

Similarly,

$$\sigma_{YY} = 4 \text{ MPa}, \quad \sigma_{ZZ} = 8 \text{ MPa}$$

Table a

	x	y	z
X	$1/\sqrt{2}$	$1/\sqrt{2}$	0
Y	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
Z	0	0	1

(cont.)

2.6 cont. Likewise by Eqs.(a) and (2.17), with Table a,

$$\sigma_{XY} = 1\ \text{MPa},\quad \sigma_{XZ} = \sqrt{2}\ \text{MPa},\quad \sigma_{YZ} = -\sqrt{2}\ \text{MPa}$$

(c) By Eqs.(2.21) and the stress components relative to axes (X, Y, Z), determined in part b,

$$I_1 = \sigma_{XX} + \sigma_{YY} + \sigma_{ZZ} = 6 + 4 + 8 = 18$$

$$I_2 = 99\ \text{MPa},\quad I_3 = 160$$

These invariants are the same as those relative to axes (x, y, z), Eqs.(b).

2.7

For body A, $\sigma_{xx} = \sigma_0$, $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$

Therefore for body A, by Eq.(2.23),

$$9\tau_{oct}^2 = 2\sigma_0^2$$

or

$$\text{For body A: } \tau_{oct} = \frac{\sqrt{2}}{3}\sigma_0 \qquad (a)$$

For body B, $\sigma_{xy} = \tau_0$, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$.

Then, by Eq.(2.23),

$$9\tau_{oct}^2 = 6\tau_0^2$$

or

$$\text{For body B: } \tau_{oct} = \frac{\sqrt{6}}{3}\tau_0 \qquad (b)$$

Equating Eqs (a) and (b), we obtain

$$\frac{\sigma_0}{\tau_0} = \sqrt{3}$$

2.8 (a) Since $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = \tau_0$, by Eqs. (2.21), the stress invariants are

$$I_1 = 0, \quad I_2 = -3\tau_0^2, \quad I_3 = 2\tau_0^3 \qquad (a)$$

By Eqs. (a) and (2.20), the principal stresses are the roots of

$$\sigma^3 - 3\tau_0^2\sigma - 2\tau_0^3 = 0$$

or

$$(\sigma - 2\tau_0)(\sigma + \tau_0)^2 = 0$$

Therefore,

$$\sigma_1 = 2\tau_0, \quad \sigma_2 = \sigma_3 = -\tau_0$$

(b) Since $\sigma_2 = \sigma_3 = -\tau_0$ (double roots), any two mutually perpendicular axes that are also perpendicular to the principal axis associated with the principal stress σ_1 serve as two principal axes. The third principal axis has direction cosines (l, m, n) where [see Eq. (2.18)]

$$-2\tau_0 l + \tau_0 m + \tau_0 n = 0$$
$$\tau_0 l - 2\tau_0 m + \tau_0 n = 0$$
$$\tau_0 l + \tau_0 m - 2\tau_0 n = 0$$
$$l^2 + m^2 + n^2 = 1$$

The roots of these equations are

$$l = \pm\frac{1}{\sqrt{3}}, \quad m = \pm\frac{1}{\sqrt{3}}, \quad n = \pm\frac{1}{\sqrt{3}}$$

2.9 By Eq. (2.22),

$$9\tau_{oct}^2 = (\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2$$

where σ_1 and σ_3 are known. To determine σ_2 such that τ_{oct} attains an extreme value, we must have

$$\frac{\partial}{\partial\sigma_2}\left(9\tau_{oct}^2\right) = 0$$

or

$$-2(\sigma_1-\sigma_2) + 2(\sigma_2-\sigma_3) = 0$$

Hence

$$\sigma_2 = \tfrac{1}{2}(\sigma_1+\sigma_3)$$

2.10 (a) With $\sigma_{xx} = -90$ MPa, $\sigma_{yy} = 50$ MPa, and $\sigma_{xy} = 6$ MPa, Eqs. (2.21) yield

$$I_1 = -40, \quad I_2 = -4536, \quad I_3 = 0 \qquad \text{(a)}$$

Then, by Eq. (2.20), $\sigma^3 + 40\sigma^2 - 4536\sigma = 0$

The roots (principal stresses) are

$$\sigma_1 = 50.257 \text{ MPa}, \quad \sigma_2 = 0, \quad \sigma_3 = -90.257 \text{ MPa} \qquad \text{(b)}$$

(b) By Eqs. (b) and (2.39),

$$\tau_{max} = \tfrac{1}{2}(\sigma_1-\sigma_3) = 70.257 \text{ MPa}$$

(c) By Eqs. (a) and (2.22),

$$9\tau_{oct}^2 = 2I_1^2 - 6I_2 = 30{,}416$$

or

$$\tau_{oct} = 58.134 \text{ MPa}$$

(cont.)

2.10 cont. (d) With $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ given and $\sigma = \sigma_1 = 50.257$ MPa Eqs. (2.18) yield

$$\begin{aligned} -140.257\,l + 6m &= 0 \\ 6\,l - 0.257m &= 0 \\ -50.257n &= 0 \qquad \text{(c)} \\ l^2 + m^2 + n^2 &= 1 \end{aligned}$$

The solution of Eqs. (c) is

$$l = 0.0427, \quad m = 0.998, \quad n = 0$$

Hence, the angle between axes x and X is given by

$$l = \cos\theta = 0.0427$$

or

$$\theta = 87.55^\circ$$

Counterclockwise from the x axis, approximately in the y direction.

2.11 By Eq. (2.6), the stress vectors $\underset{\sim}{\sigma}_x, \underset{\sim}{\sigma}_y, \underset{\sim}{\sigma}_z$ are, in terms of $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$,

$$\begin{aligned} \underset{\sim}{\sigma}_x &= \sigma_{xx}\hat{i} + \sigma_{xy}\hat{j} + \sigma_{xz}\hat{k} \\ \underset{\sim}{\sigma}_y &= \sigma_{xy}\hat{i} + \sigma_{yy}\hat{j} + \sigma_{zy}\hat{k} \qquad \text{(a)} \\ \underset{\sim}{\sigma}_z &= \sigma_{xz}\hat{i} + \sigma_{yz}\hat{j} + \sigma_{zz}\hat{k} \end{aligned}$$

By the scalar product of vectors, and Eqs. (a)

$$\begin{aligned} \underset{\sim}{\sigma}_x^2 = \underset{\sim}{\sigma}_x \cdot \underset{\sim}{\sigma}_x &= \sigma_{xx}^2 + \sigma_{xy}^2 + \sigma_{xz}^2 \\ \underset{\sim}{\sigma}_y^2 = \underset{\sim}{\sigma}_y \cdot \underset{\sim}{\sigma}_y &= \sigma_{xy}^2 + \sigma_{yy}^2 + \sigma_{zy}^2 \qquad \text{(b)} \\ \underset{\sim}{\sigma}_z^2 = \underset{\sim}{\sigma}_z \cdot \underset{\sim}{\sigma}_z &= \sigma_{xz}^2 + \sigma_{yz}^2 + \sigma_{zz}^2 \end{aligned}$$

(cont.)

2.11 cont.

Adding Eqs. (b), we obtain

$$\underset{\sim}{\sigma}_x^2+\underset{\sim}{\sigma}_y^2+\underset{\sim}{\sigma}_z^2=\sigma_{xx}^2+\sigma_{yy}^2+\sigma_{zz}^2+2\sigma_{xy}^2+2\sigma_{xz}^2+2\sigma_{yz}$$

or

$$\underset{\sim}{\sigma}_x^2+\underset{\sim}{\sigma}_y^2+\underset{\sim}{\sigma}_z^2=I_1^2-2I_2$$

where

$$I_1=\sigma_{xx}+\sigma_{yy}+\sigma_{zz}$$

$$I_2=\sigma_{xx}\sigma_{yy}+\sigma_{xx}\sigma_{zz}+\sigma_{yy}\sigma_{zz}-\sigma_{xy}^2-\sigma_{xz}^2-\sigma_{yz}^2$$

are the first and second stress invariants

Therefore, $\underset{\sim}{\sigma}_x^2+\underset{\sim}{\sigma}_y^2+\underset{\sim}{\sigma}_z^2$ is an invariant.

2.12a With the given stress components (see Table P2.12a), the stress invariants are

$$I_1=21,\quad I_2=40,\quad I_3=-705 \qquad (a)$$

Therefore,

$$\sigma^3-21\sigma^2+40\sigma+705=0 \qquad (b)$$

The roots (principal stresses) of Eq. (b) are

$$\sigma_1=15.467,\ \sigma_2=10.063,\ \sigma_3=-4.530\ [\text{MPa}] \quad (c)$$

For the principal direction (l, m, n) associated with σ_1, we have

$$\begin{aligned}-0.467l-3m&=0\\-3l-19.467m+n&=0\\m-5.467n&=0\\l^2+m^2+n^2&=1\end{aligned} \qquad (d)$$

The solution of Eqs. (d) is

$$l=\mp 0.9874,\ m=\pm 0.1537,\ n=\pm 0.0281$$

(cont.)

2.12a cont. Similarly, principal axes associated with σ_2 and σ_3 may be determined.

With Eqs. (c), the maximum shear stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 10.0 \text{ MPa}$$

also, by Eqs. (c) and (2.22), we have

$$9\tau_{oct}^2 = (\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2 = 642.04$$

or

$$\tau_{oct} = 8.45 \text{ MPa}$$

2.12b By Table P2.12b, $\sigma_{xx} = 10$ MPa, $\sigma_{yy} = -5$ MPa, $\sigma_{xy} = -5$ MPa. This is a case of plane stress relative to the xy plane. The stress invariants are

$$I_1 = \sigma_{xx} + \sigma_{yy} = 5, \quad I_2 = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = -75, \quad I_3 = 0$$

Hence, $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 5\sigma^2 - 75\sigma = 0$

or $\sigma(\sigma^2 - 5\sigma - 75) = 0$

So, $\sigma_1 = 11.514, \quad \sigma_2 = 0, \quad \sigma_3 = -6.514$ (a)

By the theory of plane stress [Eq. (2.)], the orientation of the principal axes is given by

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = -\frac{10}{15}$$

or

$$2\theta = 146.31^\circ$$

Hence, $\theta = 73.15^\circ$, $\theta + 90^\circ = 163.15^\circ$. So, the principal axes are located at $\theta_1 = 73.15^\circ$ and $\theta_3 = 163.15^\circ$, measured counterclockwise from the x axis.

(Cont.)

2.12b cont. By Eqs. (a) and (2.39),

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(11.514 + 6.514) = 9.014 \text{ MPa}$$

and by Eqs. (a) and (2.22),

$$9\tau_{oct}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 = 500.01$$

or

$$\tau_{oct} = 7.454 \text{ MPa} < \tau_{max} = 9.014 \text{ MPa}$$

2.12c For stress components $\sigma_{xx} = -10$, $\sigma_{yy} = -5$, $\sigma_{zz} = 10$, $\sigma_{xy} = 2$, $\sigma_{xz} = 3$, and $\sigma_{yz} = 4$, the stress invariants are

$$I_1 = -5, \; I_2 = -129, \; I_3 = 713$$

Therefore,

$$\sigma^3 + 5\sigma^2 - 129\sigma - 713 = 0$$

The roots (principal stresses) are

$$\sigma_1 = 11.537, \; \sigma_2 = -5.705, \; \sigma_3 = -10.832 \; [\text{MPa}] \quad (a)$$

For the principal stress σ_1, the principal axis direction cosines l, m, n are the solution of the following equations

$$-21.537l + 2m + 3n = 0$$
$$2l - 16.537m + 4n = 0$$
$$3l + 4m - 1.537n = 0$$
$$l^2 + m^2 + n^2 = 1$$

Hence,

$$l = \pm 0.156, \; m = \pm 0.250, \; n = 0.956$$

Then, by Eqs (a) and (2.39)

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 11.185 \text{ MPa}$$

(cont.)

2.12c cont. By Eqs. (a) and (2.22),

$$9\tau_{oct}^2 = (\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2 = 824$$

or

$$\tau_{oct} = 9.57 \text{ MPa}$$

2.12d With $\sigma_{xx} = 10$, $\sigma_{yy} = \sigma_{zz} = -5$, $\sigma_{xy} = \sigma_{xz} = 2$, and $\sigma_{yz} = 0$, the stress invariants are

$$I_1 = 0,\ I_2 = -83,\ I_3 = 290$$

Therefore,

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 83\sigma - 290 = 0$$

The roots (principal stresses) are

$$\sigma_1 = 10.5156,\ \sigma_2 = -5.000,\ \sigma_3 = -5.5156 \text{ [MPa]} \quad (a)$$

For principal stress σ_1, we have

$$\begin{aligned} -0.5156l + 2m + 2n &= 0 \\ 2l - 15.5156m &= 0 \\ 2l - 15.5156n &= 0 \\ l^2 + m^2 + n^2 &= 1 \end{aligned}$$

Hence,

$$l = 0.9838,\ m = 0.1268,\ n = 0.1268$$

By Eqs. (a) and (2.39),

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 8.016 \text{ MPa}$$

and by Eqs. (a) and (2.22),

$$9\tau_{oct}^2 = (\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2 = 498.0$$

or

$$\tau_{oct} = 7.439 \text{ MPa}$$

2.12e With $\sigma_{xx} = 10$ and all other stress components zero, the stress invariants are

$$I_1 = 10, \ I_2 = I_3 = 0$$

Hence,

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 10\sigma^2 = 0$$

or

$$\sigma^2(\sigma - 10) = 0$$

So, the principal stresses are

$$\sigma_1 = 10, \ \sigma_2 = \sigma_3 = 0 \qquad (a)$$

Therefore, the principal directions are the x axis and any two mutually perpendicular axes that are perpendicular to the x axis.

By Eqs. (a) and (2.39),

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 5 \text{ MPa}$$

and by Eqs. (a) and (2.22),

$$9\tau_{oct}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 = 2\sigma_1^2 = 200$$

or

$$\tau_{oct} = 4.714 \text{ MPa}$$

2.13 $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$; $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -75 \\ -75 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -55 \\ -55 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 65 \\ 65 & 0 \end{vmatrix} = -12,875$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -75 & -55 \\ -75 & 0 & 65 \\ -55 & 65 & 0 \end{vmatrix} = 536,250$$

$\sigma^3 - 12,875 - 536,250 = 0$

$\sigma_1 = \underline{130.34 \text{ MPa}}$; $\sigma_3 = \underline{-76.71 \text{ MPa}}$; $\sigma_2 = \underline{-53.63 \text{ MPa}}$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{130.34 - (-76.71)}{2} = \underline{103.53 \text{ MPa}}$$

Substitute $\sigma = \sigma_1 = 130.34$ MPa into Eqs. (2.18)

$$\left.\begin{aligned} -130.34 l_1 - 75 m_1 - 55 n_1 &= 0 \\ -75 l_1 - 130.34 m_1 + 65 n_1 &= 0 \\ -55 l_1 + 65 m_1 - 130.34 n_1 &= 0 \\ l_1^2 + m_1^2 + n_1^2 &= 1 \end{aligned}\right\} l_1 = \underline{0.5789};\ m_1 = \underline{-0.6055};\ n_1 = \underline{-0.5462}$$

Substitute $\sigma = \sigma_3 = -76.71$ MPa into Eqs. (2.18)

$$\left.\begin{aligned} 76.71 l_3 - 75 m_3 - 55 n_3 &= 0 \\ -75 l_3 + 76.71 m_3 + 65 n_3 &= 0 \\ -55 l_3 + 65 m_3 + 76.71 n_3 &= 0 \\ l_3^2 + m_3^2 + n_3^2 &= 1 \end{aligned}\right\} l_3 = \underline{0.5893};\ m_3 = \underline{0.7736};\ n_3 = \underline{-0.2330}$$

Substitute $\sigma = \sigma_2 = -53.63$ MPa into Eqs. (2.18)

$$\left.\begin{aligned} 53.63 l_2 - 75 m_2 - 55 n_3 &= 0 \\ -75 l_2 + 53.63 m_2 + 65 n_2 &= 0 \\ -55 l_2 + 65 m_2 + 53.63 n_2 &= 0 \\ l_2^2 + m_2^2 + n_2^2 &= 1 \end{aligned}\right\} l_2 = \underline{0.5637};\ m_2 = \underline{-0.1869};\ n_2 = \underline{0.8046}$$

2.14 The direction cosines are listed in Table 2.2. Substitution into Eqs. (2.15) and (2.17) gives

$\sigma_{XX} = \underline{\sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\sigma_{xy} \sin\theta \cos\theta}$

$\sigma_{YY} = \underline{\sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\sigma_{xy} \sin\theta \cos\theta}$

$\sigma_{ZZ} = \underline{\sigma_{zz}}$

$\sigma_{XY} = \underline{-(\sigma_{xx} - \sigma_{yy}) \sin\theta \cos\theta + \sigma_{xy} (\cos^2\theta - \sin^2\theta)}$

$\sigma_{XZ} = \underline{0}$

$\sigma_{YZ} = \underline{0}$

These equations are identical with Eqs. (2.30).

2.15 $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 110 - 86 + 55 = 79$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 110 & 60 \\ 60 & -86 \end{vmatrix} + \begin{vmatrix} 110 & 0 \\ 0 & 55 \end{vmatrix} + \begin{vmatrix} -86 & 0 \\ 0 & 55 \end{vmatrix} = -11{,}740$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 110 & 60 & 0 \\ 60 & -86 & 0 \\ 0 & 0 & 55 \end{vmatrix} = -718{,}300$$

$$\sigma^3 - 79\sigma^2 - 11{,}740\sigma + 718{,}300 = 0$$

$\sigma_1 = \underline{126.9\text{ MPa}}$; $\sigma_3 = \underline{-102.9\text{ MPa}}$; $\sigma_2 = \underline{55.0\text{ MPa}}$

Substitute $\sigma = \sigma_1 = 126.9$ MPa into Eqs. (2.18)

$$\left.\begin{aligned} &(110-126.9)\ell_1 + 60m_1 + (0)n_1 = 0 \\ &60\ell_1 + (-86-126.9)m_1 + (0)n_1 = 0 \\ &(0)\ell_1 + (0)m_1 + (55-126.9)n_1 = 0 \\ &\ell_1^2 + m_1^2 + n_1^2 = 1 \end{aligned}\right\} \ell_1 = \underline{0.9625};\ m_1 = \underline{0.2712};\ n_1 = \underline{0}$$

Substitute $\sigma = \sigma_3 = -102.9$ MPa into Eqs. (2.18)

$$\left.\begin{aligned} &(110+102.9)\ell_3 + 60m_3 + (0)n_3 = 0 \\ &60\ell_3 + (-86+102.9)m_3 + (0)n_3 = 0 \\ &(0)\ell_3 + (0)m_3 + (55+102.9)n_3 = 0 \\ &\ell_3^2 + m_3^2 + n_3^2 = 1 \end{aligned}\right\} \ell_3 = \underline{0.2712};\ m_3 = \underline{-0.9625};\ n_3 = \underline{0}$$

Substitute $\sigma = \sigma_2 = 55.0$ MPa into Eqs. (2.18)

$$\left.\begin{aligned} &(110-55)\ell_2 + 60m_2 + (0)n_2 = 0 \\ &60\ell_2 + (-86-55)m_2 + (0)n_2 = 0 \\ &(0)\ell_2 + (0)m_2 + (55-55)n_2 = 0 \\ &\ell_2^2 + m_2^2 + n_2^2 = 1 \end{aligned}\right\} \ell_2 = m_2 = \underline{0};\ n_2 = \underline{-1.0000}$$

$$\sigma_{NS(max)} = \frac{\sigma_1 - \sigma_3}{2} = \frac{126.9-(-102.9)}{2} = \underline{114.9\text{ MPa}}$$

2.16

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \frac{110-86}{2} + \sqrt{\left(\frac{110+86}{2}\right)^2 + 60^2} = 12 + 114.9 = \underline{126.9\text{ MPa}}$$

$\sigma_3 = 12 - 114.9 = \underline{-102.9\text{ MPa}}$

$\sigma_2 = \sigma_{zz} = \underline{55.0\text{ MPa}}$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{126.9-(-102.9)}{2} = \underline{114.9\text{ MPa}}$$

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{2(60)}{110+86} = 0.6122;\quad \theta = \underline{0.2747}$$

$$\ell_1 = \cos\theta = \underline{0.9625};\quad m_1 = \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta = \underline{0.2712};\quad n_1 = \cos\frac{\pi}{2} = \underline{0}$$

2.17 Eq. (2.11) gives, with $l = \cos(-\pi/6) = \sqrt{3}/2$, $m = \sin(-\pi/6) = -1/2$, $n = 0$,

$$\sigma_{XX} = \left(\frac{\sqrt{3}}{2}\right)^2(90) + \left(-\frac{1}{2}\right)^2(-10) + 2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)(40) = \underline{30.36 \text{ MPa}}$$

Eqs. (2.10) give

$$\sigma_{Xx} = \frac{\sqrt{3}}{2}(90) + \left(-\frac{1}{2}\right)(40) = 57.94 \text{ MPa}$$

$$\sigma_{Xy} = \frac{\sqrt{3}}{2}(40) + \left(-\frac{1}{2}\right)(-10) = 39.64 \text{ MPa}$$

$$\sigma_{Xz} = 0$$

Eq. (2.12) gives

$$\sigma_{XY} = \sqrt{\sigma_{Xx}^2 + \sigma_{Xy}^2 + \sigma_{Xz}^2 - \sigma_{XX}^2} = \sqrt{57.94^2 + 39.64^2 - 30.36^2} = \underline{63.30 \text{ MPa}}$$

2.18

$$\sigma_1 = \frac{200+100}{2} + \sqrt{\left(\frac{200-100}{2}\right)^2 + (-50)^2} = 150 + 70.7 = \underline{220.7 \text{ MPa}}$$

$$\sigma_2 = 150 - 70.7 = \underline{79.3 \text{ MPa}}$$

$$\sigma_3 = \underline{0}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{220.7 - 0}{2} = \underline{110.4 \text{ MPa}}$$

$$\tan 2\theta_1 = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y} = \frac{2(-50)}{200-100} = -1.0000$$

$$\theta_1 = \underline{-0.3927 \text{ rad}}$$

X-axis located 0.3927 rad clockwise from x-axis

2.19

$$\sigma_1 = \frac{-90+50}{2} + \sqrt{\left(\frac{-90-50}{2}\right)^2 + 60^2} = -20 + 92.2 = \underline{72.2 \text{ MPa}}$$

$$\sigma_3 = -20 - 92.2 = \underline{-112.2 \text{ MPa}}$$

$$\sigma_2 = \underline{0}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{72.2 + 112.2}{2} = \underline{92.2 \text{ MPa}}$$

$$\tan 2\theta_1 = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y} = \frac{2(60)}{-90-50} = -0.8571$$

$$\theta_1 = \underline{-0.3543 \text{ rad}}$$

X-axis located 0.3543 rad clockwise from x-axis

2.20

$$\sigma_1 = \frac{80}{2} + \sqrt{\left(\frac{80}{2}\right)^2 + 30^2} = 40 + 50 = \underline{90 \text{ MPa}}$$

$$\sigma_2 = 40 - 50 = \underline{-10 \text{ MPa}}$$

$$\sigma_3 = \underline{-60 \text{ MPa}}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{90-(-60)}{2} = \underline{75 \text{ MPa}}$$

$$\tan 2\theta_1 = \frac{2\sigma_{xy}}{\sigma_x} = \frac{2(30)}{80} = 0.75$$

$$\theta_1 = \underline{0.3216 \text{ rad}}$$

X-axis located 0.3216 rad counterclockwise from x-axis

2.21

$$\sigma_1 = \frac{150+70}{2} + \sqrt{\left(\frac{150-70}{2}\right)^2 + (-45)^2} = 110 + 60.2 = \underline{170.2 \text{ MPa}}$$

$$\sigma_2 = 110 - 60.2 = \underline{49.8 \text{ MPa}}$$

$$\sigma_3 = \underline{-80 \text{ MPa}}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{170.2-(-80)}{2} = \underline{125.1 \text{ MPa}}$$

$$\tan 2\theta_1 = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y} = \frac{2(-45)}{150-70} = -1.1250$$

$$\theta_1 = \underline{-0.4221 \text{ rad}}$$

X-axis located 0.4221 rad clockwise from x-axis

2.22

$$\sigma_{XX} = 200\cos^2(-0.5000) + 100\sin^2(-0.5000) + 2(-50)\cos(-0.5000)\sin(-0.5000)$$
$$= \underline{219.1 \text{ MPa}}$$

$$\sigma_{XY} = -(200-100)\cos(-0.5000)\sin(-0.5000) - 50\left[\cos^2(-0.5000) - \sin^2(-0.5000)\right]$$
$$= \underline{15.1 \text{ MPa}}$$

2.23

$$\sigma_{XX} = -90\cos^2(0.1500) + 50\sin^2(0.1500) + 2(60)\cos(0.1500)\sin(0.1500)$$
$$= \underline{-69.1 \text{ MPa}}$$

$$\sigma_{XY} = -(-90-50)\cos(0.1500)\sin(0.1500) + 60\left[\cos^2(0.1500) - \sin^2(0.1500)\right]$$
$$= \underline{78.0 \text{ MPa}}$$

2.24

$$\sigma_{XX} = 80\cos^2(-1.0000) + 2(30)\cos(-1.0000)\sin(-1.0000)$$

$$= \underline{-3.9\text{ MPa}}$$

$$\sigma_{XY} = -80\cos(-1.0000)\sin(-1.0000) + 30[\cos^2(-1.0000) - \sin^2(-1.0000)]$$

$$= \underline{23.9\text{ MPa}}$$

2.25

$$\sigma_{XX} = 150\cos^2(0.7000) + 70\sin^2(0.7000) + 2(-45)\cos(0.7000)\sin(0.7000)$$

$$= \underline{72.5\text{ MPa}}$$

$$\sigma_{XY} = -(150-70)\cos(0.7000)\sin(0.7000) - 45[\cos^2(0.7000) - \sin^2(0.7000)]$$

$$= \underline{-47.1\text{ MPa}}$$

2.26

$$OC = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{200+100}{2} = 150\text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{200-100}{2}\right)^2 + (-50)^2}$$

$$= 70.7\text{ MPa}$$

From Problem 2.18, $2\theta_1 = 0.7854$

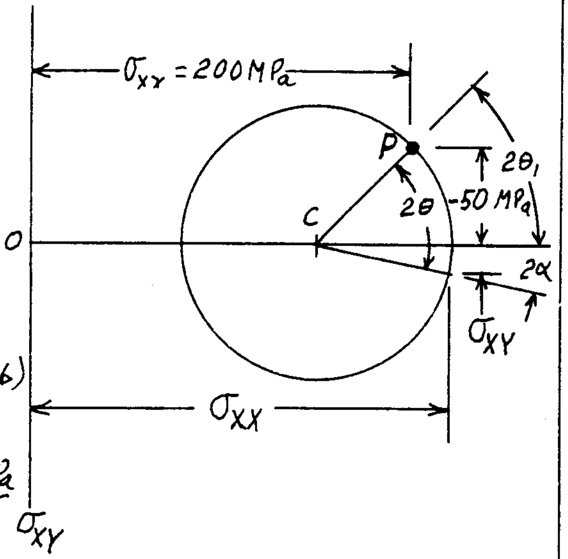
$$2\alpha = 2\theta - 2\theta_1 = 2(0.5000) - 0.7854$$

$$= 0.2146$$

$$\sigma_{XX} = OC + R\cos 2\alpha = 150 + 70.7\cos(0.2146)$$

$$= \underline{219.1\text{ MPa}}$$

$$\sigma_{XY} = R\sin 2\alpha = 70.7\sin(0.2146) = \underline{15.1\text{ MPa}}$$



2.27

$$OC = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{-90+50}{2} = -20\text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{-90-50}{2}\right)^2 + 60^2}$$

$$= 92.2\text{ MPa}$$

From Problem 2.19, $2\theta_1 = 0.7086$

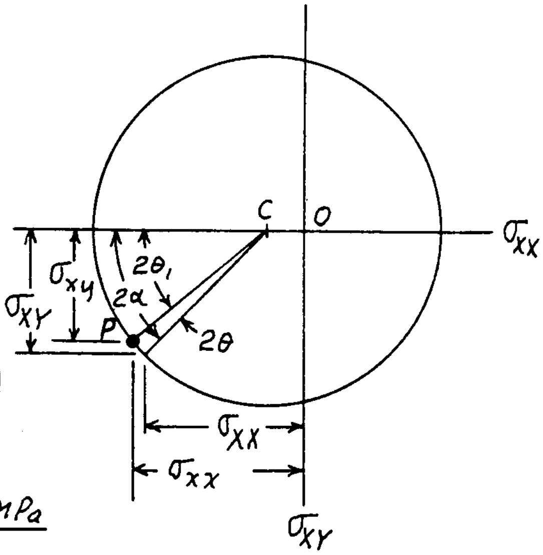
$$2\alpha = 2\theta_1 + 2\theta = 0.7086 + 2(0.1500)$$

$$= 1.0086$$

$$\sigma_{XX} = OC - R\cos 2\alpha = -20 - 92.2\cos(1.0086)$$

$$= \underline{-69.1\text{ MPa}}$$

$$\sigma_{XY} = R\sin 2\alpha = 92.2\sin(1.0086) = \underline{78.0\text{ MPa}}$$



2.28

$$OC = \frac{\sigma_{xx}}{2} = \frac{80}{2} = 40 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{80}{2}\right)^2 + 30^2}$$

$$= 50 \text{ MPa}$$

From Problem 2.20, $2\theta_1 = 0.6432$

$$2\alpha = \pi - 2\theta_1 - 2\theta = 3.1416 - 0.6432 - 2.0000$$

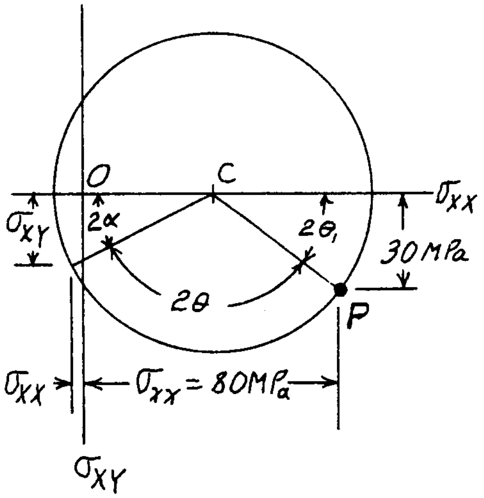
$$= 0.4984$$

$$\sigma_{XX} = OC - R\cos 2\alpha = 40 - 50\cos(0.4984)$$

$$= \underline{-3.9 \text{ MPa}}$$

$$\sigma_{XY} = R\sin 2\alpha = 50\sin(0.4984)$$

$$= \underline{23.9 \text{ MPa}}$$



2.29

$$OC = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{150 + 70}{2} = 110 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{150-70}{2}\right)^2 + (-45)^2}$$

$$= 60.2 \text{ MPa}$$

From Problem 2.21, $2\theta_1 = 0.8442$

$$2\alpha = \pi - 2\theta_1 - 2\theta = 3.1416 - 0.8442 - 1.4000$$

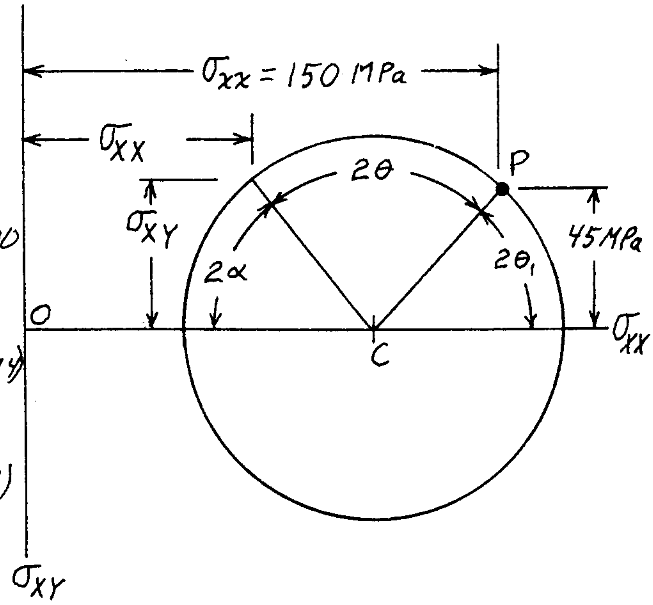
$$= 0.8974$$

$$\sigma_{XX} = OC - R\cos 2\alpha = 110 - 60.2\cos(0.8974)$$

$$= \underline{72.5 \text{ MPa}}$$

$$\sigma_{XY} = -R\sin 2\alpha = -60.2\sin(0.8974)$$

$$= \underline{-47.1 \text{ MPa}}$$

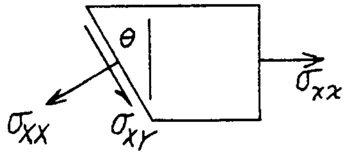


2.30

Due to free surface, $\sigma_{xy} = 0$. Since θ is positive and $\sigma_{yy} = 0$, Eqs. **(2.30)** give

$$\sigma_{XX} = \sigma_{xx}\cos^2\theta = \underline{100\cos^2\theta}$$

$$\sigma_{XY} = -(\sigma_{xx})\sin\theta\cos\theta = \underline{-100\sin\theta\cos\theta}$$



2.31 Since θ is positive, Eqs.(2.30) give

$$\sigma_{XX} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta$$

$$\sigma_{XY} = -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta$$

$$120 = \sigma_{xx}\frac{9}{25} + \sigma_{yy}\frac{16}{25}$$

$$70 = -(\sigma_{xx}-\sigma_{yy})\frac{4}{5}\,\frac{3}{5}$$

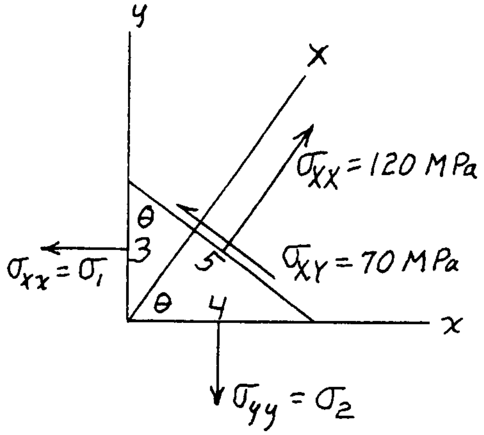
$$9\sigma_{xx} + 16\sigma_{yy} = 3000$$

$$16\sigma_{xx} - 16\sigma_{yy} = -2333.33$$

$$25\sigma_{xx} = 666.67$$

$$\sigma_{xx} = \underline{26.7\ \text{MPa}}$$

$$\sigma_{yy} = \underline{172.5\ \text{MPa}}$$



2.32 Since θ = 60°, Eqs. (2.30) give

$$\sigma_{XX} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta$$

$$\sigma_{XY} = -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$120 = 40(0.25) + 0.75\sigma_{yy} + 0.866\sigma_{xy}$$

$$-50 = -(40-\sigma_{yy})0.433 - 0.500\sigma_{xy}$$

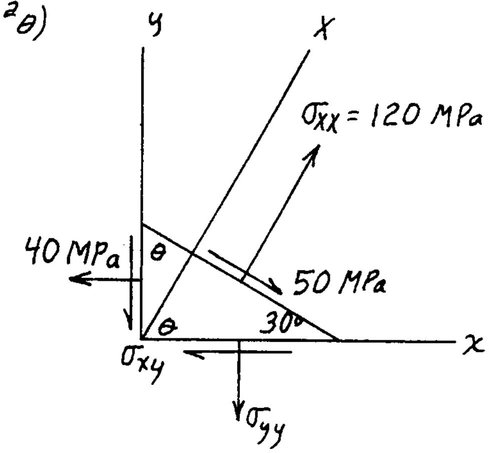
$$0.866\sigma_{yy} + \sigma_{xy} = 127.02$$

$$0.866\sigma_{yy} - \sigma_{xy} = -65.36$$

$$1.732\sigma_{yy} = 61.66$$

$$\sigma_{yy} = \underline{35.6\ \text{MPa}}$$

$$\sigma_{xy} = \underline{96.2\ \text{MPa}}$$



2.33 Since $\sin\theta = -\frac{5}{13}$ and $\cos\theta = \frac{12}{13}$, Eqs.(2.30) give

$$\sigma_{XX} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta$$

$$\sigma_{XY} = -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$-80 = \sigma_{xx}\frac{144}{169} + 40\,\frac{25}{169} + 2\sigma_{xy}\left(-\frac{60}{169}\right)$$

$$-60 = -(\sigma_{xx}-40)\left(-\frac{60}{169}\right) + \sigma_{xy}\,\frac{144-25}{169}$$

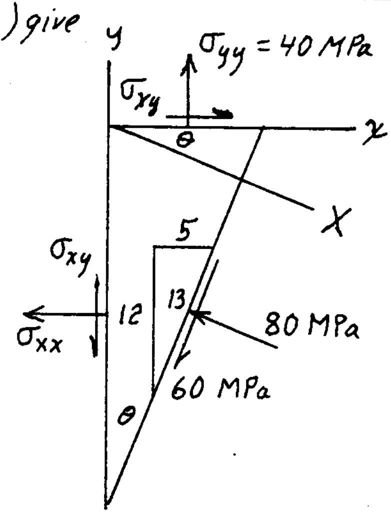
$$1.200\sigma_{xx} - \sigma_{xy} = -121.00$$

$$0.504\sigma_{xx} + \sigma_{xy} = -65.04$$

$$1.704\sigma_{xx} = -186.04$$

$$\sigma_{xx} = \underline{-109.2\ \text{MPa}}$$

$$\sigma_{xy} = \underline{-10.0\ \text{MPa}}$$



2.34

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \frac{-100+60}{2} + \sqrt{\left(\frac{-100-60}{2}\right)^2 + 50^2} = -20+94.3 = \underline{74.3\ \text{MPa}}$$

$$\sigma_3 = -20-94.3 = \underline{-114.3\ \text{MPa}}$$

$$\sigma_2 = \underline{0}$$

$$\tau_{max} = \frac{\sigma_{max}-\sigma_{min}}{2} = \frac{74.3+114.3}{2} = \underline{94.3\ \text{MPa}}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + \sigma_{xx}^2 + \sigma_{yy}^2 + 6\sigma_{xy}^2} = \frac{1}{3}\sqrt{(-100-60)^2 + (-100)^2 + 60^2 + 6(50^2)}$$

$$= \underline{77.6\ \text{MPa}}$$

2.35

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \frac{180+90}{2} + \sqrt{\left(\frac{180-90}{2}\right)^2 + 50^2} = 135+67.3 = \underline{202.3\ \text{MPa}}$$

$$\sigma_2 = 135-67.3 = \underline{67.7\ \text{MPa}}$$

$$\sigma_3 = \underline{0}$$

$$\tau_{max} = \frac{\sigma_{max}-\sigma_{min}}{2} = \frac{202.3}{2} = \underline{101.2\ \text{MPa}}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + \sigma_{xx}^2 + \sigma_{yy}^2 + 6\sigma_{xy}^2} = \frac{1}{3}\sqrt{(180-90)^2 + 180^2 + 90^2 + 6(50)^2}$$

$$= \underline{84.1\ \text{MPa}}$$

2.36

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \frac{-150-70}{2} + \sqrt{\left(\frac{-150+70}{2}\right)^2 + (-60)^2} = -110+72.1 = \underline{-37.9\ \text{MPa}}$$

$$\sigma_3 = -110-72.1 = \underline{-182.1\ \text{MPa}}$$

$$\sigma_2 = \sigma_{zz} = \underline{40\ \text{MPa}}$$

$$\tau_{max} = \frac{\sigma_{max}-\sigma_{min}}{2} = \frac{40+182.1}{2} = \underline{111.1\ \text{MPa}}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6\sigma_{xy}^2} = \frac{1}{3}\sqrt{(-150+70)^2 + (-70-40)^2 + (40+150)^2 + 6(-60)^2}$$

$$= \underline{92.0\ \text{MPa}}$$

2.37

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \frac{80-35}{2} + \sqrt{\left(\frac{80+35}{2}\right)^2 + 45^2} = 22.5+73.0 = \underline{95.5\ \text{MPa}}$$

$$\sigma_3 = 22.5-73.0 = \underline{-50.5\ \text{MPa}}$$

$$\sigma_2 = \sigma_{zz} = \underline{-50.0\ \text{MPa}}$$

$$\tau_{max} = \frac{\sigma_{max}-\sigma_{min}}{2} = \frac{95.5+50.5}{2} = \underline{73.0\ \text{MPa}}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6\sigma_{xy}^2} = \frac{1}{3}\sqrt{(80+35)^2 + (-35+50)^2 + (-50-80)^2 + 6(45)^2}$$

$$= \underline{68.7\ \text{MPa}}$$

2.38

$$\sigma_1 = \frac{\sigma_{xx}}{2} + \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2} = \frac{95}{2} + \sqrt{\left(\frac{95}{2}\right)^2 + (-55)^2} = 47.5 + 72.7 = \underline{120.2\text{ MPa}}$$

$$\sigma_3 = 47.5 - 72.7 = \underline{-25.2\text{ MPa}}$$

$$\sigma_2 = \sigma_{zz} = \underline{60\text{ MPa}}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{120.2 + 25.2}{2} = \underline{72.7\text{ MPa}}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{\sigma_{xx}^2 + \sigma_{zz}^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6\sigma_{xy}^2} = \frac{1}{3}\sqrt{95^2 + 60^2 + (60-95)^2 + 6(-55)^2}$$

$$= \underline{59.6\text{ MPa}}$$

2.39

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -120 + 140 + 66 = 86$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} -120 & 45 \\ 45 & 140 \end{vmatrix} + \begin{vmatrix} -120 & 25 \\ 25 & 66 \end{vmatrix} + \begin{vmatrix} 140 & -65 \\ -65 & 66 \end{vmatrix} = -22{,}355$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} -120 & 45 & 25 \\ 45 & 140 & -65 \\ 25 & -65 & 66 \end{vmatrix} = 969{,}200$$

$$\sigma^3 - 86\sigma^2 - 22{,}355\sigma + 969{,}200 = 0$$

$$\sigma_1 = \underline{180.2\text{ MPa}};\ \sigma_2 = \underline{40.06\text{ MPa}};\ \sigma_3 = \underline{-134.3\text{ MPa}}$$

Substitue $\sigma = \sigma_1 = 180.2$ MPa into Eqs (2.18)

$$\left.\begin{aligned} (-120-180.2)\ell_1 + 45m_1 + 25n_1 &= 0 \\ 45\ell_1 + (140-180.2)m_1 - 65n_1 &= 0 \\ 25\ell_1 - 65m_1 + (66-180.2)n_1 &= 0 \\ \ell_1^2 + m_1^2 + n_1^2 &= 1 \end{aligned}\right\} \ell_1 = \underline{0.0913};\ m_1 = \underline{0.8740};\ n_1 = \underline{-0.4773}$$

Substitue $\sigma = \sigma_2 = 40.06$ MPa into Eqs. (2.18)

$$\left.\begin{aligned} (-120-40.06)\ell_2 + 45m_2 + 25n_2 &= 0 \\ 45\ell_2 + (140-40.06)m_2 - 65n_2 &= 0 \\ 25\ell_2 - 65m_2 + (66-40.06)n_2 &= 0 \\ \ell_2^2 + m_2^2 + n_2^2 &= 1 \end{aligned}\right\} \ell_2 = \underline{0.2584};\ m_2 = \underline{0.4422};\ n_2 = \underline{0.8589}$$

Substitue $\sigma = \sigma_3 = -134.3$ MPa into Eqs (2.18)

$$\left.\begin{aligned} (-120+134.3)\ell_3 + 45m_3 + 25n_3 &= 0 \\ 45\ell_3 + (140+134.3)m_3 - 65n_3 &= 0 \\ 25\ell_3 - 65m_3 + (66+134.3)n_3 &= 0 \\ \ell_3^2 + m_3^2 + n_3^2 &= 1 \end{aligned}\right\} \ell_3 = \underline{0.9598};\ m_3 = \underline{-0.2062};\ n_3 = \underline{-0.1904}$$

2.40

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 100 \text{ MPa}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -60 \\ -60 & 100 \end{vmatrix} + \begin{vmatrix} 0 & 50 \\ 50 & 0 \end{vmatrix} + \begin{vmatrix} 100 & 35 \\ 35 & 0 \end{vmatrix}$$

$$= -7325$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -60 & 50 \\ -60 & 100 & 35 \\ 50 & 35 & 0 \end{vmatrix} = -460{,}000$$

$$\sigma^3 - 100\sigma^2 - 7325\sigma + 460{,}000 = 0$$

$$\sigma_1 = \underline{129.1 \text{ MPa}}$$

$$\sigma_2 = \underline{46.9 \text{ MPa}}$$

$$\sigma_3 = \underline{-76.0 \text{ MPa}}$$

2.41

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 120 - 55 - 85 = -20$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 120 & -55 \\ -55 & -55 \end{vmatrix} + \begin{vmatrix} 120 & -75 \\ -75 & -85 \end{vmatrix} + \begin{vmatrix} -55 & 33 \\ 33 & -85 \end{vmatrix}$$

$$= -21{,}864$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 120 & -55 & -75 \\ -55 & -55 & 33 \\ -75 & 33 & -85 \end{vmatrix} = 1{,}269{,}070$$

$$\sigma^3 + 20\sigma^2 - 21{,}864\sigma - 1{,}269{,}070 = 0$$

$$\sigma_1 = \underline{162.5 \text{ MPa}}$$

$$\sigma_2 = \underline{-68.4 \text{ MPa}}$$

$$\sigma_3 = \underline{-114.1 \text{ MPa}}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{162.5 + 114.1}{2} = \underline{138.3 \text{ MPa}}$$

2.42

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -90 - 60 + 40 = -110$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} -90 & 70 \\ 70 & -60 \end{vmatrix} + \begin{vmatrix} -90 & -55 \\ -55 & 40 \end{vmatrix} + \begin{vmatrix} -60 & -40 \\ -40 & 40 \end{vmatrix}$$

$$= -10{,}125$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} -90 & 70 & -55 \\ 70 & -60 & -40 \\ -55 & -40 & 40 \end{vmatrix} = 653{,}500$$

$$\sigma^3 + 110\sigma^2 - 10{,}125\sigma - 653{,}500 = 0$$

$\sigma_1 = \underline{88.34 \text{ MPa}}$

$\sigma_2 = \underline{-49.80 \text{ MPa}}$

$\sigma_3 = \underline{-148.54}$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{88.34 + 148.54}{2} = \underline{118.44 \text{ MPa}}$$

2.43

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -150 + 80 = -70$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} -150 & -40 \\ -40 & 0 \end{vmatrix} + \begin{vmatrix} -150 & 50 \\ 50 & 80 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 80 \end{vmatrix}$$

$$= -16{,}100$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xy} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} -150 & -40 & 50 \\ -40 & 0 & 0 \\ 50 & 0 & 80 \end{vmatrix} = 128{,}000$$

$$\sigma^3 + 70\sigma^2 - 16{,}100\sigma - 128{,}000$$

$\sigma_1 = \underline{91.2 \text{ MPa}}$

$\sigma_2 = \underline{8.3 \text{ MPa}}$

$\sigma_3 = \underline{-169.5 \text{ MPa}}$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{91.2 + 169.5}{2} = \underline{130.3 \text{ MPa}}$$

2.44 $\sigma_{xx} = -10,\ \sigma_{yy} = 30,\ \sigma_{xy} = 15,\ \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$

By Fig. 2.12, $OC = \dfrac{\sigma_{xx} + \sigma_{yy}}{2} = 10$ (Fig. a).

By Eq. (2.34), $R = \sqrt{\left(\dfrac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = 25$. By Fig. (a),

(a) $\sigma_1 = 10 + 25 = 35,\ \sigma_2 = 10 - 25 = -15$. By Eq. (2.36),

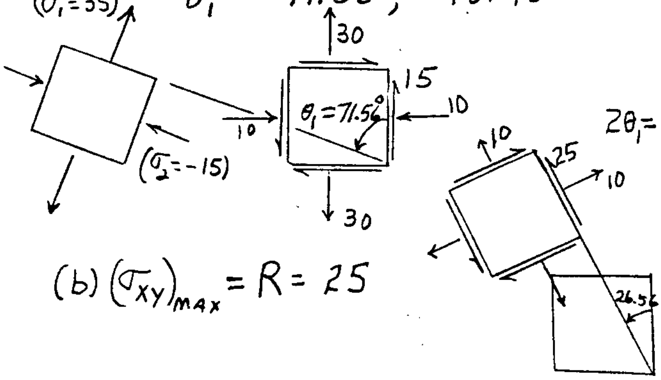
$$\tan 2\theta_1 = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = -0.75$$

$$2\theta_1 = 2.49809,\ -0.64350$$

$$= 143.13^\circ,\ -36.87^\circ$$

and

$$\theta_1 = 71.56^\circ,\ -18.43^\circ$$



(b) $(\sigma_{xy})_{MAX} = R = 25$

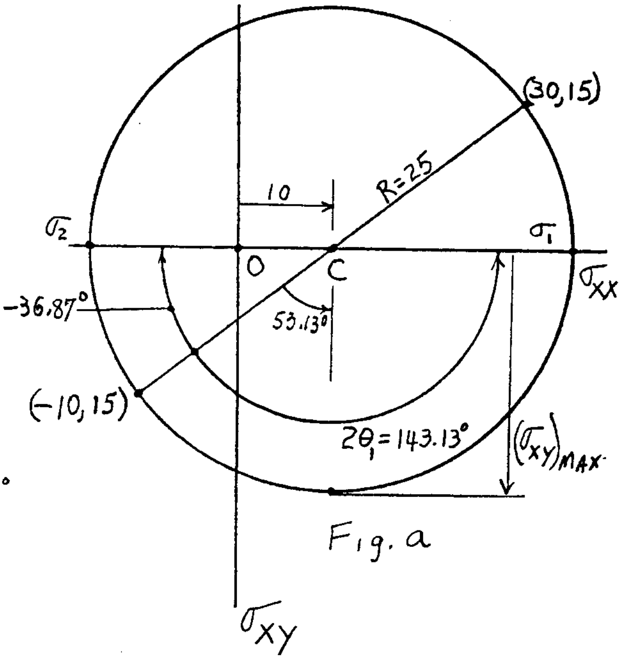


Fig. a

2.45 In MPa, $\sigma_{xx} = 240,\ \sigma_{yy} = 100,\ \sigma_{xy} = -80,\ \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$

(a) By Eq. (2.21), $I_1 = 340,\ I_2 = 17600,\ I_3 = 0$.

By Eq. (2.20), $\sigma^3 - 340\sigma^2 + 17600\sigma = 0$

$\therefore\ \sigma = 0,\ \sigma^2 - 340\sigma + 17600 = 0 \longrightarrow \sigma_{1,2} = 170 \pm 106.30$

$\therefore\ \sigma_1 = 276.30,\ \sigma_2 = 63.70,\ \sigma_3 = 0$

By Eqs. (2.37), with $\sigma_{zz} = \sigma_3 = 0$,

$$\sigma_1 = \frac{240 + 100}{2} + \sqrt{\left(\frac{240 - 100}{2}\right)^2 + 80^2} = 170 + 106.30 = 276.30$$

$$\sigma_2 = 170 - 106.30 = 63.70$$

$$\sigma_3 = 0$$

(b) By Fig. 2.12 and Eq. (2.34), $OC = \dfrac{\sigma_{xx} + \sigma_{yy}}{2} = 170,\ R = 106.30$

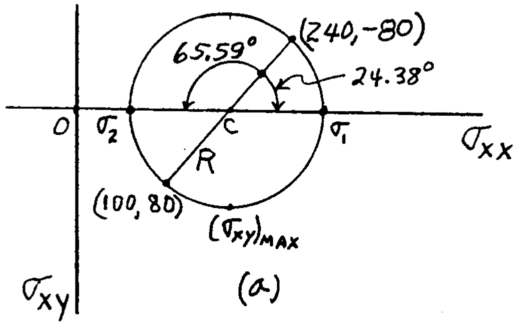
$\therefore\ \sigma_1 = OC + R = 276.30,\ \sigma_2 = OC - R = 63.70$, Fig. a.

(Cont.)

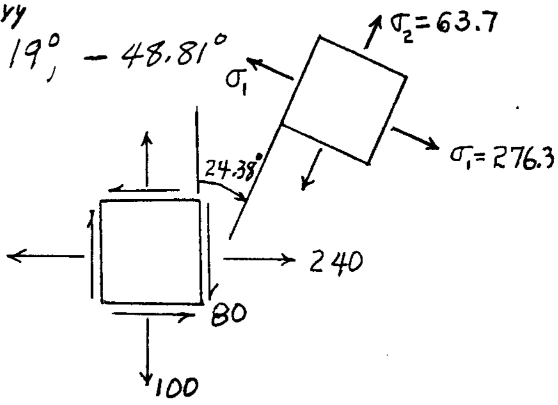
2.45 continued By Eq.(2.36), $\tan 2\theta_1 = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = -1.1428.$

$$\therefore\ 2\theta_1 = 2.2896,\ -0.8520 \text{ rad} = 131.19^\circ,\ -48.81^\circ$$

$$\theta_1 = 65.59^\circ,\ -24.38^\circ \quad (\text{Fig. b})$$



(a)



(b)

(c) By Fig. a, $(\sigma_{XY})_{MAX} = R = 106.30$

By Eq.(2.22), $9\tau_{oct}^2 = (\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2$

$$= (276.3-63.7)^2 + (276.3)^2 + (63.7)^2 = 125598.2$$

$$\therefore\ \tau_{oct} = 118.13$$

2.46 With $x = r$, $y = \theta$, $z = z$, we have $\alpha = 1$, $\beta = r$, $\gamma = 1$. Then, the first of Eqs.(2.46) yields

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial}{\partial \theta}(\sigma_{\theta r}) + \frac{\partial}{\partial z}(r\sigma_{zr}) - \sigma_{\theta\theta}\frac{\partial r}{\partial r} + rB_r = 0$$

or

$$r\frac{\partial \sigma_{rr}}{\partial r} + \sigma_{rr}\frac{\partial r}{\partial r} + \frac{\partial}{\partial \theta}(\sigma_{\theta r}) + r\frac{\partial \sigma_{zr}}{\partial z} - \sigma_{\theta\theta} + rB_r = 0$$

or

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + B_r = 0$$

which is the first of Eqs.(2.50).

Similar substitutions yield the second and third equations in Eqs.(2.50).

2.47 With $x = r$, $y = \theta$, $z = \phi$, we get $\alpha = 1$, $\beta = r$, $\gamma = r\sin\theta$. Then, the first of Eqs. (2.46) yields

$$\frac{\partial}{\partial r}\left[r^2(\sin\theta)\sigma_{rr}\right] + \frac{\partial}{\partial\theta}\left[r(\sin\theta)\sigma_{\theta r}\right] + \frac{\partial}{\partial\phi}(r\sigma_{\phi r}) - r(\sin\theta)\sigma_{\theta\theta}\frac{\partial r}{\partial r}$$
$$- r\sigma_{\phi\phi}\frac{\partial}{\partial r}(r\sin\theta) + r^2(\sin\theta)B_r = 0$$

or

$$2r(\sin\theta)\sigma_{rr} + r^2(\sin\theta)\frac{\partial\sigma_{rr}}{\partial r} + r(\cos\theta)\sigma_{\theta r} + r(\sin\theta)\frac{\partial\sigma_{\theta r}}{\partial\theta}$$
$$+ r\frac{\partial\sigma_{\phi r}}{\partial\phi} - r(\sin\theta)\sigma_{\theta\theta} - r(\sin\theta)\sigma_{\phi\phi} + r^2(\sin\theta)B_r = 0$$

Hence,

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi r}}{\partial\phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}$$
$$+ \sigma_{\theta r}\cot\theta) + B_r = 0$$

which is the first of Eqs. (2.53).

Similar substitutions yield the second and third of Eqs. (2.53).

2.48 With $x = r$, $y = \theta$, $z = z$, $\alpha = 1$, $\beta = r$, $\gamma = 1$, and $\sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = \partial/\partial z = 0$, the first of Eqs. (2.46) yields

$$\frac{\partial(r\sigma_{rr})}{\partial r} + \frac{\partial\sigma_{\theta r}}{\partial\theta} - \sigma_{\theta\theta}\frac{\partial r}{\partial r} + rB_r = 0$$

or

$$\sigma_{rr} + r\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{\theta r}}{\partial\theta} - \sigma_{\theta\theta} + rB_r = 0$$

or

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + B_r = 0$$

which is the first of Eqs. (2.54).

Similarly, the second of Eqs. (2.46) yields the second of Eqs. (2.54)

The third of Eqs. (2.46) yields $B_z = 0$.

2.49 By the first of Eqs. (2.65),

$$l^* = \frac{dx^*}{ds} \cdot \frac{ds}{ds^*} \quad \text{or} \quad \frac{dx^*}{ds} = l^* \frac{ds^*}{ds} \qquad (a)$$

By the first of Eqs. (2.66),

$$\frac{dx^*}{ds} = \left(1 + \frac{\partial u}{\partial x}\right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \qquad (b)$$

By Eqs. (a) and (b), we have

$$l^* \frac{ds^*}{ds} = \left(1 + \frac{\partial u}{\partial x}\right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \qquad (c)$$

By Eq. (2.67),

$$\frac{ds^*}{ds} = 1 + \epsilon_E \qquad (d)$$

Hence, by Eqs. (c) and (d), we obtain

$$(1 + \epsilon_E) l^* = \left(1 + \frac{\partial u}{\partial x}\right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \qquad (e)$$

Equation (e) is the first of Eqs. (2.68). In a similar manner, the second and third of Eqs. (2.68) may be derived.

2.53 Before deformation, volume = $Lbh = (5000)(100)(200)$. After deformation, let width dimension be increased by δ. The depth dimension becomes $(h + 2\delta)$. The final volume $= (5020)(100+\delta)(200+2\delta)$ Setting the two volumes equal gives

$$5020\,(100+\delta)(200+2\delta) = 5000\,(100)(200)$$

$$\delta^2 + 200\,\delta + 39.8406 = 0$$

$$\delta = -0.1994 \text{ mm}$$

The displacements take the form

$$u = \underline{-0.001994\ x}$$

$$v = \underline{-0.001994\ y}$$

$$w = \underline{0.004000\ z}$$

(a) For small displacements

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \underline{-0.001994}; \quad \epsilon_{yy} = \frac{\partial v}{\partial y} = \underline{-0.001994}; \quad \epsilon_{zz} = \frac{\partial w}{\partial z} = \underline{0.004000}$$

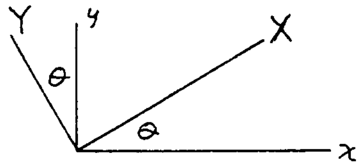
$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = \underline{0}$$

(b) For large displacements

$$\epsilon_{z} = \frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \underline{0.004008}$$

2.54 The direction cosines of X-axis are

$$\ell_1 = \cos\theta; \quad m_1 = \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta; \quad n_1 = \cos\tfrac{\pi}{2} = 0$$



Equation (**2.61**) gives

$$\epsilon_{XX} = \epsilon_{xx}\,\ell_1^2 + \epsilon_{yy}\,m_1^2 + 2\,\epsilon_{xy}\,\ell_1 m_1$$

$$= \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + 2\,\epsilon_{xy}\cos\theta\,\sin\theta$$

$$= \underline{\frac{\epsilon_{xx} + \epsilon_{yy}}{2} + \frac{\epsilon_{xx} - \epsilon_{yy}}{2}\cos 2\theta + \epsilon_{xy}\sin 2\theta}$$

The direction cosines of Y-axis are

$$\ell_2 = \cos\left(\tfrac{\pi}{2} + \theta\right) = -\sin\theta; \quad m_2 = \cos\theta; \quad n_2 = \cos\tfrac{\pi}{2} = 0$$

Equation (**2.71**) gives

$$\epsilon_{XY} = \frac{1}{2}\gamma_{xy} = \epsilon_{xx}\,\ell_1\ell_2 + \epsilon_{yy}\,m_1 m_2 + \epsilon_{xy}\,(\ell_1 m_2 + \ell_2 m_1)$$

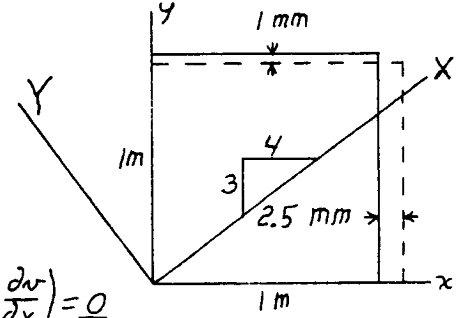
$$= -\epsilon_{xx}\sin\theta\cos\theta + \epsilon_{yy}\sin\theta\cos\theta + \epsilon_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= \underline{-\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)\sin 2\theta + \epsilon_{xy}\cos 2\theta}$$

For principal direction $\epsilon_{XY} = 0$; therefore

$$\tan 2\theta = \underline{\frac{2\,\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}}}$$

2.55



For X-axis, $\ell_1 = 0.8$; $m_1 = 0.6$; $n_1 = 0$

For Y-axis, $\ell_2 = -0.6$; $m_2 = 0.8$; $n_2 = 0$

(a) $u = \underline{0.0025\, x}$

$v = \underline{-0.0010\, y}$

(b) $\epsilon_{xx} = \frac{\partial u}{\partial x} = \underline{0.0025}$; $\epsilon_{yy} = \frac{\partial v}{\partial y} = \underline{-0.0010}$; $\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \underline{0}$

$\epsilon_{XX} = \ell_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + 2\epsilon_{xy}\ell_1 m_1$

$= 0.0025(0.8)^2 + (-0.0010)(0.6)^2$

$= \underline{0.00124}$

$\epsilon_{YY} = \epsilon_{xx} + \epsilon_{yy} - \epsilon_{XX}$

$= 0.0025 - 0.0010 - 0.00124$

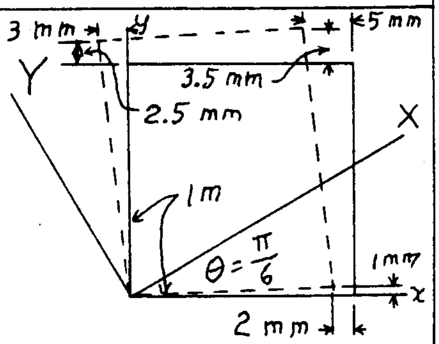
$= \underline{0.00026}$

$\gamma_{XY} = 2\epsilon_{XY} = 2\ell_1\ell_2\,\epsilon_{xx} + 2m_1m_2\,\epsilon_{yy}$

$= 2\left[(0.0025)(0.8)(-0.6) + (-0.0010)(0.6)(0.8)\right]$

$= \underline{-0.00336}$

2.56



For X-axis, $\ell_1 = 0.866$; $m_1 = 0.500$; $n_1 = 0$

For Y-axis, $\ell_2 = -0.500$; $m_2 = 0.866$; $n_2 = 0$

$u = \underline{-0.0020\, x - 0.0030\, y}$

$v = \underline{0.0010\, x + 0.0025\, y}$

(a) $\epsilon_{xx} = \frac{\partial u}{\partial x} = \underline{-0.0020}$; $\epsilon_{yy} = \frac{\partial v}{\partial y} = \underline{0.0025}$

$\gamma_{xy} = 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \underline{-0.0020}$

(b) $\epsilon_{XX} = \ell_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + 2\epsilon_{xy}\ell_1 m_1$

$= -0.0020(0.866)^2 + 0.0025(0.500)^2 + (-0.0020)(0.866)(0.500)$

$= \underline{-0.00174}$

$\epsilon_{XY} = \epsilon_{xx} + \epsilon_{yy} - \epsilon_{XX} = -0.0020 + 0.0025 + 0.00174$

$= \underline{0.00224}$

$\gamma_{XY} = 2\epsilon_{XY} = 2\ell_1\ell_2\,\epsilon_{xx} + 2m_1m_2\,\epsilon_{yy} + 2(\ell_1 m_2 + \ell_2 m_1)\epsilon_{xy}$

$= 2(-0.0020)(0.866)(-0.500) + 2(0.0025)(0.500)(0.866) + 2(-0.0010)\left[(0.866)^2 + (0.5)(-0.5)\right]$

$= \underline{0.00290}$

2.57

$$\tan 2\theta = \frac{2\epsilon_{xy}}{\epsilon_{xx}-\epsilon_{yy}} = \frac{2(-0.0010)}{-0.0020-0.0025} = 0.4444$$

$$\theta = \underline{0.2091 \text{ rad}}$$

$$\epsilon_1 = \epsilon_{XX} = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + 2\epsilon_{xy}\sin\theta\cos\theta$$

$$= -0.0020(0.9782)^2 + 0.0025(0.2076)^2 + 2(-0.0010)(0.2076)(0.9782)$$

$$= \underline{-0.00221}$$

$$\epsilon_2 = \epsilon_{xx} + \epsilon_{yy} - \epsilon_1 = -0.0020 + 0.0025 + 0.00185$$

$$= \underline{0.00271}$$

2.58

(a) $u = \underline{0.000667\,xy}$

$v = \underline{0.001333\,xy}$

At point B

$$\epsilon_{xx} = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=1.5\\ y=3}} = 0.000667(3) = \underline{0.00200}$$

$$\epsilon_{yy} = \left.\frac{\partial v}{\partial y}\right|_{\substack{x=1.5\\ y=3}} = 0.001333(1.5) = \underline{0.00200}$$

$$\gamma_{xy} = 2\epsilon_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{\substack{x=1.5\\ y=3}} = 0.000667(1.5) + 0.001333(3)$$

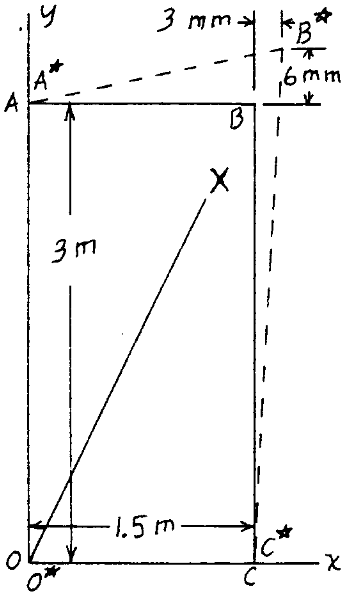
$$= \underline{0.00500}$$

(b) For X-axis $\ell_1 = \frac{1}{\sqrt{5}}$; $m_1 = \frac{2}{\sqrt{5}}$; $n_1 = 0$

$$\epsilon_{XX} = \epsilon_{xx}\ell_1^2 + \epsilon_{yy}m_1^2 + 2\epsilon_{xy}\ell_1 m_1$$

$$= 0.00200\left(\frac{1}{5}\right) + 0.00200\left(\frac{4}{5}\right) + 0.00500\left(\frac{2}{5}\right)$$

$$= \underline{0.00400}$$



2.59

$$\tan 2\theta = \frac{2\epsilon_{xy}}{\epsilon_{xx}-\epsilon_{yy}} = \frac{-0.00220}{0.00180-(-0.00108)} = -0.7639$$

$$\theta = \underline{-0.3262 \text{ rad}}$$

$$\epsilon_1 = \epsilon_{XX} = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + 2\epsilon_{xy}\sin\theta\cos\theta$$

$$= 0.00180(0.9473)^2 - 0.00108(-0.3204)^2 - 0.00220(0.9473)(-0.3204)$$

$$= \underline{0.00217}$$

$$\epsilon_2 = \epsilon_{xx} + \epsilon_{yy} - \epsilon_{XX} = 0.00180 + (-0.00108) - 0.00217$$

$$= \underline{-0.00145}$$

$\boxed{2.60}$ $I_1 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0.00180 + (-0.00108) + 0 = 0.00072$

$$\bar{I}_2 = \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{xx}\epsilon_{zz} - \epsilon_{xy}^2 - \epsilon_{yz}^2 - \epsilon_{zx}^2$$

$$= 0.00180(-0.00108) - (-0.00110)^2 = -0.000003154$$

$$\bar{I}_3 = \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{vmatrix} = \begin{vmatrix} 0.00180 & -0.00110 & 0 \\ -0.00110 & -0.00108 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\epsilon^3 - 0.00072\epsilon^2 - 0.000003154\epsilon = 0$$

$\epsilon_1 = \underline{0.00217}$; $\epsilon_2 = \underline{0}$; $\epsilon_3 = -0.00145$

$\boxed{2.61}$ $I_1 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0.002667 + 0.002000 + (-0.00200) = 0.002667$

$$\bar{I}_2 = \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx} - \epsilon_{xy}^2 - \epsilon_{yz}^2 - \epsilon_{zx}^2$$

$$= 0.002667(0.002000) + 0.002000(-0.002000) + (-0.002000)(0.002667)$$

$$-(0.002667)^2 - (-0.001500)^2 - (-0.000033)^2 = -0.0000133640$$

$$\bar{I}_3 = \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{vmatrix} = \begin{vmatrix} 0.002667 & 0.002667 & -0.000033 \\ 0.002667 & 0.002000 & -0.001500 \\ -0.000033 & -0.001500 & -0.00200 \end{vmatrix} = -0.000000021811$$

$$\epsilon^3 - 0.002667\epsilon^2 - 0.0000133640\epsilon + 0.000000021811 = 0$$

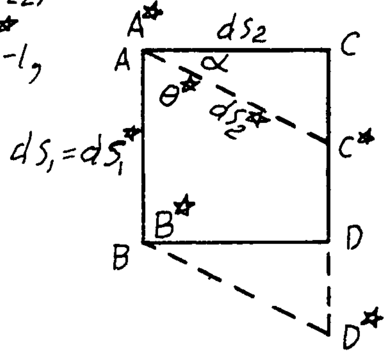
$\epsilon_1 = \underline{0.005170}$; $\epsilon_2 = 0.000158$; $\epsilon_3 = -0.002662$

$\boxed{2.62}$ Equation (2.71) gives $\gamma_{zx} = (1+\epsilon_{E1})(1+\epsilon_{E2})\cos\theta^*$

where $\epsilon_{E1} = \dfrac{ds_1^* - ds_1}{ds_1} = 0$, $\epsilon_{E2} = \dfrac{ds_2^* - ds_2}{ds_2} = \dfrac{A^*C^*}{AC} - 1$,

and $\cos\theta^* = \dfrac{CC^*}{A^*C^*}$.

$$\gamma_{zx} = \frac{A^*C^*}{AC}\,\frac{CC^*}{A^*C^*} = \frac{CC^*}{AC} = \underline{\tan\alpha}$$



$\boxed{2.63}$ In units of $\mu = 10^{-6}$, $\epsilon_{xx} = -2000$, $\epsilon_{yy} = 400$, $\epsilon_{xy} = -900$

(a) $\epsilon_{1,2} = \dfrac{\epsilon_{xx}+\epsilon_{yy}}{2} \pm \dfrac{1}{2}\sqrt{(\epsilon_{xx}-\epsilon_{yy})^2 + 4\epsilon_{xy}^2} = \dfrac{-2000+400}{2} \pm \dfrac{1}{2}\sqrt{(-2000-400)^2+4(-900)^2}$

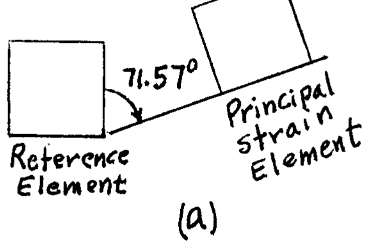
$= -800 \pm 1500$

$\therefore\ \epsilon_1 = 700,\ \epsilon_2 = -2300\ ;\ R = 1500,\ OC = -800$

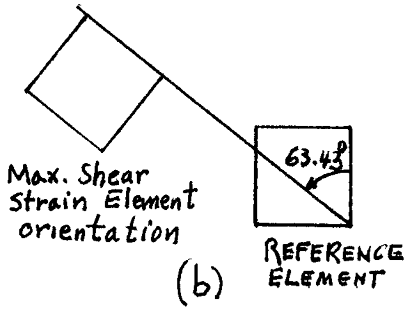
$\tan 2\theta_1 = \dfrac{2\epsilon_{xy}}{\epsilon_{xx}-\epsilon_{yy}} = \dfrac{-2(900)}{-2000-400} = 0.75$

$2\theta_1 = 0.6435, -2.4981$ rad $= 36.87°, -143.13°$

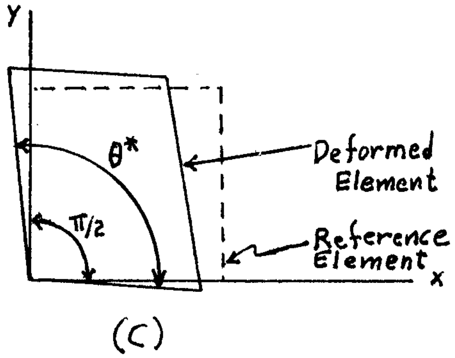
$\therefore\ \theta_1 = 18.43°, -71.57°$



(b) $(\epsilon_{xy})_{MAX} = R = 1500$ @ $\theta = 18.43° + 45° = 63.43°$



(c)



Since ϵ_{xx} is negative, line elements in the x-direction contract, and since ϵ_{yy} is positive, line elements in the y-direction elongate. Also, since $2\epsilon_{xy} = \gamma_{xy} = \frac{\pi}{2} - \theta^*$ is negative, the angle θ^* is greater than $\pi/2$, where θ^* is the final angle between the line element initially along the x-axis and the line element initially along the y-axis (Fig. c). [see Boresi and Chong, 2000, Eq. (2-8.6), page 89]

2.64 The square plate, 1-m long on a side, is shown in its deformed plane strain state (Fig. a), $0^*A^*B^*C^*0^*$.

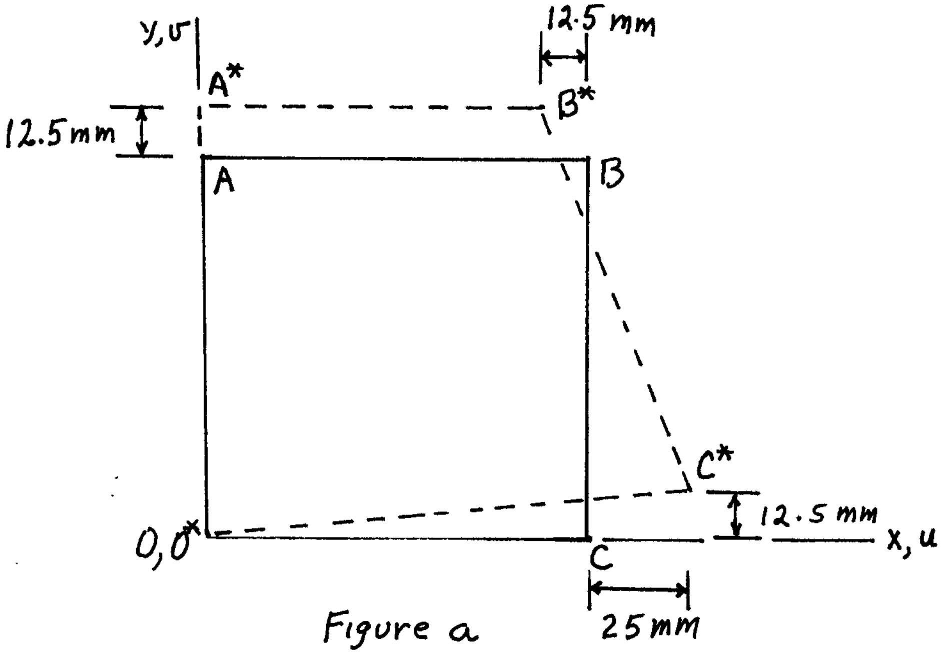


Figure a

(a) Since under the deformation straight lines in the plate remain straight, to satisfy the displacement components u, v at the four corners $0^*, A^*, B^*, C^*$, we take

$$u = a_0 + a_1 x + a_2 y + a_3 xy, \quad v = b_0 + b_1 x + b_2 y + b_3 xy \qquad (a)$$

The four constants in u, namely a_0, a_1, a_2, a_3 and in v, b_0, b_1, b_2, b_3 are determined by Eqs. (a) and the displacements of points 0, A, B, and C. Thus, by Eq. (a) and Fig. a,

$$u(0,0) = 0 = a_0 \qquad u(0,1) = 0 = a_2$$
$$u(1,0) = 0.025 = a_1 \qquad u(1,1) = -0.0125 = 0.025 + a_3 \qquad (b)$$
$$\text{or } a_3 = -0.0375$$

(cont.)

2.64 cont.

$$v(0,0) = 0 = b_0 \,, \qquad v(0,1) = 0.0125 = b_2$$

$$v(1,0) = 0.0125 = b_1 \,, \quad v(1,1) = 0.0125 = 0.0250 + b_3$$

$$\text{or, } b_3 = -0.0125$$

Therefore,

$$\begin{aligned} u(x,y) &= 0.025x - 0.0375xy \\ v(x,y) &= 0.0125x + 0.0125y - 0.0125xy \end{aligned} \qquad (c)$$

(b) By Eqs.(c),

$$\begin{aligned} \frac{\partial u}{\partial x} &= 0.025 - 0.0375y \,, & \frac{\partial v}{\partial x} &= 0.0125 - 0.0125y \\ \frac{\partial u}{\partial y} &= -0.0375x \,, & \frac{\partial v}{\partial y} &= 0.0125 - 0.0125x \end{aligned} \qquad (d)$$

By Eqs.(d) and (2.62), the Green strain components are

$$\epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0.025390625 - 0.03859375y + 0.00078125y^2$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] = 0.012578125 - 0.01265625x + 0.00078125x^2$$

$$(e)$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) = 0.006328125 - 0.019296875x - 0.006328125y + 0.00078125xy$$

$$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$$

(c) For line OB (Fig. a), the direction cosines are

$$l = \frac{\sqrt{2}}{2}\,, \quad m = \frac{\sqrt{2}}{2}\,, \quad n = 0 \qquad (f)$$

By Eqs. (e), (f) and (2.61), with $x = y = 1$ m at point B,

$$M = l^2\epsilon_{xx} + m^2\epsilon_{yy} + 2lm\epsilon_{xy} = -0.024375 \qquad (g)$$

(Cont.)

2.64 cont.

(d) By Eqs.(d) and (2.81), the strain components for small strains are

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = 0.025 - 0.0375y$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = 0.0125 - 0.0125x \qquad (h)$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0.00625 - 0.01875x - 0.00625y$$

at point B, $x = y = 1$, and Eqs.(h) yield

$$\epsilon_{xx} = -0.0125, \ \epsilon_{yy} = 0, \ \epsilon_{xy} = -0.01875 \qquad (i)$$

Compared to the Green strain components [Eqs.(e)],

$$\epsilon_{xx} = -0.012422, \epsilon_{yy} = 0.000703, \epsilon_{xy} = -0.018516$$

(e) By Eqs.(f) and (i), the magnification factor for small strain is

$$M = l^2\epsilon_{xx} + m^2\epsilon_{yy} + 2lm\,\epsilon_{xy} = -0.025 \qquad (j)$$

Hence, the difference between the Green strain at point B [Eq.(g)] and the small strain magnification [Eq.(j)] is

$$\frac{(-0.024375) - (-0.025)}{(-0.024375)} \times 100 = -2.56\%$$

2.65

(a) False

(b) False

(c) True

(d) True

2.66 The strain components of Example 2.11 are

$$\epsilon_{xx} = Cy(L-x),\ \epsilon_{yy} = Dy(L-x),\ \gamma_{xy} = 2\epsilon_{xy} = -(C+D)(A^2-y^2) \quad (a)$$

also, by the first of Eqs. (2.83), the strain compatibility relation for small displacement plane strain is

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \qquad (b)$$

Substitution of Eqs. (a) into Eq. (b) yields

$$0 + 0 = 0$$

Hence, the strain components are compatible.

2.67 By the given strain components and the strain-displacement relations for small-displacement plane strain, we have

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = A(L-x) \qquad (a)$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = B(L-x) \qquad (b)$$

$$2\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad (c)$$

Integration of Eqs. (a) and (b) yields

$$u = A(Lx - \tfrac{1}{2}x^2) + Y(y) \qquad (d)$$

$$v = B(Ly - XY) + X(x) \qquad (e)$$

Where $X(x)$ and $Y(y)$ are functions of x and y, respectively.

Substitution of Eqs. (d) and (e) into Eq. (c) yields

(Cont.)

2.67 cont.

$$\frac{dX}{dx} - By + \frac{dY}{dY} = 0$$

or

$$\frac{dY}{dy} - By = -\frac{dX}{dx} = C = \text{constant}$$

Therefore,

$$\frac{dY}{dy} - By = C, \quad \frac{dX}{dx} = -C$$

Integration yields

$$Y(y) = \frac{1}{2}By^2 + Cy + D \qquad (f)$$

$$X(x) = -Cx + E$$

where D and E are constants. Hence, by Eqs. (d), (e), and (f), we obtain

$$u = A(Lx - \tfrac{1}{2}x^2) + \tfrac{1}{2}By^2 + Cy + D \qquad (g)$$

$$v = B(Ly - xy) - Cx + E \qquad (h)$$

By Eqs. (g) and (h) and the conditions $u = v = 0$ for $x = y = 0$, we find

$$D = E = 0 \qquad (i)$$

and by Eqs. (g), (h), and (i), and the conditions $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ for $x = y = 0$, we find

$$C = 0 \qquad (j)$$

So, by Eqs. (g), (h), (i), and (j),

$$u = A(Lx - \tfrac{1}{2}x^2) + \tfrac{1}{2}By^2$$

$$v = B(Ly - xy)$$

2.68 For cylindrical coordinates, $x=r$, $y=\theta$, $z=z$, and $\alpha=1$, $\beta=r$, $\gamma=1$. Therefore, the first three of Eqs. (2.84) yield the first three of Eqs. (2.85) as follows:

$$\epsilon_{rr} = \frac{1}{1}\left[\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial(1)}{\partial\theta} + \frac{w}{1}\frac{\partial(1)}{\partial z}\right] = \frac{\partial u}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\left[\frac{\partial v}{\partial\theta} + \frac{w}{1}\frac{\partial r}{\partial z} + \frac{u}{1}\frac{\partial(r)}{\partial r}\right] = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}$$

$$\epsilon_{zz} = \frac{1}{1}\left[\frac{\partial w}{\partial z} + \frac{u}{1}\frac{\partial(1)}{\partial r} + \frac{v}{r}\frac{\partial(1)}{\partial\theta}\right] = \frac{\partial w}{\partial z}$$

Similarly, the last three of Eqs. (2.84) yield the last three of Eqs. (2.85).

2.69 For spherical coordinates, $x=r$, $y=\theta$, $z=\phi$ and $\alpha=1$, $\beta=r$, $\gamma=r\sin\theta$. Then the first three of Eqs. (2.84) yield the first three of Eqs. (2.86) as follows:

$$\epsilon_{rr} = \frac{1}{1}\left[\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial(1)}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial(1)}{\partial\phi}\right] = \frac{\partial u}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\left[\frac{\partial v}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial r}{\partial\phi} + \frac{u}{1}\frac{\partial r}{\partial r}\right] = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}$$

$$\epsilon_{zz} = \frac{1}{r\sin\theta}\left[\frac{\partial w}{\partial\phi} + \frac{u}{1}\frac{\partial(r\sin\theta)}{\partial r} + \frac{v}{r}\frac{\partial(r\sin\theta)}{\partial\theta}\right] = \frac{u}{r} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi}$$

Similarly the last three of Eqs. (2.84) yield the last three of Eqs. (2.86)

2.70 For plane polar coordinates, $x=r$, $y=\theta$, $z=z$, $\alpha=1$, $\beta=r$, $\gamma=1$, $w=\partial/\partial z=0$, and $u=u(r,\theta)$, $v=v(r,\theta)$. Then, Eqs. (2.84) yield

$$\epsilon_{rr} = \frac{1}{1}\left[\frac{\partial u}{\partial r} + \frac{v}{1}\frac{\partial(1)}{\partial\theta} + \frac{w}{1}\frac{\partial(1)}{\partial z}\right] = \frac{\partial u}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\left[\frac{\partial v}{\partial\theta} + \frac{w}{1}\frac{\partial(1)}{\partial z} + \frac{u}{1}\frac{\partial r}{\partial r}\right] = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}$$

$$\epsilon_{zz} = \frac{1}{1}\left[\frac{\partial w}{\partial z} + \frac{u}{1}\frac{\partial(1)}{\partial r} + \frac{v}{r}\frac{\partial(1)}{\partial\theta}\right] = 0$$

$$2\epsilon_{r\theta} = \left[\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{1}{1}\frac{\partial v}{\partial r} - \frac{v}{r}\frac{\partial r}{\partial r} - \frac{u}{r}\frac{\partial(1)}{\partial\theta}\right] = \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

Likewise, $\epsilon_{rz} = \epsilon_{\theta z} = 0$

2.71 (a) Given that the (x, y, z) displacements are

$$u = c_1 x z, \quad v = c_2 y z, \quad w = c_3 z \qquad (a)$$

for small strains, Eqs. (a) and (2.81) yield

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = c_1 z, \quad \epsilon_{yy} = \frac{\partial v}{\partial y} = c_2 z, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} = c_3 \qquad (b)$$

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad \gamma_{xz} = 2\epsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = c_1 x, \quad \gamma_{yz} = 2\epsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = c_2 y$$

At point E, by Fig. E2.8,

$$x = 1.5\text{ m}, \quad y = 1.0\text{ m}, \quad z = 2.0\text{ m} \qquad (c)$$

By Eqs. (b) and (c),

$$\epsilon_{xx} = 2c_1, \quad \epsilon_{yy} = 2c_2, \quad \epsilon_{zz} = c_3 \qquad (d)$$

$$\gamma_{xy} = 2\epsilon_{xy} = 0, \quad \gamma_{xz} = 2\epsilon_{xz} = 1.5c_1, \quad \gamma_{yz} = 2\epsilon_{yz} = c_2$$

(b) Let the X axis lie along the line from E to C. The direction cosines of EC are

$$l_1 = -0.6, \quad m_1 = 0, \quad n_1 = -0.8 \qquad (e)$$

By Eqs. (d), (e), (2.61), and (2.82), the magnitude of ϵ_{XX} is

$$\begin{aligned} \epsilon_{XX} &= \epsilon_{xx} l_1^2 + \epsilon_{zz} n_1^2 + 2\epsilon_{xz} l_1 n_1 \\ &= 2c_1(-0.6)^2 + c_3(-0.8)^2 + 1.5c_1(-0.6)(-0.8) \\ &= 1.44c_1 + 0.64c_3 \end{aligned}$$

(c) Let the X axis lie along the line EF and the Y axis lie along line ED (Fig. E2.8). For line EF, $l_1 = -1$, $m_1 = n_1 = 0$, and for line ED, $l_2 = n_2 = 0$, $m_2 = -1$. Then, by Eqs. (d) and (2.76d),

$$\gamma_{XY} = 2\epsilon_{XY} = 2\epsilon_{xy} l_1 m_2 = 2\epsilon_{xy} = 0$$

(cont.)

2.71 cont. (d) For $c_1 = 0.002\ m^{-1}$, $c_2 = 0.004\ m^{-1}$, $c_3 = -0.004$, by parts (a), (b), and (c), we obtain

$$\epsilon_{xx} = 0.004,\ \epsilon_{yy} = 0.008,\ \epsilon_{zz} = -0.004$$

$$\gamma_{xy} = 2\epsilon_{xy} = 0,\ \gamma_{xz} = 2\epsilon_{xz} = 0.003,\ \gamma_{yz} = 2\epsilon_{yz} = 0.004$$

$$\epsilon_{XX} = 0.00032,\ \gamma_{XY} = 2\epsilon_{XY} = 0$$

2.72 Given the small strain components,

$$\epsilon_{xx} = Az^3,\ \epsilon_{yy} = Bx^2,\ \epsilon_{zz} = Cx^2$$
$$\epsilon_{xy} = Dxy,\ \epsilon_{xz} = Exz^2,\ \epsilon_{yz} = Fxz \qquad (a)$$

Eqs. (a) and (2.83) yield

$$\frac{\partial^2 \epsilon_{yy}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial y^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \Rightarrow 2B + 0 = 2D;\ B = D$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{xz}}{\partial x \partial z} \Rightarrow 2C + 6Az = 4Ez;\ C = 0,\ 3A = 2E$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} \Rightarrow 0 + 0 = 0$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial z^2} = \frac{\partial^2 \epsilon_{yz}}{\partial z \partial x} + \frac{\partial^2 \epsilon_{zx}}{\partial y \partial z} \Rightarrow 0 + 0 = F + 0;\ F = 0$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} + \frac{\partial^2 \epsilon_{xz}}{\partial y^2} = \frac{\partial^2 \epsilon_{xy}}{\partial y \partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x \partial y} \Rightarrow 0 + 0 = 0 + 0$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x^2} = \frac{\partial^2 \epsilon_{xz}}{\partial x \partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial x \partial z} \Rightarrow 0 + 0 = 0 + 0$$

Hence, compatibility is possible, if and only if $B = D$, $E = 1.5A$, and $C = F = 0$.

2.73 By Eqs. (b) of Example 2.12, for $\theta = 45°$,

$$\epsilon_{xx} = \epsilon_a, \quad \epsilon_{yy} = \epsilon_c, \quad \epsilon_{xy} = \frac{1}{2}(2\epsilon_b - \epsilon_a - \epsilon_c) \qquad \text{(a)}$$

By Eqs. (a) and Eqs. (e) and (f) of Example 2.10, we find

$$\epsilon_1 = \frac{1}{2}(\epsilon_a + \epsilon_c) + \frac{1}{2}\left[(\epsilon_a - \epsilon_c)^2 + (2\epsilon_b - \epsilon_a - \epsilon_c)^2\right]^{1/2}$$

$$\epsilon_2 = \frac{1}{2}(\epsilon_a - \epsilon_c) - \frac{1}{2}\left[(\epsilon_a - \epsilon_c)^2 + (2\epsilon_b - \epsilon_a - \epsilon_c)^2\right]^{1/2}$$

By Eqs. (a) and Eq. (d) of Example 2.10, we have

$$\tan 2\phi = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = \frac{2\epsilon_b - \epsilon_a - \epsilon_c}{\epsilon_a - \epsilon_c}$$

2.74 By Eqs. (b) of Example 2.12, for $\theta = 60°$,

$$\epsilon_{xx} = \epsilon_a, \quad \epsilon_{yy} = \frac{2(\epsilon_b + \epsilon_c) - \epsilon_a}{3}, \quad \epsilon_{xy} = \frac{\epsilon_b - \epsilon_c}{\sqrt{3}} \qquad \text{(a)}$$

By Eqs. (a) and Eqs. (e) and (f) of Example 2.10, we obtain

$$\epsilon_1 = \frac{\epsilon_a + \epsilon_b + \epsilon_c}{3} + \frac{1}{3}\left[(2\epsilon_a - \epsilon_b - \epsilon_c)^2 + 3(\epsilon_b - \epsilon_c)^2\right]^{1/2}$$

$$\epsilon_2 = \frac{\epsilon_a + \epsilon_b + \epsilon_c}{3} - \frac{1}{3}\left[(2\epsilon_a - \epsilon_b - \epsilon_c)^2 + 3(\epsilon_b - \epsilon_c)^2\right]^{1/2}$$

By Eqs. (a) and Eq. (d) of Example 2.10, we have

$$\tan 2\phi = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = \frac{\sqrt{3}(\epsilon_b - \epsilon_c)}{2\epsilon_a - \epsilon_b - \epsilon_c}$$

2.75 For the rectangular strain rosette, with ϵ_a directed along the positive x axis of (x, y) axes,

(a) $\epsilon_{xx} = \epsilon_a$, $\epsilon_{yy} = \epsilon_c$. By Eq.(2.76b) with $l^2 = m^2 = \frac{1}{2}$ and $n = 0$,

$$\epsilon_b = \frac{1}{2}\epsilon_a + \frac{1}{2}\epsilon_c + 2\left(\frac{1}{2}\right)\epsilon_{xy}, \text{ or } \epsilon_{xy} = \epsilon_b - \frac{1}{2}(\epsilon_a + \epsilon_c).$$

Hence,
$$\tan 2\theta = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = \frac{2\epsilon_b - \epsilon_a - \epsilon_c}{\epsilon_a - \epsilon_c}$$

(b) By Eq. (c) of Example 2.10,

$$R^2 = \frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2 = \frac{1}{4}\left[(\epsilon_a - \epsilon_c)^2 + (2\epsilon_b - \epsilon_a - \epsilon_c)^2\right]$$

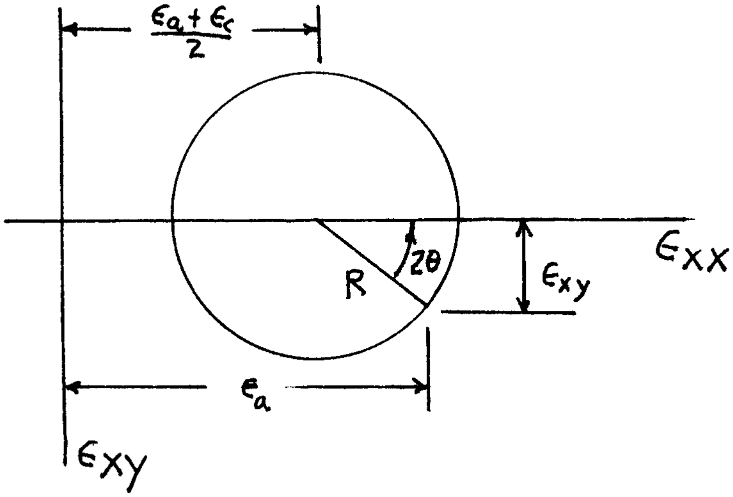
$$\therefore \quad R = \frac{1}{2}\left[(\epsilon_a - \epsilon_c)^2 + (2\epsilon_b - \epsilon_a - \epsilon_c)^2\right]^{1/2}$$

By Fig. E2.10a,

$$\epsilon_1 = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) + R = \frac{1}{2}(\epsilon_a + \epsilon_c) + R$$

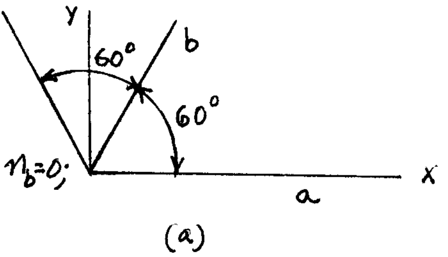
$$\epsilon_2 = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) - R = \frac{1}{2}(\epsilon_a + \epsilon_c) - R$$

(c)



2.76

(a) By Fig. a, the direction cosines of rosette arms a, b, c are $l_a = 1$, $m_a = n_a = 0$; $l_b = \frac{1}{2}$, $m_b = \frac{\sqrt{3}}{2}$, $n_b = 0$; $l_c = -\frac{1}{2}$, $m_c = \frac{\sqrt{3}}{2}$, $n_c = 0$.



Therefore, by Eqs. (2.76a, b, c),

$$\epsilon_a = \epsilon_{xx} \qquad (a)$$

$$\epsilon_b = \frac{1}{4}\epsilon_a + \frac{3}{4}\epsilon_{yy} + \frac{\sqrt{3}}{2}\epsilon_{xy} \qquad (b)$$

$$\epsilon_c = \frac{1}{4}\epsilon_a + \frac{3}{4}\epsilon_{yy} - \frac{\sqrt{3}}{2}\epsilon_{xy} \qquad (c)$$

By Eqs. (b) and (c),

$$\epsilon_{yy} = \frac{1}{3}(2\epsilon_b + 2\epsilon_c - \epsilon_a), \quad \epsilon_{xy} = \frac{1}{\sqrt{3}}(\epsilon_b - \epsilon_c) \qquad (d)$$

Therefore,

$$\tan 2\theta = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = \frac{\sqrt{3}(\epsilon_b - \epsilon_c)}{2\epsilon_a - \epsilon_b - \epsilon_c} \qquad (e)$$

(b) By Eq. (C), Example 2.10 and Eqs. (a) and (d) above,

$$R = \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \epsilon_{xy}^2} = \frac{1}{3}\left[(2\epsilon_a - \epsilon_b - \epsilon_c)^2 + 3(\epsilon_b - \epsilon_c)^2\right]^{1/2} \qquad (f)$$

By Eqs. (c), (e) and (f) of Example 2.10, and Eqs. (a) and (d) above,

$$\epsilon_1 = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) + R = \frac{1}{3}(\epsilon_a + \epsilon_b + \epsilon_c) + R \qquad (g)$$

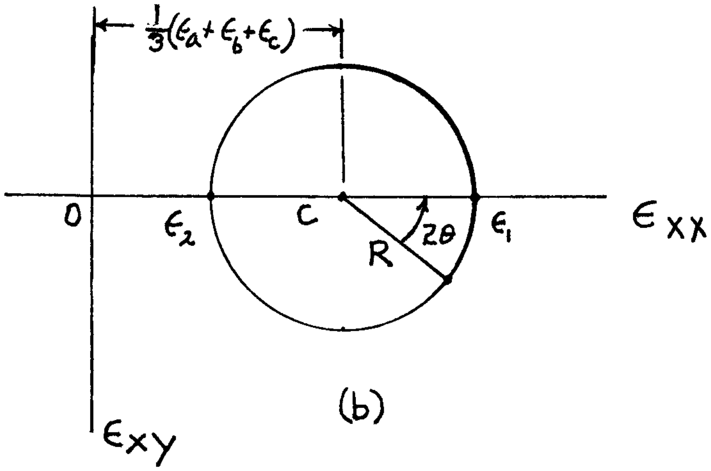
$$\epsilon_2 = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) - R = \frac{1}{3}(\epsilon_a + \epsilon_b + \epsilon_c) - R \qquad (h)$$

where R is given by Eq. (f) above.

(Cont.)

2.76 continued

Since $\frac{1}{2}(\epsilon_{xx}+\epsilon_{yy}) = \frac{1}{3}(\epsilon_a+\epsilon_b+\epsilon_c)$ and R is given by Eq.(f) above, Mohr's circle for the delta rosette is shown in Fig. b.



(b)

2.77 Given $\epsilon_a = 2450\mu$, $\epsilon_b = 1360\mu$, $\epsilon_c = -1310\mu$, $\mu = 10^{-6}$, in μ units, Eq.(f), Prob. 2.76 yields

$$R = \frac{1}{3}\left[(2\times 2450 - 1360 + 1310)^2 + 3(1360+1310)^2\right]^{1/2} = 2233.81 \quad (a)$$

By Eqs.(g) and (h) of Prob. 2.76, with Eq.(a) above, we have

$$\epsilon_1 = \frac{1}{3}(2450+1360-1310) + 2233.81 = 3067.14$$

$$\epsilon_2 = \frac{1}{3}(2450+1360-1310) - 2233.81 = -1400.48$$

By Eq.(e), Prob. 2.76,

$$\tan 2\theta = \frac{\sqrt{3}(1360+1310)}{2(2450)-1360+1310} = 0.9535$$

$$\therefore\ 2\theta = 0.7616 \text{ rad} = 43.64^\circ \text{ (See Fig. b, Prob. 2.76)}$$

$$\theta = 21.82^\circ$$

$$(\epsilon_{xy})_{\text{MAX}} = R = 2233.81$$

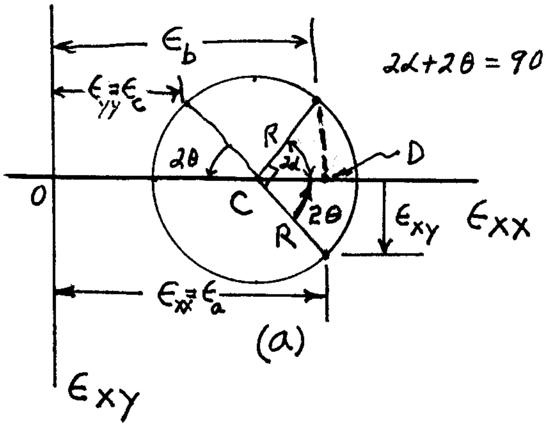
2.78

When arm a of the rectangular rosette is directed along the positive x-axis, the arm c is directed along the positive y-axis. Therefore, $\epsilon_{xx} = \epsilon_a$ and $\epsilon_{yy} = \epsilon_c$. Then $OC = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) = \frac{1}{2}(\epsilon_a + \epsilon_c)$. By Mohr's circle (Fig. a),

$$\epsilon_{xy} = R \sin 2\theta = \overset{R\sin(90^\circ - 2\alpha) = R\cos 2\alpha =}{\wedge} CD = OD - OC = \epsilon_b - \frac{1}{2}(\epsilon_a + \epsilon_c)$$

$$= \frac{1}{2}(2\epsilon_b - \epsilon_a - \epsilon_c)$$

$$\therefore \gamma_{xy} = 2\epsilon_{xy} = 2\epsilon_b - \epsilon_a - \epsilon_c$$



(a)

3.1 By Table A.1, for AISI-3140 steel, $E = 200\,GPa$, $\nu = 0.29$.

By Eqs. (3.30), the principal strain-principal stress relations are

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \qquad (a)$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right] \qquad (b)$$

$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right] = 0$$

or

$$\sigma_3 = \nu(\sigma_1 + \sigma_2) \qquad (c)$$

Substitution of Eq. (c) into Eqs. (a) and (b) yields

$$\begin{aligned} E\epsilon_1 &= (1-\nu^2)\sigma_1 - \nu(1+\nu)\sigma_2 \\ E\epsilon_2 &= (1-\nu^2)\sigma_2 - \nu(1+\nu)\sigma_1 \end{aligned} \qquad (d)$$

Solving Eqs. (d) for σ_1 and σ_2 in terms of ϵ_1 and ϵ_2, we obtain

$$\begin{aligned} \sigma_1 &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_1 + \nu\epsilon_2\right] \\ \sigma_2 &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\epsilon_1 + (1-\nu)\epsilon_2\right] \end{aligned} \qquad (e)$$

For $E = 200\,GPa$ and $\nu = 0.29$, Eqs. (e) and (c) become

$$\begin{aligned} \sigma_1 &= (262.09\epsilon_1 + 107.05\epsilon_2)\times 10^9 \quad [Pa] \\ \sigma_2 &= (107.05\epsilon_1 + 262.09\epsilon_2)\times 10^9 \quad [Pa] \\ \sigma_3 &= 107.05(\epsilon_1 + \epsilon_2)\times 10^9 \quad [Pa] \end{aligned} \qquad (f)$$

For point 1 in Table P3.1, $\epsilon_1 = 0.08$ and $\epsilon_2 = -0.002$. Then, Eqs. (f) yield

$$\sigma_1 = 1882.62\,MPa, \quad \sigma_2 = 332.22\,MPa, \quad \sigma_3 = 642.30\,MPa$$

(cont.)

3.1 cont. For point 2, $\epsilon_1 = 0.006$, $\epsilon_2 = -0.003$. Then, Eqs.(f) yield

$$\sigma_1 = 1251.39 \text{ MPa}, \ \sigma_2 = -143.97 \text{ MPa}, \ \sigma_3 = 321.15 \text{ MPa}$$

Likewise for points 3, 4, and 5, we have

Point 3: $\epsilon_1 = -0.007$, $\epsilon_2 = -0.008$

$$\sigma_1 = -2691.03 \text{ MPa}, \ \sigma_2 = -2846.07 \text{ MPa}, \ \sigma_3 = -1605.75 \text{ MPa}$$

Point 4: $\epsilon_1 = 0.004$, $\epsilon_2 = -0.005$

$$\sigma_1 = 513.11 \text{ MPa}, \ \sigma_2 = -882.25 \text{ MPa}, \ \sigma_3 = -107.05 \text{ MPa}$$

Point 5: $\epsilon_1 = 0.009$, $\epsilon_2 = 0.002$

$$\sigma_1 = 2572.91 \text{ MPa}, \ \sigma_2 = 1487.63 \text{ MPa}, \ \sigma_3 = 1177.55 \text{ MPa}$$

3.2 By Table A.1, for aluminum alloy 7075 T6, $E = 72$ GPa, $\nu = 0.33$. Also, by Eq. (3.32), the principal stress–principal strain relations are

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_1 + \nu\epsilon_2 + \nu\epsilon_3\right] \qquad (a)$$

$$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\epsilon_1 + (1-\nu)\epsilon_2 + \nu\epsilon_3\right] \qquad (b)$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\epsilon_1 + \nu\epsilon_2 + (1-\nu)\epsilon_3\right] \qquad (c)$$

Since there is no normal pressure on the surface of the wing (the wing is being bent), $\sigma_3 = 0$. Hence, Eq. (c) yields the principal strain ϵ_3 as

$$\epsilon_3 = -\frac{\nu}{(1-\nu)}(\epsilon_1 + \epsilon_2) \qquad (d)$$

(cont.)

3.2 cont. Substituting Eq. (d) into Eqs. (a) and (b), we find

$$\sigma_1 = \frac{E}{1-\nu^2}\left(\epsilon_1 + \nu\epsilon_2\right), \quad \sigma_2 = \frac{E}{1-\nu^2}\left(\nu\epsilon_1 + \epsilon_2\right) \qquad (e)$$

For $E = 72$ GPa and $\nu = 0.33$, Eqs (e) and (d) become

$$\begin{aligned} \sigma_1 &= (80.80\,\epsilon_1 + 26.66\,\epsilon_2)\times 10^9 \ [\mathrm{Pa}] \\ \sigma_2 &= (26.66\,\epsilon_1 + 80.80\,\epsilon_2)\times 10^9 \ [\mathrm{Pa}] \end{aligned} \qquad (f)$$

and

$$\epsilon_3 = -0.4925(\epsilon_1 + \epsilon_2) \qquad (g)$$

For point 1, Table P3.2, $\epsilon_1 = -0.004$ and $\epsilon_2 = -0.006$. Then, Eqs. (f) and (g) yield

$$\sigma_1 = -483.2 \text{ MPa}, \ \sigma_2 = -591.4 \text{ MPa}, \ \epsilon_3 = 0.0049.$$

Similarly for points 2, 3, 4, and 5, we have

Point 2: $\epsilon_1 = 0.008$, $\epsilon_2 = 0.002$

$$\sigma_1 = 699.7 \text{ MPa}, \ \sigma_2 = 374.9 \text{ MPa}, \ \epsilon_3 = -0.0049.$$

Point 3: $\epsilon_1 = 0.006$, $\epsilon_2 = 0.002$

$$\sigma_1 = 538.1 \text{ MPa}, \ \sigma_2 = 321.6 \text{ MPa}, \ \epsilon_3 = -0.0039$$

Point 4: $\epsilon_1 = -0.005$, $\epsilon_2 = -0.008$

$$\sigma_1 = -617.3 \text{ MPa}, \ \sigma_2 = -779.7 \text{ MPa}, \ \epsilon_3 = 0.0064$$

Point 5: $\epsilon_1 = 0.002$, $\epsilon_2 = -0.002$

$$\sigma_1 = 108.3 \text{ MPa}, \ \sigma_2 = -108.3 \text{ MPa}, \ \epsilon_3 = 0$$

3.3

Since $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = \sigma_{xy} = \epsilon_{yy} = 0$,

$$\epsilon_{yy} = 0 = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx}$$

$$\sigma_{yy} = \nu\sigma_{xx} = 0.29(500) = \underline{145.0 \text{ MPa}}$$

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} = \frac{500}{200{,}000} - \frac{0.29(145)}{200{,}000} = \underline{0.002290}$$

$$\epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{0.29(500+145)}{200{,}000} = \underline{-0.000935}$$

$$L_x = 800(1.002290) = \underline{801.83 \text{ mm}}$$

$$L_y = \underline{800 \text{ mm}}$$

$$L_z = 10(1-0.000935) = \underline{9.99 \text{ mm}}$$

3.4

Since $\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = \sigma_{xy} = 0$ and $\epsilon_{xx} = 2\epsilon_{yy}$

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} = 2\epsilon_{yy}$$

$$\epsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{zz}$$

$$\epsilon_{zz} = 0 = \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}$$

$$\sigma_{yy} = \frac{\sigma_{xx}(1+2\nu+\nu^2)}{2+\nu-\nu^2} = \frac{500[1+2(0.29)+(0.29)^2]}{2+0.29-(0.29)^2}$$

$$= \underline{377.2 \text{ MPa}}$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = 0.29(500+377.2)$$

$$= \underline{254.4 \text{ MPa}}$$

3.5 With $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$, the stress-strain relations are, by Eqs. (3.28),

$$
\begin{aligned}
-p &= \lambda e + 2G\epsilon_{xx}, \quad e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \\
-p &= \lambda e + 2G\epsilon_{yy} \qquad\qquad\qquad (a) \\
-p &= \lambda e + 2G\epsilon_{zz}
\end{aligned}
$$

adding Eqs. (a), we have

$$-3p = (3\lambda + 2G)e$$

Therefore,

$$p = -Ke$$

where

$$K = \frac{3\lambda + 2G}{3} = \frac{E}{3(1-2\nu)}$$

since $\lambda = \nu E / [(1+\nu)(1-2\nu)]$ and $G = E/[2(1+\nu)]$

3.6 Initially the undeformed unit cube has volume $V_0 = 1$
The volume of the deformed cube is

$$
\begin{aligned}
V_1 &= (1+\epsilon_{xx})(1+\epsilon_{yy})(1+\epsilon_{zz}) \\
&= 1 + \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} + \text{higher order terms}
\end{aligned}
$$

The change in volume is, with Eqs. (3.30)

$$\Delta V = V_1 - V_0 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \frac{1-2\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

Therefore, $\Delta V = 0$ for $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$; $\nu \neq 1/2$.
Note: If $\nu = \frac{1}{2}$, $\Delta V = 0$ for all values of $\sigma_{xx} + \sigma_{yy} + \sigma_{zz}$.

3.7

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{2(-150)}{250+50} = -1.000$$

$$\theta = \underline{-0.3927 \text{ rad}}$$

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \frac{250-50}{2} + \sqrt{\left(\frac{250+50}{2}\right)^2 + (-150)^2}$$

$$= 100 + 212.1 = \underline{312.1 \text{ MPa}}$$

$$\sigma_2 = 100 - 212.1 = \underline{-112.1 \text{ MPa}}$$

$$\sigma_3 = \underline{0}$$

$$\epsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 = \frac{312.1}{72{,}000} - \frac{0.33(-112.1)}{72{,}000} = \underline{0.00485}$$

$$\epsilon_2 = \frac{1}{E}\sigma_2 - \frac{\nu}{E}\sigma_1 = -\frac{112.1}{72{,}000} - \frac{0.33(312.1)}{72{,}000} = \underline{-0.00299}$$

3.8

$$G = \frac{E}{2(1+\nu)} = \frac{82{,}600}{2(1+0.35)} = 30{,}590 \text{ MPa}$$

$$\epsilon_{zz} = 0 = \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy})$$

$$\sigma_{zz} = 0.35(90-50) = \underline{14.0 \text{ MPa}}$$

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{yy}+\sigma_{zz}) = \frac{90}{82{,}600} - \frac{0.35(-50+14.0)}{82{,}600} = \underline{0.00124}$$

$$\epsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{zz}) = -\frac{50}{82{,}600} - \frac{0.35(90+14.0)}{82{,}600} = \underline{-0.00105}$$

$$\gamma_{xy} = \frac{1}{G}\sigma_{xy} = \frac{70}{30{,}590} = \underline{0.00229}$$

3.9

$$E\epsilon_{xx} = 2E\epsilon_{yy} = \sigma_{xx} - \nu\sigma_{yy} = 2(\sigma_{yy} - \nu\sigma_{xx})$$

$$\sigma_{yy} = \frac{\sigma_{xx}(1+2\nu)}{2+\nu} = \frac{500[1+2(0.29)]}{2+0.29} = \underline{345.0 \text{ MPa}}$$

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} = \frac{500}{200{,}000} - \frac{0.29(345.0)}{200{,}000} = \underline{0.00200}$$

$$\epsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx} = \frac{345.0}{200{,}000} - \frac{0.29(500)}{200{,}000} = \underline{0.00100}$$

$$\epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}) = -\frac{0.29}{200{,}000}(500+345.0) = \underline{-0.00123}$$

$$L_x = 800(1+0.00200) = \underline{801.60 \text{ mm}}$$

$$L_y = 800(1+0.00100) = \underline{800.80 \text{ mm}}$$

$$L_z = 10(1-0.00123) = \underline{9.99 \text{ mm}}$$

3.10

From Prob. (2.78), $\gamma_{xy} = 2\epsilon_b - \epsilon_a - \epsilon_c$. With $\epsilon_a = 0.00250\ \epsilon_{xx}$
$\epsilon_b = 0.00140$, $\epsilon_c = -0.00125 = \epsilon_{yy}$, $\gamma_{xy} = 0.00155$. By Table 1, Appendix A, $E = 72.0$ GPa, $\nu = 0.33$.

(a) Since $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, Eqs. (3.32) reduce to

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy});\ \sigma_{yy} = \frac{E}{1-\nu^2}(\epsilon_{yy} + \nu\epsilon_{xx});\ \sigma_{xy} = G\gamma_{xy}.$$

Hence,
$$\sigma_{xx} = \frac{72{,}000}{1-0.33^2}\left[0.0025 + 0.33(-0.00125)\right] = 168.67 \text{ MPa}$$

$$\sigma_{yy} = \frac{72{,}000}{1-0.33^2}\left[-0.00125 + 0.33(.00250)\right] = -34.34 \text{ MPa}$$

$$\sigma_{xy} = \frac{72{,}000}{2(1+0.33)}(0.00155) = 41.95 \text{ MPa}$$

By Eqs. (2.37),

$$\sigma_1 = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2}$$

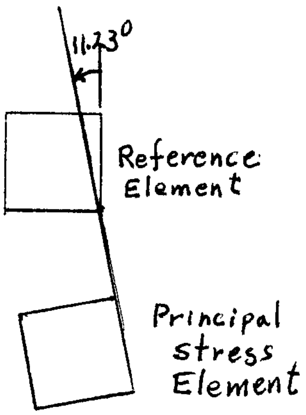
$$= \frac{1}{2}(168.67 - 34.34) + \sqrt{\left(\frac{168.67 + 34.34}{2}\right)^2 + 41.95^2}$$

$$= 67.16 + 109.83 = 176.99 \text{ MPa}$$

$$\sigma_2 = 67.16 - 109.83 = -42.67 \text{ MPa}$$

(b) $\tan 2\theta = \dfrac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = 0.4133$; $2\theta = 0.3919 \text{ rad} = 22.45°$

$\therefore\ \theta = 11.23°$



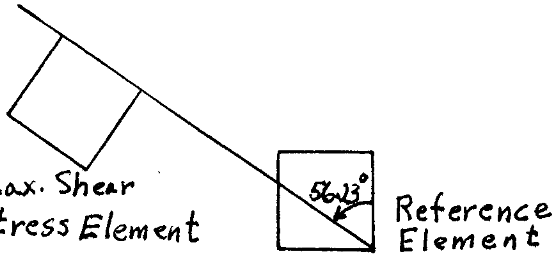
(Cont.)

3.10 continued

(c) $\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}(176.99 + 42.67)$

or $\tau_{max} = 109.83$ MPa.

(d) θ for $\tau_{max} = 11.23° + 45° = 56.23°$



3.11 With $\sigma_{xx} = 80$ MPa, $\sigma_{yy} = 120$ MPa and $\sigma_{xy} = 50$ MPa, Eqs. (3.30) yield

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) = \frac{1}{200{,}000}(80 - 0.29 \times 120) = 0.000226$$

$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) = \frac{1}{200{,}000}(120 - 0.29 \times 80) = 0.000484$$

$$\epsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} = \frac{1.29}{200{,}000}(50) = 0.000323$$

By Eqs. (e) and (f) of Example 2.10 the principal strains relative to (x, y) axes are

$$\epsilon_{1,2} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} \pm \sqrt{\frac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2}$$

$$= \frac{0.000226 + 0.000484}{2} \pm \sqrt{\left(\frac{0.000226 - 0.000484}{2}\right)^2 + 0.000323^2}$$

$$= 0.000355 \pm 0.000348$$

$\therefore \quad \epsilon_1 = 0.000703, \quad \epsilon_2 = 0.000007$

The third principal strain is $\epsilon_{zz} = \epsilon_3 = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$

or $\epsilon_3 = -\frac{0.29}{200{,}000}(80 + 120) = -0.00029$

3.12 For -30°, $l = \frac{\sqrt{3}}{2}$, $m = \frac{-1}{2}$, $n = 0$. Then by Eq. (2.61)

$$\epsilon_{30^\circ} = l^2 \epsilon_{xx} + m^2 \epsilon_{yy} + 2 l m \epsilon_{xy}$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 (0.000226) + \left(\frac{1}{2}\right)^2 (0.000484) + 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right)(0.000323)$$

or

$$\epsilon_{30^\circ} = 0.0000108$$

3.13 By Eq. (2.37), the principal stresses relative to (x, y) axes are given by

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$= \frac{-80 + 100}{2} \pm \sqrt{\left(\frac{-80 - 100}{2}\right)^2 + 50^2}$$

$$= 10 \pm 102.96 = 112.96, -92.96 \text{ MPa}$$

The third principal stress is $\sigma_{zz} = 0$. Ordering the principal stresses, we have

$$\sigma_1 = 112.96, \quad \sigma_2 = 0, \quad \sigma_3 = -92.96 \quad [\text{MPa}]$$

By Eqs. (3.30), the corresponding principal strains are

$$\epsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) = \frac{1}{200{,}000}(112.96 + 0.29 \times 92.96) = 0.000700$$

$$\epsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3) = \frac{1}{200{,}000}(-0.29 \times 112.96 + 0.29 \times 92.96) = -0.000029$$

$$\epsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2) = \frac{1}{200{,}000}(-92.96 - 0.29 \times 112.96) = -0.0006286$$

3.14 From Prob. 3.13, the stress components are, in MPa,

$\sigma_{xx} = -80$, $\sigma_{yy} = 100$, $\sigma_{xy} = 50$, $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$.

By Eqs. (3.30), the corresponding strain components are

$$\epsilon_{xx} = \frac{1}{200{,}000}(-80 - 0.29 \times 100) = -0.000545$$

$$\epsilon_{yy} = \frac{1}{200{,}000}(100 + 0.29 \times 80) = 0.000616$$

$$\epsilon_{zz} = -\frac{0.29}{200{,}000}(-80 + 100) = -0.000029$$

$$\epsilon_{xy} = \frac{1.29}{200{,}000}(50) = 0.000323$$

$$\epsilon_{xz} = \epsilon_{yz} = 0$$

For the direction 20° counterclockwise from the x-axis,

$$l = \cos 20^\circ = 0.9397, \quad m = \sin 20^\circ = 0.3420, \quad n = 0.$$

Therefore, by Eq. (2.61),

$$\epsilon_{20^\circ} = (0.9397)^2(-0.000545) + (0.3420)^2(0.000616) + 2(0.9397)(0.3420)(0.000323)$$

$$\epsilon_{20^\circ} = -2.016 \times 10^{-4}$$

3.15 (a) The fact that b does not change means that

$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) = 0$. $\therefore$ $\sigma_{y} = \nu\sigma_{xx} = 0.33 \times 200 = 66$ MPa

(b) $\epsilon_{xx} = \frac{\Delta a}{a} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) = \frac{1}{72{,}000}(200 - 0.33 \times 66) = 0.00248$

$\therefore$ The change in a is $\Delta a = 0.00248a$.

(c) The change in cross sectional area is due to the strain in the z-direction (⊥ to the plane of the figure). Therefore,

$\epsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}) = \frac{1}{72{,}000}(0 - 0.33 \times 200 - 0.33 \times 66) = -0.00122$

The reduction in dimension in the z-direction is $0.00122 \times 200 = 0.244$ mm, and the change in area is

$0.244 \times 200 = 48.8$ mm^2 (reduction).

3.16 In Example 3.7, $\Delta T = 100°C$ was an increase (+) in temperature. Now $\Delta T = 100°C$ is a decrease (-), or $\Delta T = -100°C$. Similarly, since the pressure is external, $p = -689.4$ kPa. With these changes, the formulas derived in Example 3.7 hold approximately for thin-wall cylinders. Thus, by the formulas developed in Example 3.7, we have

$$\sigma_{\theta S} = 37.16p + 0.8639(\Delta T) = 37.16(-0.6894) + 0.8639(-100) = -112 \text{ MPa}$$

$$\sigma_{\theta A} = 12.48p - 0.8639(\Delta T) = 12.48(-0.6894) - 0.8639(-100) = 77.5 \text{ MPa}$$

$$\sigma_{LA} = 4.28p - 1.779(\Delta T) = 4.28(-0.6894) - 1.779(-100) = 175 \text{ MPa}$$

$$\sigma_{LS} = 10.40p - 1.994(\Delta T) = 10.40(-0.6894) - 1.994(-100) = 192 \text{ MPa}$$

All stresses change signs. Other cases may also be considered. For example, $p = 689.4$, $\Delta T = -100°C$.

3.17 By Table A.1, for class 30 gray cast iron,

$$E = 103 \text{ GPa}, \quad \nu = 0.20 \quad \text{(a)}$$

Let the (x, y) axes coincide with arms a and c, respectively (Fig. a). Then in terms of the strain components ϵ_{xx}, ϵ_{yy}, ϵ_{xy}, the extensional strains in the direction of the arms a, b, and c are [see Eqs. (2.61)]

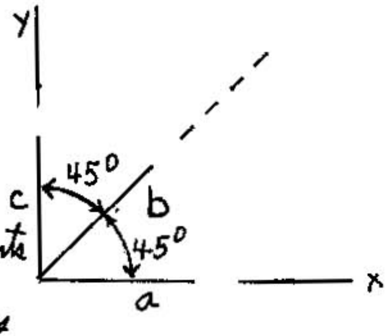


Figure a

$$\epsilon_a = \epsilon_{xx} = 0.00080$$

$$\epsilon_b = \epsilon_{xx}\left(\tfrac{1}{\sqrt{2}}\right)^2 + \epsilon_{yy}\left(\tfrac{1}{\sqrt{2}}\right)^2 + 2\epsilon_{xy}\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right) = 0.00010 \quad \text{(b)}$$

$$\epsilon_c = \epsilon_{yy} = 0.00040$$

Solving Eqs (b) for ϵ_{xy}, we obtain

$$\epsilon_{xy} = \epsilon_b - \tfrac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) = 0.00010 - \tfrac{1}{2}(0.00080 + 0.00040) = -0.005$$

(Cont.)

3.17 cont. Hence, the (x, y) strain components are

$$\epsilon_{xx} = 0.00080, \quad \epsilon_{yy} = 0.00040, \quad \epsilon_{xy} = -0.0005 \qquad \text{(c)}$$

Since the surface is free of applied forces,

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \qquad \text{(d)}$$

where z is the axis perpendicular to the surface.

So, the state of stress at the strain gage is one of plane stress, with nonzero stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$.

Then by Eqs. (a), (c), and (3.32a),

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy}) = 94.42 \text{ MPa}$$
$$\sigma_{yy} = \frac{E}{1-\nu^2}(\nu\epsilon_{xx} + \epsilon_{yy}) = 60.08 \text{ MPa}$$
$$\sigma_{xy} = \frac{E}{1+\nu}\epsilon_{xy} = -42.92 \text{ MPa}$$

3.18 (a) Since the strains are measured on a free surface, the state of stress is one of plane stress in the (x, y) plane;

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \qquad \text{(a)}$$

Then, by Eqs. (3.32a), the nonzero stress components are

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy})$$
$$\sigma_{yy} = \frac{E}{1-\nu^2}(\nu\epsilon_{xx} + \epsilon_{yy}) \qquad \text{(b)}$$
$$\sigma_{xy} = \frac{E}{1+\nu}\epsilon_{xy}$$

Since $E = 72\text{GPa}$, $\nu = 0.33$, $\epsilon_{xx} = 0.0020$, $\epsilon_{yy} = 0.0010$, and $\epsilon_{xy} = 0.0010$, Eqs. (b) yield

$$\sigma_{xx} = 188.26 \text{ MPa}, \quad \sigma_{yy} = 134.13 \text{ MPa}, \quad \sigma_{xy} = 54.14 \text{ MPa} \qquad \text{(c)}$$

(cont.)

3.18 cont. (b) By Eqs. (C) and the third of Eqs. (2.37),

$$\tau_{max} = \sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^{\ 2}} = 60.53 \text{ MPa}$$

So, the design requirement that τ_{max} not exceed 70 MPa is satisfied.

3.19 Since the strain measurements are on a free surface, plane stress exists there. By Eqs. (3.32a),

$$\sigma_1 = \frac{E}{1-\nu^2}(\epsilon_1+\nu\epsilon_2)$$
$$\sigma_2 = \frac{E}{1-\nu^2}(\nu\epsilon_1+\epsilon_2) \qquad (a)$$

By Table A.1, $E = 110$ GPa, $\nu = 0.35$. Then, with the given data, $\epsilon_1 = 0.0015$, $\epsilon_2 = 0.0005$, Eqs. (a) yield

$$\sigma_1 = 209.97 \text{ MPa}, \quad \sigma_2 = 128.49 \text{ MPa}$$

Hence, $\sigma_{max} = \sigma_1 = 209.97 \text{ MPa} > 200 \text{ MPa}$. Therefore, the design criterion that $\sigma_{max} \leq 200$ MPa is not satisfied.

3.20 (a) First we calculate the quantities E, ν, and β as follows [see Eqs. (g) of Example 3.6], with $E_F = 72.4$ GPa, $E_R = 3.50$ GPa, $\nu_F = 0.30$, $\nu_R = 0.30$, and $f = 0.70$,

$$E = fE_F + (1-f)E_R = 0.70(72.4) + 0.30(3.50) = 51.73 \text{ GPa}$$

$$\nu = f\nu_F + (1-f)\nu_R = 0.70(0.30) + 0.30(0.30) = 0.30 \qquad (a)$$

$$\beta = f(1-f)\left[(1-\nu_R^2)\frac{E_F}{E_R} + (1-\nu_F^2)\frac{E_R}{E_F} + 2\nu_F\nu_R + \frac{1-f}{f} + \frac{f}{1-f}\right]$$

$$= 0.70(0.30)\left[(1-0.30^2)\frac{72.4}{3.50} + (1-0.30^2)\left(\frac{3.50}{72.4}\right) + 2(0.30)(0.30) + \frac{0.30}{0.70} + \frac{0.70}{0.30}\right]$$

$$= 4.580$$

(cont.)

3.20 cont. Then, by Eqs.(a) and Eqs.(l) and (n) of Example 3.6, with $G_F = 27.8$ GPa and $G_R = 1.35$ GPa,

$$C_{11} = \frac{\beta E}{\beta - \nu^2} = \frac{4.580(51.73)}{4.580 - 0.30^2} = 52.77 \text{ GPa}$$

$$C_{22} = \frac{E}{\beta - \nu^2} = \frac{51.73}{4.580 - 0.30^2} = 11.52 \text{ GPa}$$

$$C_{12} = \frac{\nu E}{\beta - \nu^2} = \frac{0.30(51.73)}{4.580 - 0.30^2} = 3.46 \text{ GPa} \qquad (b)$$

$$C_{33} = G = \frac{G_F G_R}{f G_R + (1-f) G_F} = \frac{27.8(1.35)}{0.70(1.35) + 0.30(27.8)} = 4.04 \text{ GPa}$$

(b) By Eqs.(b) and Eqs.(m) of Example 3.6, the stress components are, with $\epsilon_{xx} = 500\mu$, $\epsilon_{yy} = 350\mu$, $\gamma_{xy} = 1000\mu$,

$$\sigma_{xx} = C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} = 52.77(500) + 3.46(350) = 27.60 \text{ MPa}$$

$$\sigma_{yy} = C_{12}\epsilon_{xx} + C_{22}\epsilon_{yy} = 3.46(500) + 11.52(350) = 5.76 \text{ MPa} \qquad (c)$$

$$\sigma_{xy} = C_{33}\gamma_{xy} = 4.04(1000) = 4.04 \text{ MPa}$$

By Eqs.(c), the stress invariants are

$$I_1 = \sigma_{xx} + \sigma_{yy} = 33.36, \; I_2 = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = 142.65, \; I_3 = 0 \qquad (d)$$

The principal stresses are the roots of the equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma(\sigma^2 - 33.36\sigma + 142.65) = 0$$

or

$$\sigma_1 = 28.32 \text{ MPa}, \; \sigma_2 = 5.036 \text{ MPa}, \; \sigma_3 = 0$$

The principal stress axes are the roots l_i, m_i, n_i $i = 1, 2, 3$, of the equations

(Cont.)

3.20 cont.

$$l_i(\sigma_{xx}-\sigma_i)+m_i\sigma_{xy}+n_i\sigma_{xz}=0$$

$$l_i\sigma_{xy}+m_i(\sigma_{yy}-\sigma_i)+n_i\sigma_{yz}=0$$

$$l_i\sigma_{xz}+m_i\sigma_{yz}+n_i(\sigma_{zz}-\sigma_i)=0$$

$$l_i^2+m_i^2+n_i^2=1$$

$$i=1,\ 2,\ 3.$$

For $\sigma_i=\sigma_1$, we have with $\sigma_{xz}=\sigma_{yz}=\sigma_{zz}=0$

$$l_1(27.6-28.32)+m_1(4.04)=0$$

$$l_1(4.04)+m_1(5.76-28.32)=0$$

$$n_1(-28.32)=0$$

$$l_1^2+m_1^2+n_1^2=1$$

These equations yield the direction cosines

$$l_1=0.9845,\quad m_1=0.1754,\ n_1=0$$

Similarly for $\sigma=\sigma_2$ and $\sigma=\sigma_3$, we find

$$l_2=-0.1754,\ m_2=0.9845,\ n_2=0$$

$$l_3=m_3=0,\quad n_3=1$$

3.21 (a) Since $\epsilon_{xz} = \epsilon_{yz} = 0$, the z-axis is a principal strain axis. The other principal strain axes lie in the (x, y) plane. The counterclockwise rotation from (x, y) axes to principal strain axes is given by, with $\epsilon_{xx} = 0.0003$, $\epsilon_{yy} = 0.0002$, and $\epsilon_{xy} = 0.00005$,

$$\theta_{strain} = \frac{1}{2} \arctan \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = 0.3927 \text{ rad}$$

or

$$\theta_{strain} = 22.5^0$$

(b) By the given stress-strain relations, with $C_1 = 103$ GPa, $C_2 = 55$ GPa, and $C_3 = 27.6$ GPa, we have

$$\sigma_{xx} = C_1 \epsilon_{xx} + C_2(\epsilon_{yy} + \epsilon_{zz}) = 103(0.0003) + 55(0.0002 + 0.0001) = 47.4 \text{MPa}$$

$$\sigma_{yy} = C_2(\epsilon_{xx} + \epsilon_{zz}) + C_1 \epsilon_{yy} = 55(0.0003 + 0.0001) + 103(0.0002) = 42.6 \text{MPa}$$

$$\sigma_{zz} = C_2(\epsilon_{xx} + \epsilon_{yy}) + C_1 \epsilon_{zz} = 55(0.0003 + 0.0002) + 103(0.0001) = 37.8 \text{MPa}$$

$$\sigma_{xy} = 2C_3 \epsilon_{xy} = 2(27.6)(0.00005) = 2.76 \text{ MPa}$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

(c) Since $\sigma_{xz} = \sigma_{yz} = 0$, the z-axis is a principal stress axis. The counterclockwise rotation from the (x, y) axes to principal stress axes is given by

$$\theta_{stress} = \frac{1}{2} \arctan \frac{2(2.76)}{47.4 - 42.6} = 0.42753 \text{ rad}$$

or

$$\theta_{stress} = 24.5^0$$

Hence, principal axes of stress and strain do not coincide.

(d) As in part a, $\theta_{strain} = 22.5^0$. For the isotropic case, the stress-strain relations are [see Eqs. (3.28)]

$$\sigma_{xx} = \lambda e + 2G\epsilon_{xx}, \quad \sigma_{yy} = \lambda e + 2G\epsilon_{yy}, \quad \sigma_{zz} = \lambda e + 2G\epsilon_{zz}$$

$$\sigma_{xy} = 2G\epsilon_{xy}, \quad \sigma_{xz} = 2G\epsilon_{xz}, \quad \sigma_{yz} = 2G\epsilon_{yz} \qquad (a)$$

(cont.)

3.21 cont. where, with $E = 72\,GPa$ and $\nu = 0.33$,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{0.33(72)}{1.33(1-0.66)} = 52.54\,GPa$$

$$G = \frac{E}{2(1+\nu)} = \frac{72}{2(1.33)} = 27.07\,GPa \qquad (b)$$

and

$$e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0.0003 + 0.0002 + 0.0001 = 0.0006 \,. \quad (c)$$

Hence, by Eqs. (a), (b), and (c),

$$\sigma_{xx} = 52.54(0.0006) + 2(27.07)(0.0003) = 47.77\ MPa$$

$$\sigma_{yy} = 52.54(0.0006) + 2(27.07)(0.0002) = 42.35\,MPa$$

$$\sigma_{zz} = 52.54(0.0006) + 2(27.07)(0.0001) = 36.94\,MPa$$

$$\sigma_{xy} = 2(27.07)(0.00005) = 2.707\ MPa$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

Therefore, the z axis is a principal stress axis. In the (x, y) plane, the principal stress axes are rotated counterclockwise by the angle

$$\theta_{stress} = \frac{1}{2} \arctan \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{1}{2} \arctan \frac{2(2.707)}{47.77 - 42.35}$$

or

$$\theta_{stress} = 22.5^0$$

Therefore, the principal axes of stress and strain coincide.

3.22 As in Problem 3.21, the z axis is a principal axis for both stress and strain.

(a) In the (x, y) plane, for stress, the angle of rotation from (x, y) axes to principal axes is, with $\sigma_{xx} = 7\,MPa$, $\sigma_{yy} = 2.1\ MPa$, $\sigma_{xy} = 1.4\,MPa$

$$\theta_{stress} = \frac{1}{2} \arctan \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{1}{2} \arctan \frac{2(1.4)}{7 - 2.1} = 14.87^0 \quad (a)$$

Counterclockwise

(Cont.)

3.22 cont. (b). With $E_x = 15{,}920\text{ MPa}$, $E_y = 1195\text{ MPa}$, $E_z = 765\text{ MPa}$, $\nu_{xy} = 0.426$, $\nu_{xz} = 0.451$, $\nu_{yz} = 0.697$, (b)

Eqs. (3.52) yield

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x} = \frac{0.426}{15{,}290} = 2.786\times10^{-5}, \quad \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x} = \frac{0.451}{15{,}290} = 2.950\times10^{-5}$$

$$\frac{\nu_{zy}}{E_z} = \frac{\nu_{yz}}{E_y} = \frac{0.697}{1195} = 5.833\times10^{-4} \qquad \text{(c)}$$

With Eqs(b) and (c) and $\sigma_{xx} = 7\text{ MPa}$, $\sigma_{yy} = 2.1\text{ MPa}$, $\sigma_{zz} = 2.8\text{ MPa}$, $\sigma_{xy} = 1.4\text{ MPa}$, $G_{xy} = 1130\text{ MPa}$, $G_{xz} = 1040\text{ MPa}$, $G_{yz} = 260\text{ MPa}$ and $\sigma_{xz} = \sigma_{yz} = 0$, Eqs. (3.51) yield

$$\epsilon_{xx} = \frac{1}{E_x}\sigma_{xx} - \frac{\nu_{yx}}{E_y}\sigma_{yy} - \frac{\nu_{zx}}{E_z}\sigma_{zz} = \frac{7}{15290} - (2.786\times10^{-5})(2.1) - (2.95\times10^{-5})(-2.8) = 0.0004819$$

$$\epsilon_{yy} = -\frac{\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_{yy} - \frac{\nu_{zy}}{E_z}\sigma_{zz} = -(2.786\times10^{-5})(7) + \frac{2.1}{1195} - (5.833\times10^{-4})(-2.8) = 0.003195$$

$$\epsilon_{zz} = -\frac{\nu_{xz}}{E_x}\sigma_x - \frac{\nu_{yz}}{E_y}\sigma_{yy} + \frac{1}{E_z}\sigma_{zz} = -(2.95\times10^{-5})(7) - (5.833\times10^{-4})(2.1) + \frac{(-2.1)}{765} = -0.005092 \qquad \text{(d)}$$

$$\gamma_{xy} = \frac{1}{G_{xy}}\sigma_{xy} = \frac{1.4}{1130} = 0.001239$$

$$\gamma_{xz} = \gamma_{yz} = 0$$

(c) With Eqs.(d), the angle of rotation in the (x, y) plane from the (x, y) axes to the principal axes is

$$\theta_{\text{strain}} = \frac{1}{2}\arctan\frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = \frac{1}{2}\arctan\frac{0.001239}{0.0004819 - 0.003195}$$

or

$$\theta_{\text{strain}} = -12.27^\circ \text{ (}12.27^\circ\text{ clockwise)} \qquad \text{(e)}$$

Comparison of Eqs.(a) and (e) shows that

$$\theta_{\text{stress}} \neq \theta_{\text{strain}}$$

4.1

BY FIG. E4.1a, THE LENGTH OF THE ADDED MEMBER BD IS 2400 mm. THE LENGTHS OF MEMBERS AD AND CD ARE EQUAL TO 3000 mm. FOR A GIVEN DISPLACEMENT u, THE ELONGATION OF MEMBER BD IS u, AND THE ELONGATIONS OF MEMBERS AD AND CD ARE EQUAL TO $u\cos\theta = 0.80u$. THEREFORE, IN GENERAL THE STRAINS IN MEMBERS BD, AD, AND CD ARE

$\epsilon_{BD} = u/2400$, $\epsilon_{AD} = \epsilon_{CD} = 0.8u/3000 = 0.64\epsilon_{BD}$.

BY SUMMATION OF FORCES ACTING ON PIN D, THE LOAD P IS:

$$P = (\sigma_{BD} + 2\sigma_{AD}\cos\theta)A = (\sigma_{BD} + 1.6\sigma_{AD})A,$$

WHERE $\cos\theta = 0.8$, σ_{BD} AND σ_{AD} ARE STRESSES IN MEMBERS BD AND AD, AND $A = 645\ mm^2$ IS THE CROSS-SECTIONAL AREA OF EACH BAR.

FOR $u = 2mm$, $\epsilon_{BD} = 0.000833 < \epsilon_Y$ ($\epsilon_Y = 0.001195$), $\epsilon_{AD} = \epsilon_{CD} = 0.000533 < \epsilon_Y$.

THEREFORE ALL BARS ARE ELASTIC AND

$$\sigma_{BD} = 211400(.000833) = 176.17\ MPa$$

$$\sigma_{AD} = \sigma_{CD} = 211400(.000533) = 112.75\ MPa$$

HENCE, $P = (176.17 + (1.6)112.75)645 = \underline{\underline{229.98\ kN}}$

FOR $u = 4mm$, $\epsilon_{BD} = 0.001667 > \epsilon_Y$, $\epsilon_{AD} = \epsilon_{CD} = 0.001067 < \epsilon_Y$.

THUS, BAR BD HAS YIELDED, BUT BARS AD AND CD ARE ELASTIC, AND

$$\sigma_{BD} = 232.4 + 16900(.001667) = 260.57\ MPa.$$

$$\sigma_{AD} = \sigma_{CD} = 211400(0.001067) = 225.49\ MPa.$$

$$P = (260.57 + (1.6)225.49)645 = \underline{\underline{400.78\ kN}}.$$

FOR $u = 4.481\ mm$, $\epsilon_{BD} = 0.001867$, $\epsilon_{AD} = \epsilon_{CD} = 0.001195 = \epsilon_Y$

$\sigma_{BD} = 232.4 + 16900(.001867) = 263.95\ MPa$; $\sigma_{AD} = \sigma_{CD} = Y = 252.6\ MPa$

$P = (263.95 + (1.6)252.6)645 = \underline{\underline{430.93\ kN}}$.

(cont.)

4.1 (CONTINUED)

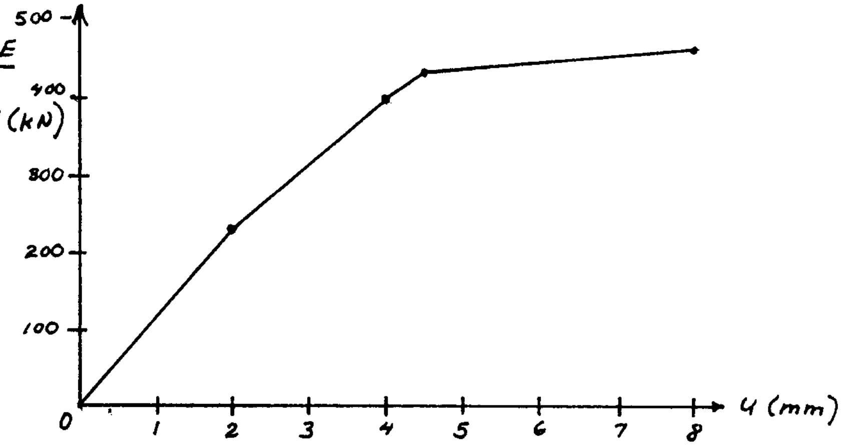
FOR $u = 8mm$, $\epsilon_{BD} = 0.003333$, $\epsilon_{AD} = \epsilon_{CD} = 0.002133$

$$\sigma_{BD} = 232.4 + 16900(.003333) = 288.73\ MPa$$

$$\sigma_{AD} = \sigma_{CD} = 232.4 + 16900(.002133) = 268.45\ MPa$$

$$P = (288.73 + (1.6)268.45)645 = \underline{\underline{463.27\ kN}}.$$

LOAD-DEFLECTION CURVE



4.2 BY FIG. E4.1b, SUMMATION OF FORCES IN THE DIRECTION OF P GIVES:

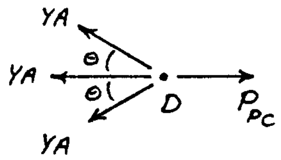
$$P_{PC} = 2F\cos\theta = 2YA\cos\theta = 2(252.6)645(0.8) = \underline{\underline{260.68\ kN}}$$

4.3 AT PLASTIC COLLAPSE, THE STRESS IN THE THREE BARS IS Y.

THE FORCE IN EACH BAR IS YA AND FOR EQUILIBRIUM

$$P_{PC} = YA + 2YA\cos\theta = YA(1 + 2\cos\theta)$$

$$= (252.6)(645)(1 + 2(0.8)) = \underline{\underline{423.61\ kN}}$$



4.4 Let subscript S denote steel and subcript A denote aluminum. Then from Appendix A, $E_S = 200\ GPa$, $Y_S = 250\ MPa$, $E_A = 72\ MPa$, and $Y_A = 500\ MPa$. For a given vertical displacement u of beam ABC, the strains in the steel and aluminum bars are, respectively, $\epsilon_S = u/L_S$ and $\epsilon_A = u/L_A$. Thus, the stresses in the bars are

$$\sigma_S = E_S\epsilon_S = E_S u/L_S \quad \text{and} \quad \sigma_A = E_A\epsilon_A = E_A u/L_A.$$

(Cont.)

4.4 Continued: For yield of the steel bars, $u = Y_S L_S / E_S$ or $u = 250(1.5)/200 = 1.875$ mm. For yield of the aluminun bar $u = Y_A L_A / E_A = 500(1.2)/72 = 8.333$ mm. Therefore, the steel bars yield first. With $u = 1.875$ mm, the stress in the steel bars is $Y_S = 250$ MPa, and the stress in the aluminum bar is $\sigma_A = E_A u / L_A = \frac{72(1.875)}{1.2} = 112.5$ MPa

(a) At yield of the steel bars, summation of forces in the vertical direction yields the result

$$P = P_Y = 2 Y_S A + \sigma_A A = [2(250) + 112.5](100) = 61.25 \text{ kN}$$

(b) Similarly at yield of the aluminum bar,

$$P = P_P = 2 Y_S A + Y_A A = [2(250) + 500](100) = 100 \text{ kN}$$

4.5 Consider the joint B, Fig. a, subject to displacement u.

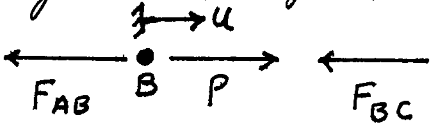


Fig. a

By equilibrium, $P = F_{AB} + F_{BC}$. Also, the strain in each bar is $\epsilon = u/L$ ($L = 1.0$ m). Therefore, $F_{AB} = \sigma_{AB} A = E \epsilon A = EAu/L$ and $F_{BC} = \sigma_{BC} A = E\epsilon A = EAu/L = F_{AB}$ (as long as AB remains elastic). At initiation of yield in bar AB, $F_{AB} = Y_{AB} A = F_{BC}$. Thus, at initial yield (in tension),

$$F_{AB} = F_{BC} = Y_{AB} A = 250 \times 25 = 6.25 \text{ kN}. \text{ Therefore,}$$

$$P = P_{YT} = F_{AB} + F_{AC} = 2(6.25) = 12.5 \text{ kN, and}$$

$$u = Y_{AB} L / E = 250(1)/200 = 1.25 \text{ mm}.$$

When $P = 20$ kN, $F_{AB} = 6.25$ kN and $F_{BC} = P - F_{AB} = 13.75$ kN.

(Cont.)

4.5 | Continued: Then $u = F_{BC}L/(EA) = 13.75(1)/(200\times25) = 2.75$ mm.

Upon unloading, member AB has zero stress when u is reduced by 1.25 mm, and it yields in compression when u is further reduced by 1.25 mm; that is, when $u = 2.75$ mm $- 2.50$ mm $= 0.25$ mm. Then, $F_{AB} = -6.25$ kN and $\sigma_{BC} = E\epsilon = 200(0.25)/1.0 = 50$ MPa. Thus, at yield of bar AB in compression, $P = P_{YC} = -6.25 + 50\times25 = -5.0$ kN.

When $P = -20$ kN, $P = -20 = -250\times25 + F_{BC}$. $\therefore F_{BC} = -13.73$ kN, and $u = F_{BC}L/(EA) = -13.75(1.0)/(200\times25) = -2.75$ mm.

As P increases from -20 kN to 20 kN, yield in bar AB (in tension) occurs at $u = -0.25$ mm (at an increase in u of 2.50 mm). Then, $F_{BC} = EAu/L = 200(25)(-0.25)/1.0 = 1.25$ kN. Thus, $P = 6.25 - 1.25 = 5.0$ kN.

For $P = 20$ kN, $F_{BC} = P - F_{AB} = 20 - 6.25 = 13.75$ kN, and again, $u = F_{BC}L/(EA) = 13.75(1.0)/(200\times25) = 2.75$ kN

SUMMARY

P(kN)	u (mm)
12.5	1.25
20.0	2.75
−5.0	0.25
−20.0	−2.75
5.0	−0.25
20.0	2.75

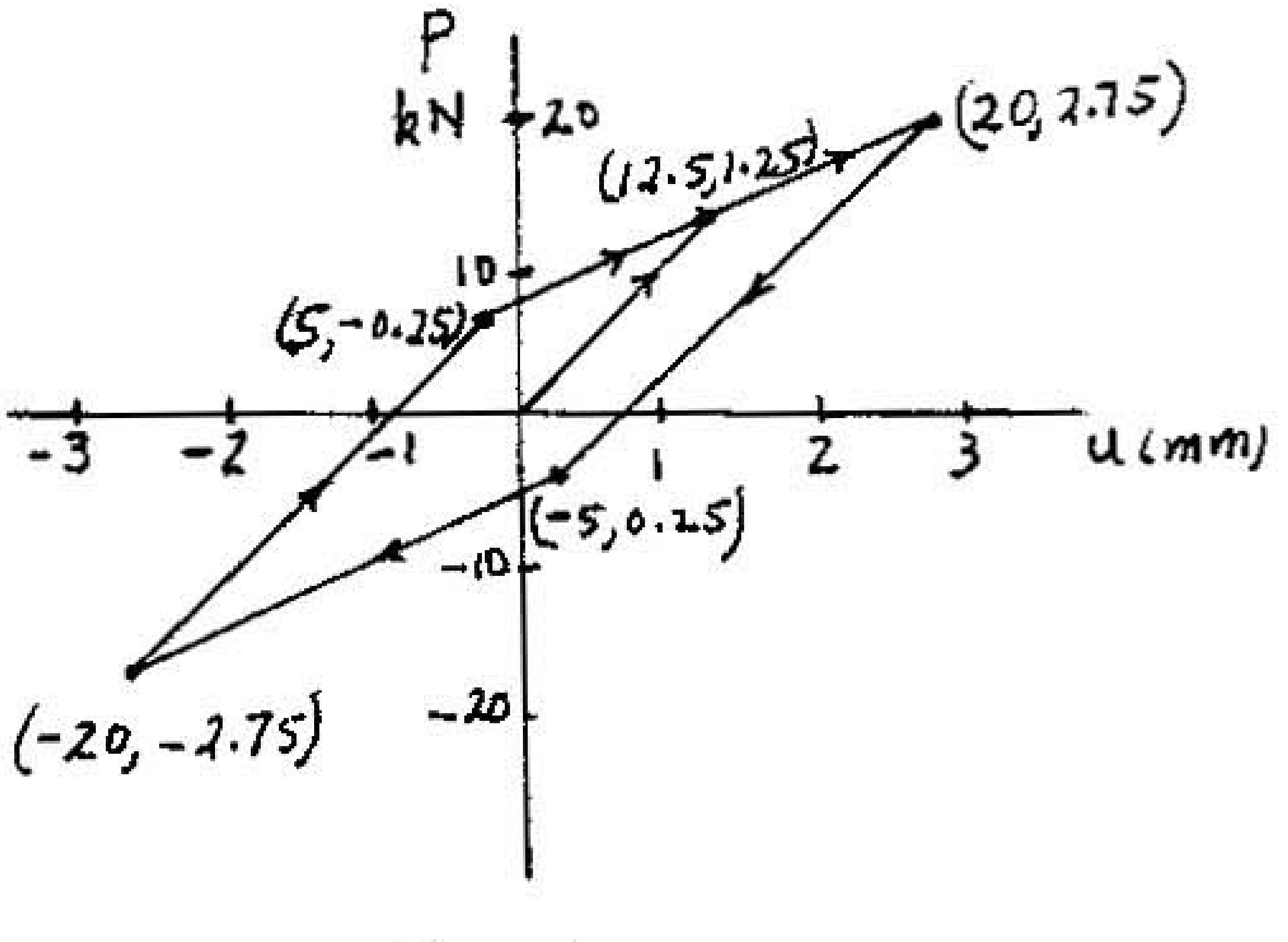


Fig. b

4.6 In the torsion test of a circular cross section bar, $\sigma_1 = \tau_u$, $\sigma_2 = 0$, $\sigma_3 = -\tau_u$. In the tension test, $\sigma_1 = \sigma_u$, $\sigma_2 = \sigma_3 = 0$. The maximum principal stress theory of failure by fracture requires

$$f = \max(|\sigma_1|, |\sigma_2|, |\sigma_3|) - \sigma_u = 0 \qquad \text{or}$$

$$\sigma_u = \tau_u$$

4.7 Since $\sigma_{xz} = \sigma_{yz} = 0$, $\sigma_{zz} = 0$ is a principal stress. By Eq.(2.37), the remaining two principal stresses are

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} = 10 + 70\sqrt{2} = 108.99 \text{ MPa}$$

$$\sigma_3 = 10 - 70\sqrt{2} = -88.99 \text{ MPa}, \text{ with } \sigma_{xx} = -60\text{MPa}, \sigma_{yy} = 80\text{MPa}, \sigma_{xy} = 70\text{MPa}.$$

(a) The maximum principal stress criterion requires

$$(SF)\sigma_{max} = (SF)\sigma_1 = \sigma_u \quad \text{or} \quad SF = \frac{\sigma_u}{\sigma_1} = \frac{460}{108.99} = 4.22$$

(b) By Eqs.(4.5) and (4.6), with $\sigma_u = 460\text{MPa}$,

$$f = (SF)(\sigma_1 - \nu\sigma_3) - \sigma_u = 0. \quad \text{Therefore}$$

$$SF = \frac{\sigma_u}{\sigma_1 - \nu\sigma_3} = \frac{460}{108.99 - 0.2(-88.99)} = 3.63$$

4.8 As in Prob 4.7, $\sigma_{zz} = 0$ is a principal stress, and with $\sigma_{xx} = 160$ MPa, $\sigma_{yy} = 0$, $\sigma_{xz} = 60$ MPa, the other two principal stresses are

$$\sigma_1 = \frac{160}{2} + \sqrt{\left(\frac{160}{2}\right)^2 + 60^2} = 80 + 100 = 180 \text{ MPa},$$

$$\sigma_3 = 80 - 100 = -20\text{MPa}. \quad \text{Also, } \sigma_2 = \sigma_{zz} = 0.$$

$$\therefore (SF)(\sigma_{max}) = (SF)\sigma_1 = \sigma_u \quad \therefore SF = \frac{\sigma_u}{\sigma_1} = \frac{590}{180} = 3.28$$

4.9 Given $\sigma_{xx} = 150$ MPa, $\sigma_{xy} = 65$ MPa, $\sigma_{yy} = \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$; therefore $\sigma_{zz} = 0$ is a principal stress. As in Prob. 4.7, the other two principal stresses are

$$\sigma_1 = \frac{150}{2} + \sqrt{\left(\frac{150}{2}\right)^2 + 65^2} = 75 + 99.25 = 174.25 \text{ MPa}$$

$$\sigma_3 = 75 - 99.25 = -24.25 \text{ MPa}; \text{ also } \sigma_2 = \sigma_{zz} = 0.$$

$$\therefore \tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{1}{2}(174.25 + 24.25) = 99.25 \text{ MPa}$$

(a) The Tresca criterion requires

$$(SF)\,\tau_{max} = \frac{1}{2}Y \quad \therefore SF = \frac{Y}{2\tau_{max}} = \frac{450}{2(99.25)} = 2.267$$

(b) The von Mises criterion requires [see Eq. (4.21)]

$$\frac{(SF)}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] = \frac{1}{3}Y^2$$

Therefore,

$$SF = \frac{\sqrt{2}\,Y}{\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]^{1/2}} = \frac{\sqrt{2}\,(450)}{\left[174.25^2 + (24.25)^2 + 198.5^2\right]^{1/2}}$$

or

$$SF = 2.399$$

4.10 With $\epsilon_{zz} = 0$, we have $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$. Also $\sigma_{xx} = 60$ MPa, $\sigma_{yy} = 240$ MPa, and $\sigma_{xy} = -80$ MPa. Since $\epsilon_{xz} = \epsilon_{yz} = 0$, $\sigma_{xz} = \sigma_{yz} = 0$. Therefore σ_{zz} is a principal stress; $\sigma_{zz} = 0.29(60 + 240) = 87$ MPa. Also as in Prob. 4.7, the other two principal stresses are

$$\sigma_{1,2} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = 150 \pm 120.42 = 270.42, 29.58$$

or ordering the stresses $\sigma_1 = 270.42$ MPa, $\sigma_2 = 87$ MPa, $\sigma_3 = 29.58$ MPa. Now, $\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{1}{2}(\sigma_1 - \sigma_3) = 120.42$ MPa. Therefore,

$$(SF)\,\tau_{max} = \frac{1}{2}Y \quad \text{or} \quad SF = \frac{Y}{2\tau_{max}} = \frac{490}{2(120.42)} = 2.035.$$

4.11 With $\epsilon_{zz}=0$, $\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$. Also, $\sigma_{xx} = 60$ MPa, $\sigma_{yy} = 240$ MPa, and $\sigma_{xy} = -80$ MPa. Since $\epsilon_{xz} = \epsilon_{yz} = 0$, $\sigma_{xz} = \sigma_{yz} = 0$. Therefore, σ_{zz} is a principal stress; with $\nu = 0.29$

$$\sigma_{zz} = 0.29(60+240) = 87 \text{ MPa} \qquad (a)$$

The other two principal stresses are given by

$$\sigma = \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

or

$$\sigma = 150 \pm 120.42 = 270.42,\ 29.58\ [\text{MPa}] \quad (b)$$

By Eqs. (a) and (b), ordering the principal stresses, we have

$$\sigma_1 = 270.42 \text{ MPa},\ \sigma_2 = 87 \text{ MPa},\ \sigma_3 = 29.58 \text{ MPa} \quad (c)$$

By the octahedral shear-stress criterion [see Eqs. (4.20) and (4-21)],

$$|J_2| = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = \frac{1}{3}Y^2$$

Introducing a safety factor, SF, we have

$$\frac{(SF)^2}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = \frac{1}{3}Y^2 \qquad (d)$$

By Eqs. (c) and (d), with $Y = 490$ MPa, we obtain

$$SF = \frac{\sqrt{2}\,Y}{[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} = 2.249$$

So, the octahedral shear stress criterion predicts a safety factor that is larger by [see Problem 4.10]

$$\left[\frac{2.249-2.035}{2.035}\right]\times 100 = 10.5\%$$

than that predicted by the maximum shear-stress criterion.

4.12 In the z direction, $\sigma_{zz} = 0$, is a principal stress. In the (x, y) plane, the two principal stresses are given by

$$\sigma = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \qquad (a)$$

With $\sigma_{xx} = 120$ MPa, $\sigma_{yy} = 50$ MPa, and $\sigma_{xy} = 50$ MPa, Eq (a) yields

$$\sigma = 146.03,\ 23.97$$

Ordering the principal stresses, we obtain

$$\sigma_1 = 146.03 \text{ MPa},\ \sigma_2 = 23.97 \text{ MPa},\ \sigma_3 = 0 \qquad (b)$$

(a) The maximum shear stress is, with Eq. (a),

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 73.02 \text{ MPa} \qquad (c)$$

The maximum shear-stress criterion is, with safety factor SF

$$(SF)\,\tau_{max} = \frac{1}{2} Y \qquad (d)$$

By Eqs. (c) and (d), with $Y = 300$ MPa, we obtain

$$SF = 2.05$$

(b) The octahedral shear-stress criterion is, with safety factor SF

$$\frac{(SF)^2}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{1}{3} Y^2 \qquad (e)$$

Substituting Eqs. (b) into Eqs. (e) and solving for SF, we find by the octahedral shear-stress criterion

$$SF = 2.21$$

which is 7.8% larger than the safety factor predicted by the maximum shear-stress criterion.

4.13 (a) Since $\sigma_2 = \sigma_3 = \sigma_{min}$, $\sigma_1 = \sigma_{max}$. Then,

$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{1}{2}(\sigma_1 - \sigma_2) \qquad (a)$$

The maximum shear-stress criterion, with safety factor SF, is with Eq.(a)

$$(SF)\tau_{max} = \frac{SF}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}Y \qquad (b)$$

Therefore, by Eq.(b),

$$SF = \frac{Y}{\sigma_1 - \sigma_2} \qquad (c)$$

(b) By the octahedral shear-stress criterion, with safety factor SF,

$$\frac{(SF)^2}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{1}{3}Y^2 \qquad (d)$$

Since $\sigma_2 = \sigma_3$, Eq.(d) yields

$$SF = \sqrt{\frac{Y^2/3}{(\sigma_1 - \sigma_2)^2/3}} = \frac{Y}{\sigma_1 - \sigma_2} \qquad (e)$$

Thus, the same safety factor is obtained in parts (a) and (b); Eqs.(c) and (e).

4.14 First, consider the rectangular cross section. The elastic-plastic stress state is shown in Fig. a.

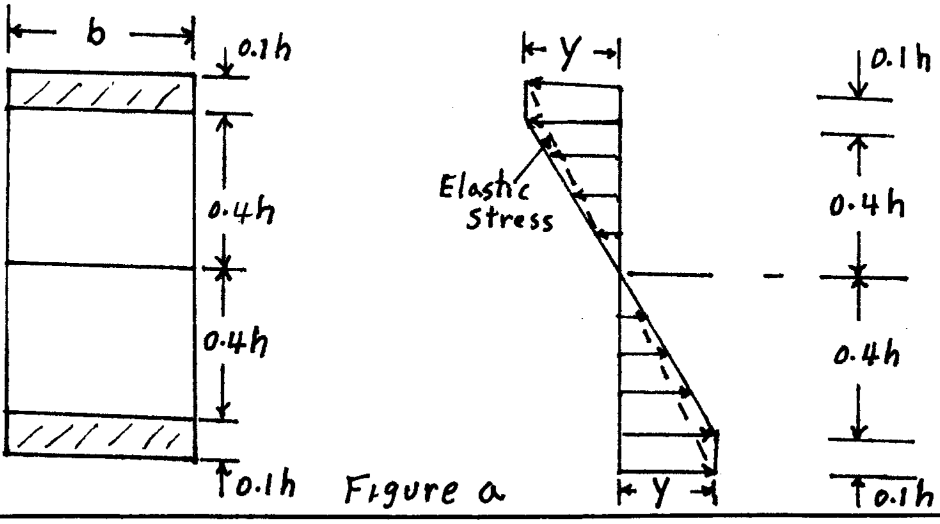


Figure a

(cont.)

4.14 cont. By Fig. a, the elastic-plastic moment is

$$M = 2Y[(0.1h)(b)(0.45h)] + 2Y[\tfrac{1}{2}(0.4h)(b)(\tfrac{2}{3})(0.4h)]$$

or since $b = \frac{1}{2}h$, in terms of h and Y,

$$M = 0.0983h^3 Y \qquad (a)$$

Also, by Fig. a, the maximum elastic moment is

$$M_Y = 2Y[\tfrac{1}{2}(0.5h)(b)(\tfrac{2}{3})(0.5h)$$

or in terms of h and Y,

$$M_Y = 0.0833h^3 Y \qquad (b)$$

By Eqs. (a) and (b),

$$\frac{M}{M_Y} = 1.240$$

For the I-shaped beam, the elastic-plastic stress state is shown in Fig. b.

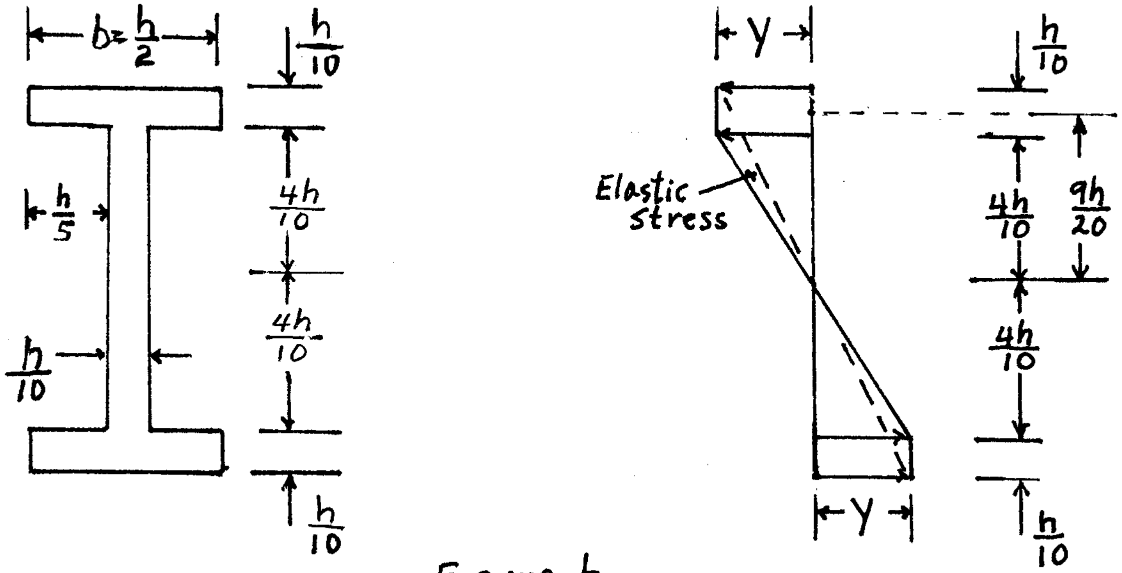


Figure b

(cont.)

4.14 cont. By Fig. b, the elastic - plastic moment is

$$M = 2Y\left[\left(\frac{h}{10}\right)\left(\frac{h}{2}\right)\left(\frac{9h}{20}\right) + \frac{1}{2}\left(\frac{4h}{10}\right)\left(\frac{h}{10}\right)\left(\frac{2}{3}\right)\left(\frac{4h}{10}\right)\right]$$

or

$$M = 0.0557 h^3 Y \qquad (c)$$

also, by Fig. b, the maximum elastic moment is

$$M_Y = 2Y\left[\frac{1}{2}\left(\frac{h}{2}\right)\left(\frac{h}{2}\right)\left(\frac{2}{3}\,\frac{h}{2}\right)\right] - 4(0.8Y)\left[\frac{1}{2}\left(\frac{4h}{10}\right)\left(\frac{h}{5}\right)\left(\frac{2}{3}\right)\left(\frac{4h}{10}\right)\right].$$

or

$$M_Y = 0.0492 h^3 Y \qquad (d)$$

By Eqs. (c) and (d),

$$\frac{M}{M_Y} = 1.13$$

4.15 The fully-plastic stress state is shown in Fig. a, for the rectangular cross section. By Fig. a, the fully-plastic moment is

$$M_P = 2Y\left(\frac{h}{2}\right)\left(\frac{h}{2}\right)\left(\frac{h}{4}\right)$$

$$= 0.125\, h^3 Y \qquad (a)$$

The maximum elastic moment is [see Prob. 4.14]

$$M_Y \doteq 0.0833\, h^3 Y \quad (b)$$

By Eqs. (a) and (b), for the rectangular cross section,

$$\frac{M_P}{M_Y} = 1.50$$

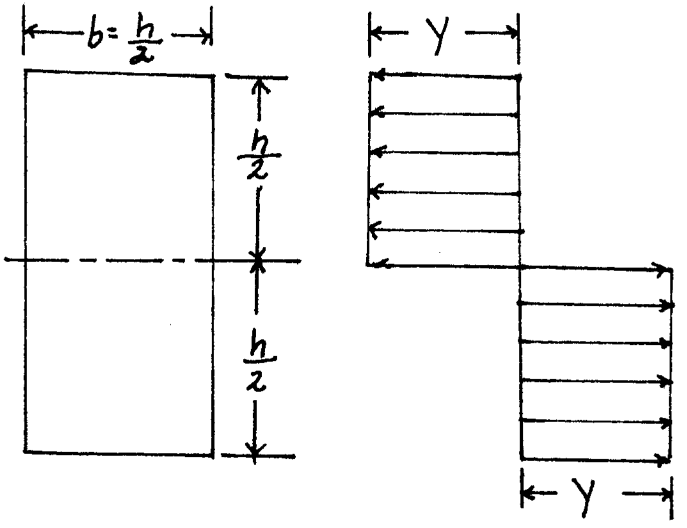


Figure a

(cont.)

4.15 cont. The fully-plastic stress state for the I-shaped beam is shown in Fig. b. By Fig. b, the fully-plastic moment is

$$M_p = 2Y\left(\frac{h}{10}\right)\left(\frac{h}{2}\right)\left(\frac{9h}{20}\right) + 2Y\left(\frac{4h}{10}\right)\left(\frac{h}{10}\right)\left(\frac{2h}{10}\right)$$

or

$$M_p = 0.061h^3Y \quad (c)$$

The maximum elastic moment is (see Prob. 4.14)

$$M_Y = 0.0492h^3Y \quad (d)$$

By Eqs. (c) and (d),

$$\frac{M_P}{M_Y} = 1.24$$

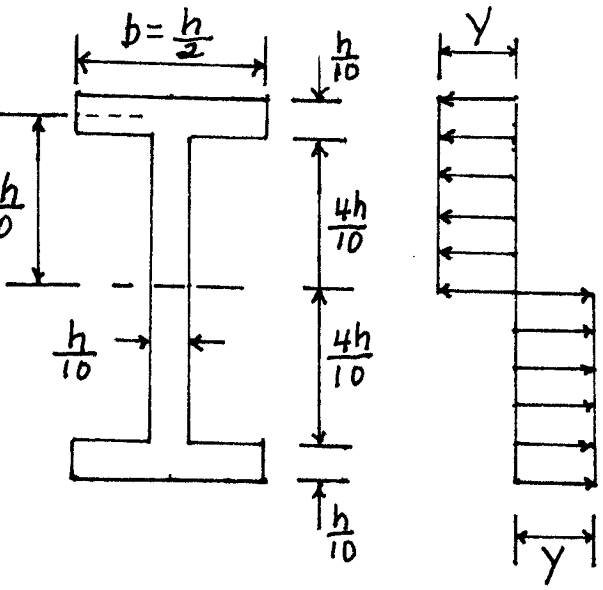


Figure b

4.16 For the rectangular cross section, the fully-plastic moment is $M_P = 0.125h^3Y$ (see Prob. 4.15). Then, the elastic stress distribution, with $M = M_P$ and y positive upward (Fig. a), is

$$\sigma = -\frac{My}{I_x} = -\frac{M_P y}{I_x}$$

where

$$-\frac{h}{2} \le y \le \frac{h}{2} \quad \text{and}$$

$$I_x = \frac{1}{12}bh^3 = \frac{1}{24}h^4$$

So, the elastic stress distribution is

$$\sigma = -\frac{0.125h^3Y}{\frac{1}{24}h^4}y = -\frac{3Y}{h}y \quad (a)$$

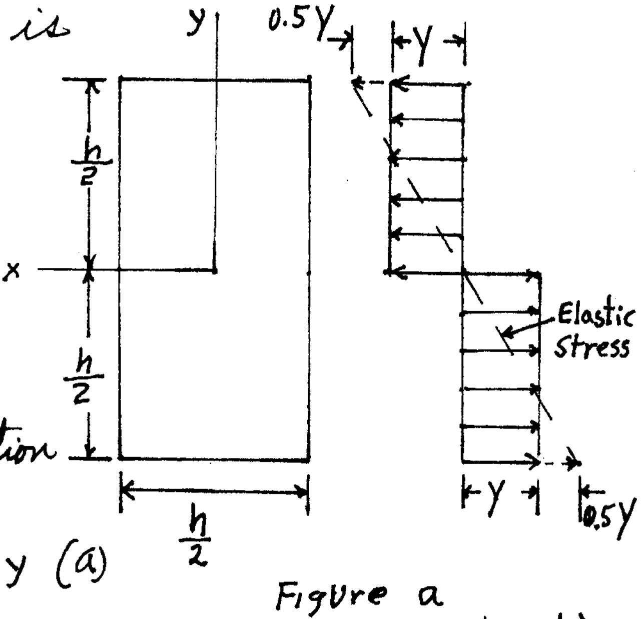


Figure a

(cont.)

4.16 cont. The fully-plastic stress state and the elastic stress state for $M = M_P$ are shown in Fig. a. The difference between the fully-plastic and the elastic stress states is shown in Fig. b. By Fig b, it is seen that the residual stress at the bottom of the beam is $\sigma_R = 0.5Y$, in compression. Since the stress $\sigma \leq Y$ over the cross section, the stress distribution is an accurate representation of the residual stress distribution.

For the I-shaped beam, the fully-plastic moment is $M_P = 0.0610 h^3 Y$ (see Prob. 4.15). The elastic stress distribution, with $M = M_P$ and y positive upward (Fig. c), is

$$\sigma = -\frac{M_P y}{I_x}$$

where $-\frac{h}{2} \leq y \leq \frac{h}{2}$ and

$$I_x = 2\left[\frac{1}{12}\left(\frac{h}{2}\right)\left(\frac{h}{10}\right)^3 + \left(\frac{h}{2}\right)\left(\frac{h}{10}\right)\left(\frac{9h}{20}\right)^2\right] + \frac{1}{12}\left(\frac{h}{10}\right)\left(\frac{8h}{10}\right)^3$$

or

$$I_x = 0.0246\, h^4$$

Hence, the elastic stress distribution is

$$\sigma = -\frac{0.0610 h^3 Y}{0.0246 h^4} = -\frac{2.480 Y}{h} y$$

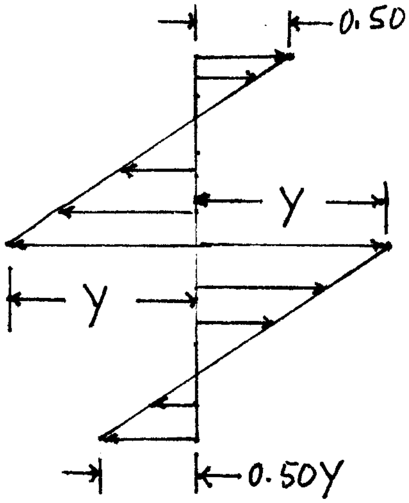


Figure b

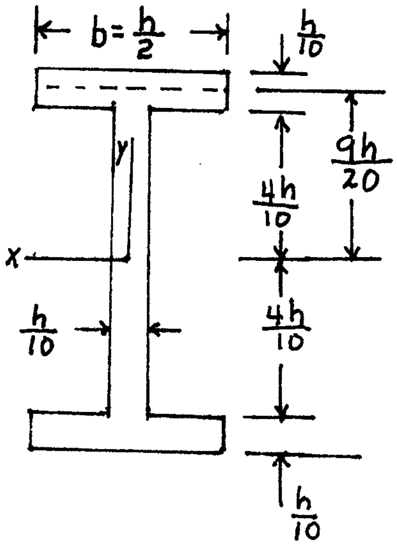


Figure c

(cont.)

4.16 cont. The fully-plastic stress state and the elastic stress state for $M = M_P$ are shown in Fig. d. The residual stress (the fully-plastic stress minus the elastic stress is shown in Fig. e.

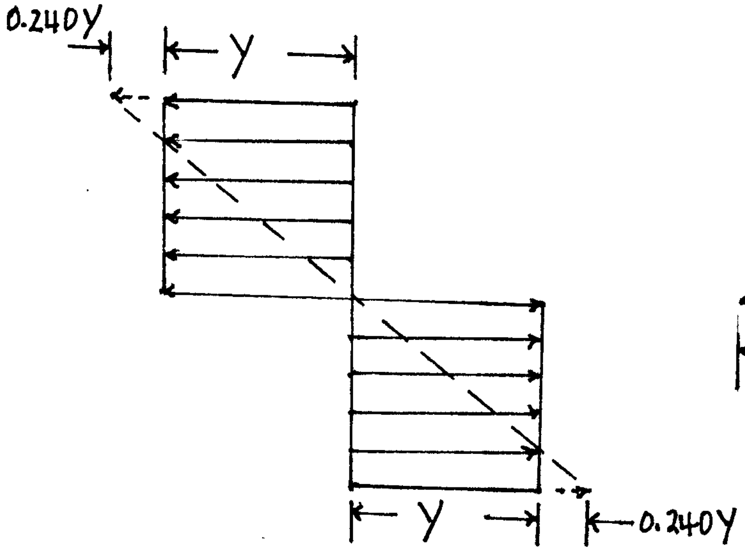


Figure d

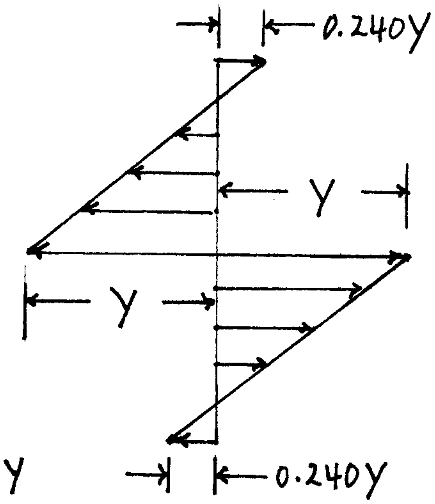


Figure e

Since the stress $\sigma \leq Y$ over the cross section, the distribution is an accurate representation of the residual stress.

4.17 Consider the axis N-N that divides the cross section into two-equal areas (Fig. a). Taking moments about the N-N axis, we obtain the fully-plastic moment, with $Y = 260$ MPa,

$$M_p = Y[(0.200)(0.050)(0.025) + 0.200(0.050)(0.100)]$$

or

$$M_p = 325 \text{ kN}\cdot\text{m}$$

(Cont.)

4.17 cont.

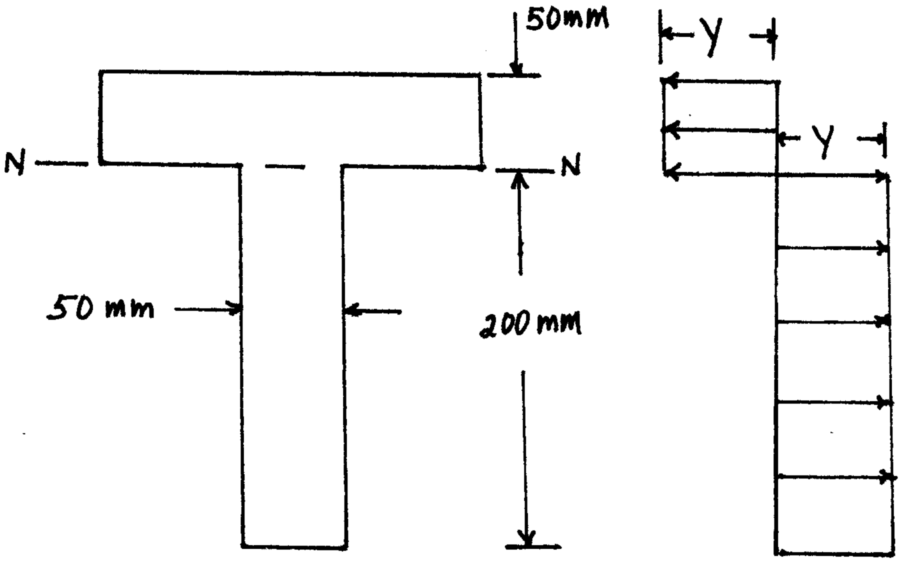


Figure a

4.18 Since the beam is unloaded elastically from the fully-plastic stress, the elastic stress of unloading is

$$\sigma_E = -\frac{M_P y}{I_x} \qquad (a)$$

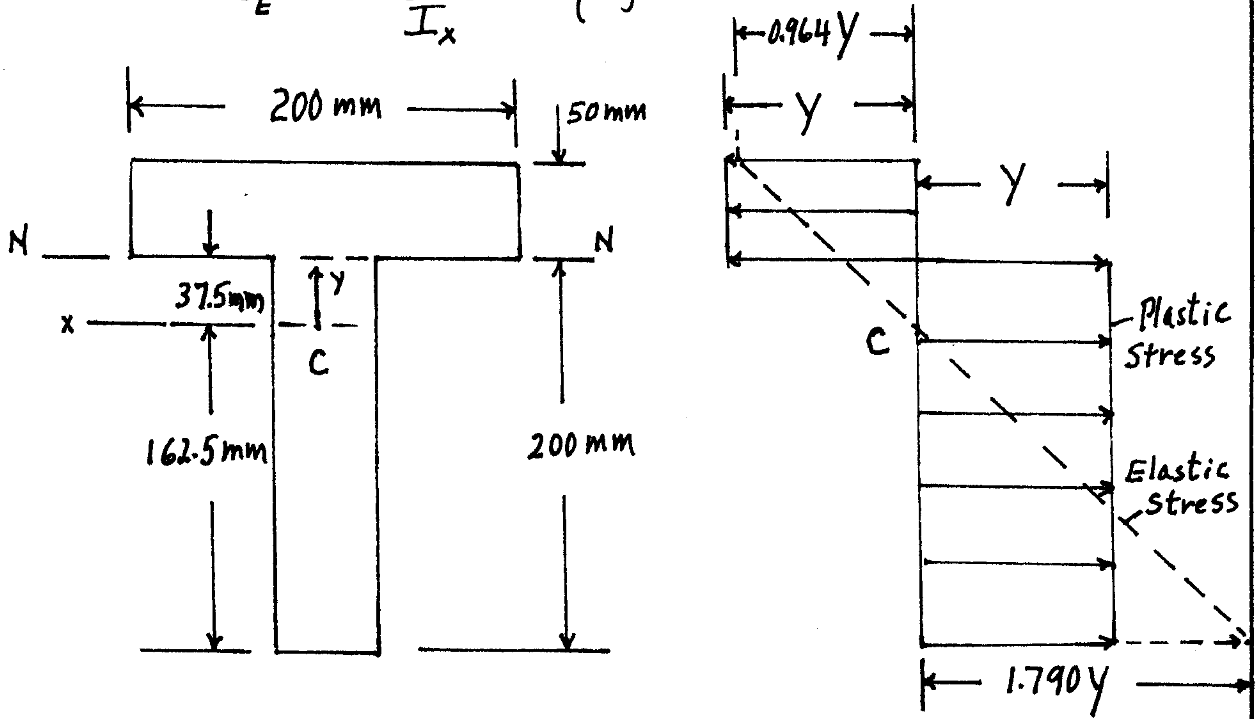


Figure a; Y = 260 MPa

(cont.)

4.18 cont. where $+y$, directed upward, is the distance from the centroidal axis (Fig. a) and M_P is the fully-plastic moment. By Prob. 4.17 and Fig. a,

$$M_P = 325 \text{ kN}\cdot\text{m} \qquad (b)$$

$$A = \text{Area} = 2(200)(50) = 20{,}000 \text{ mm}^2$$

$$A\bar{y} = (200)(50)(100) + (200)(50)(225) = 3.25\times 10^6 \text{ mm}^3$$

$$\text{or} \quad \bar{y} = 162.5 \text{ mm} \qquad (c)$$

$$I_x = \frac{1}{12}(50)(200)^3 + (200)(50)(62.5)^2 + \frac{1}{12}(200)(50)^3 + (200)(50)(62.5)^2$$

$$\text{or} \quad I_x = 1.135\times 10^8 \text{ mm}^4 = 1.135\times 10^{-4} \text{ m}^4 \qquad (d)$$

By Eqs. (a), (b) and (d), the elastic stress distribution is

$$\sigma_E = -286.34\times 10^7 y$$

For $y = 0.0875$ m (the top of the beam),

$$\sigma_E = -250.55 \text{ MPa} = -0.964 Y$$

since $Y = 260$ MPa. For $y = -0.1625$ m (the bottom of the beam),

$$\sigma_E = 465.31 \text{ MPa} = 1.790 Y$$

Subracting the elastic stress from the plastic stress, we obtain the residual stress distribution σ_R shown in Fig. b.

At the top of the beam $\sigma_R = 0.036 Y = 9.36$ MPa, and

(Cont.)

4.18 cont. at the bottom of the beam, $\sigma_R = 0.790Y = 205.4\ \text{MPa}$, both in compression.

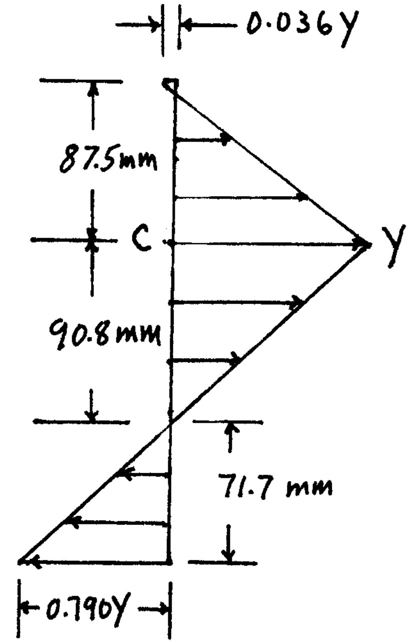


Figure b

4.19 The centroidal axis of the beam is located 100 mm above the base (Fig. a).

By Table B.1, the moment of inertia of the area of the beam about the centroidal axis is

$$I_x = \frac{1}{36} bh^3 = \frac{1}{36}(200)(300)^3$$

or

$$I_x = 150 \times 10^6\ \text{mm}^4 = 150 \times 10^{-6}\ \text{m}^4 \quad (a)$$

By the flexure formula, with $\sigma = Y = 240\ \text{MPa}$, the maximum elastic moment is

$$M_Y = \frac{Y I_x}{c} = \frac{(240 \times 10^6)(150 \times 10^{-6})}{0.200} = 180\ \text{kN·m} \quad (b)$$

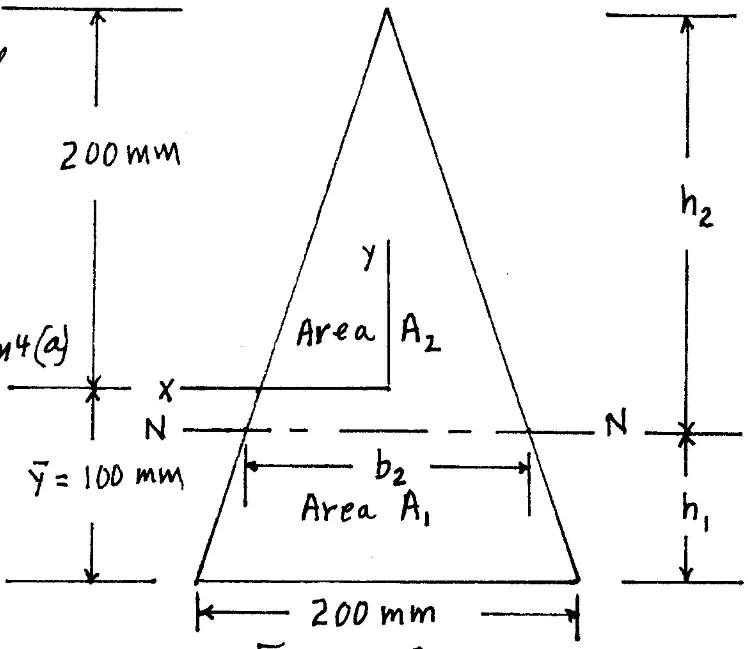


Figure a

(cont.)

4.19 cont. assume that the top of the beam is in compression. Then the compression area, above the axis N-N, must be equal to the tension area, below the axis N-N, when the cross section is fully plastic. In other words, by Fig. a,

$$A_1 = \frac{1}{2}(200)(300) - \frac{1}{2} b_2 h_2$$

$$A_2 = \frac{1}{2} b_2 h_2 = A_1 = 30{,}000 - \frac{1}{2} b_2 h_2 \qquad (c)$$

$$\frac{b_2}{h_2} = \frac{2}{3}$$

By Eqs.(c),

$$h_1 = 87.868 \text{ mm}, \; h_2 = 212.132 \text{ mm}, \; b_2 = 141.421 \text{ mm} \qquad (d)$$

$$A_1 = A_2 = 15{,}000 \text{ mm}^2 = 15 \times 10^{-3} \text{ m}^2$$

With Eqs.(d), we plot the fully-plastic stress distribution shown in Fig. b. By Fig. b, with $Y = 240$ MPa, the fully plastic moment is, with $A_1 = A_2$, $F_1 = F_2$,

$$M_p = A_1 Y (0.070711 + 0.046447)$$

$$= (15 \times 10^{-3})(240 \times 10^6)(0.11716)$$

or

$$M_p = 421.77 \text{ kN}\cdot\text{m} \quad (e)$$

Hence, by Eqs.(a) and (e), we obtain

$$\frac{M_p}{M_Y} = 2.34$$

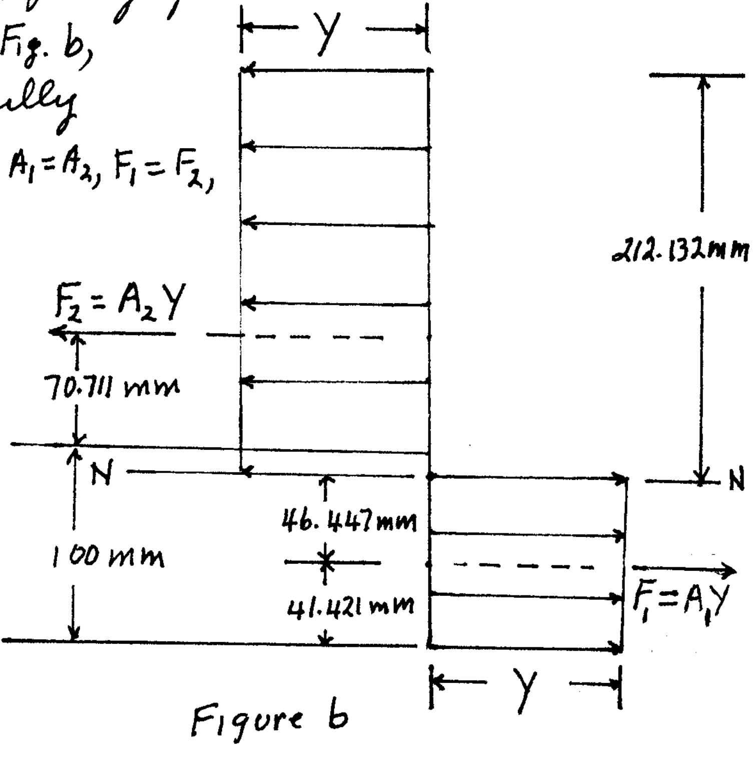


Figure b

4.20 The centroidal axis of the beam cross section is located 133.33 mm from the base of the trapezoid (Fig. a). The moment of inertia of the cross-sectional area relative to the centroidal axis is

$$I_x = \frac{1}{12}(100)(300)^3 + (100)(300)(16.67)^2 + \frac{2}{36}(50)(300)^3 + \frac{2}{2}(50)(300)(33.33)^2$$

or

$$I_x = 325 \times 10^6\ \text{mm}^4 = 325 \times 10^{-6}\ \text{m}^4 \quad (a)$$

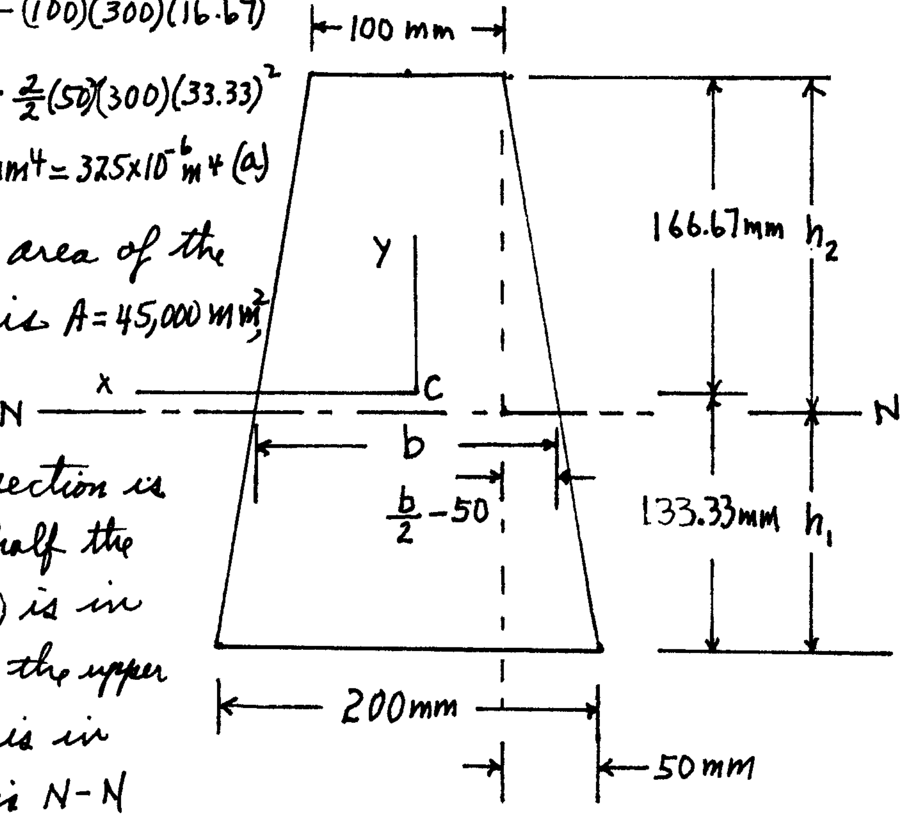


Figure a

Since the total area of the cross section is $A = 45{,}000\ \text{mm}^2$, when the cross section is fully plastic, half the area (22,500 mm²) is in compression (say the upper area) and half is in tension. The axis N-N divides the cross section into equal areas (Fig. a). By Fig. a, the area above the N-N axis is

$$100h_2 + \frac{(b-100)}{2}h_2 = \frac{1}{2}bh_2 + 50h_2 = 22{,}500 \quad (b)$$

Also, by Fig. a,

$$\frac{100}{300} = \frac{(b-100)}{h_2} \quad (c)$$

The solution of Eqs. (b) and (c) is

(cont.)

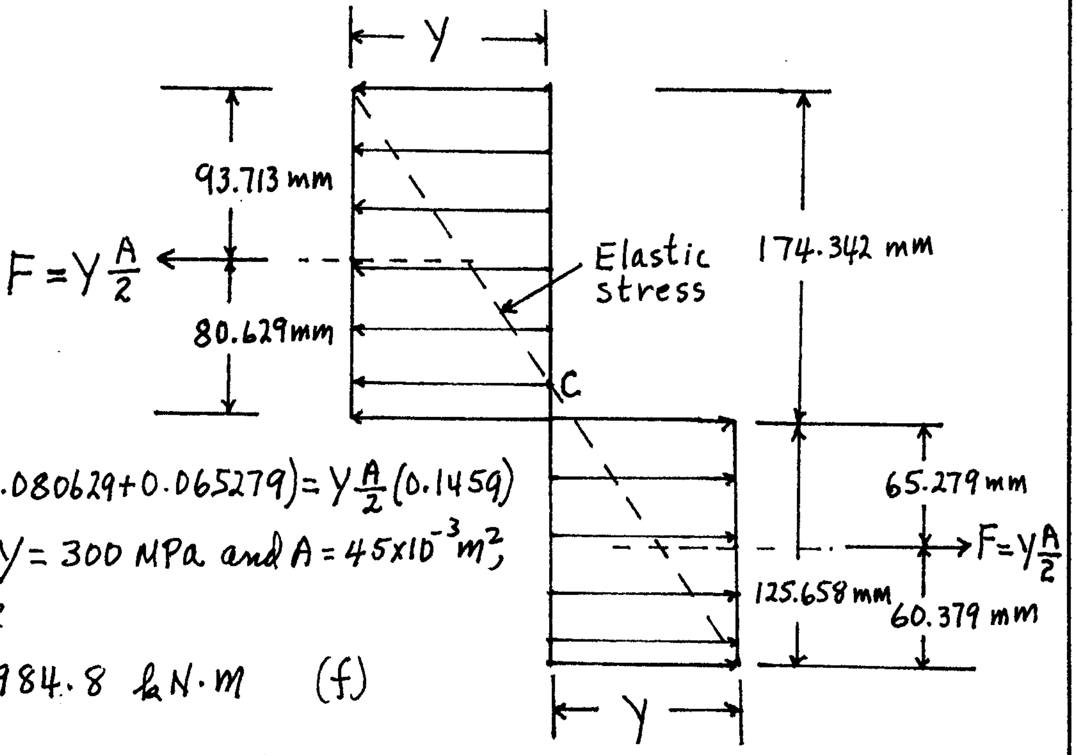
4.20 cont.

$$b = 158.114 \text{ mm}, \quad h_2 = 174.342 \text{ mm} \qquad (d)$$

Hence,

$$h_1 = 300 - h_2 = 125.658 \text{ mm} \qquad (e)$$

The stress distribution for the fully-plastic cross section is shown in Fig. b. By Fig. b, the fully-plastic moment is



$$M_p = F(0.080629 + 0.065279) = Y\frac{A}{2}(0.1459)$$

or with $Y = 300$ MPa and $A = 45 \times 10^{-3}\,\text{m}^2$, we have

$$M_p = 984.8 \text{ kN}\cdot\text{m} \qquad (f)$$

Figure b

By Fig. a, Eq. (a), and the flexure formula, the maximum elastic moment, with $\sigma = Y$ (see Fig. b), is

$$M_Y = \frac{Y I_x}{c} = \frac{(300 \times 10^6)(325 \times 10^{-6})}{0.1667} = 584.9 \text{ kN}\cdot\text{m} \qquad (g)$$

Hence, by Eqs. (f) and (g),

$$\frac{M_P}{M_Y} = 1.68$$

4.21 Since the cross section has two axes of symmetry, the centroidal axis is 40 mm from the bottom and from the top of the cross section (Fig. a). By Fig. a, the area A of the cross section is

$$A = 60(80) - 30(40) = 3600 \text{ mm}^2 = 3.6\times10^{-3} \text{ m}^2 \quad (a)$$

Also, by Fig. a, the moment of inertia of the cross section is, about the centroidal axis x,

$$I_x = \frac{1}{12}(60)(80)^3 - \frac{1}{12}(30)(40)^3$$

or

$$I_x = 2.4\times10^6 \text{ mm}^4 = 2.4\times10^{-6} \text{ m}^4 \quad (b)$$

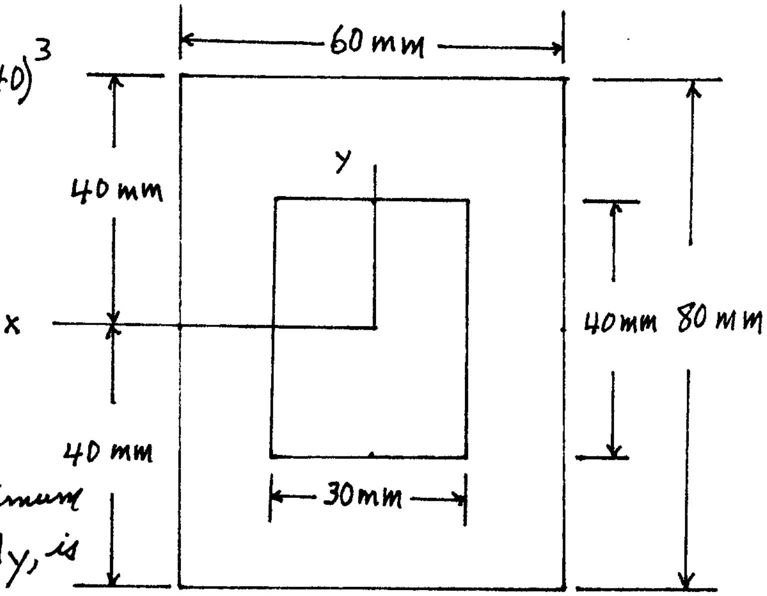


Figure a

By the flexure formula, Fig. a, and Eq. (b), the maximum elastic moment, M_Y, is with $\sigma = Y = 250$ MPa,

$$M_Y = \frac{Y I_x}{c} = \frac{(250\times10^6)(2.4\times10^{-6})}{0.040}$$

or

$$M_Y = 15.0 \text{ kN}\cdot\text{m} \quad (c)$$

The fully-plastic stress distribution is shown in Fig. b. The centroidal distance of the areas above and below the x axis is, with Eq. (a),

$$\bar{y} = \frac{60(40)(20) - 30(20)(10)}{1800} = 23.33 \text{ mm} = 0.02333 \text{ m} \quad (d)$$

(cont.)

4.21 cont.

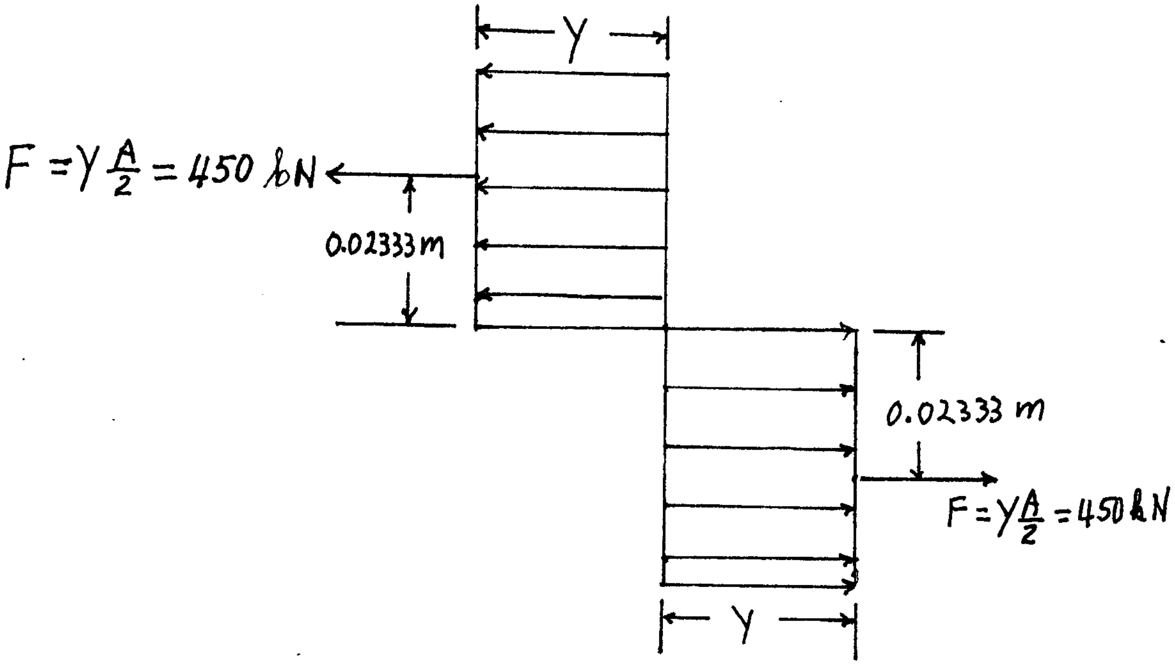


Figure b

Hence, by Fig. b and Eq. (d), the fully-plastic moment M_P is

$$M_P = 2(450)(0.02333) = 21.0 \text{ kN}\cdot\text{m} \quad \text{(e)}$$

So, by Eqs. (c) and (e),

$$\frac{M_P}{M_E} = 1.40$$

4.22 By Fig. a, the area of the cross section is

$$A = 150(200) - 100(100) = 20{,}000\ \text{mm}^2 = 20 \times 10^{-3}\ \text{m}^2 \quad (a)$$

The centroidal axis is located at a distance $\bar{y}$ from the bottom of the cross section, where

$$A\bar{y} = 150(200)(100) - 100(100)(125) = 1{,}750{,}000\ \text{mm}^3$$

or

$$\bar{y} = 1{,}750{,}000/20{,}000 = 87.5\ \text{mm} = 0.0875\ \text{m}$$

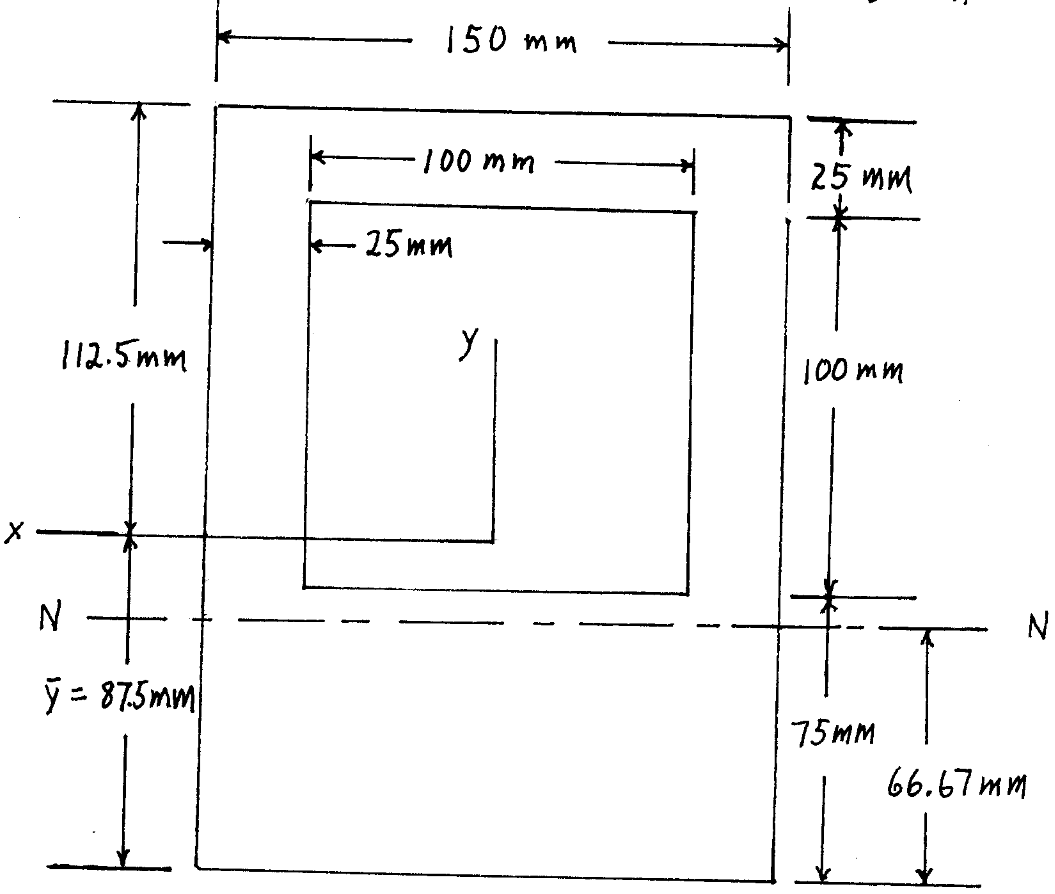


Figure a

Then, by Fig. a,

$$I_x = \frac{1}{12}(150)(200)^3 + 150(200)(100-87.5)^2 - \frac{1}{12}(100)(100)^3 - 100(100)(125-87.5)^2$$

or

$$I_x = 82.292 \times 10^6\ \text{mm}^4 = 82.292 \times 10^{-6}\ \text{m}^4 \quad (b)$$

(cont.)

4.22 cont. Then, by Fig. a and Eq. (b), with $\sigma = Y = 280 \text{ MPa}$, the flexure formula yields the maximum elastic moment M_Y as

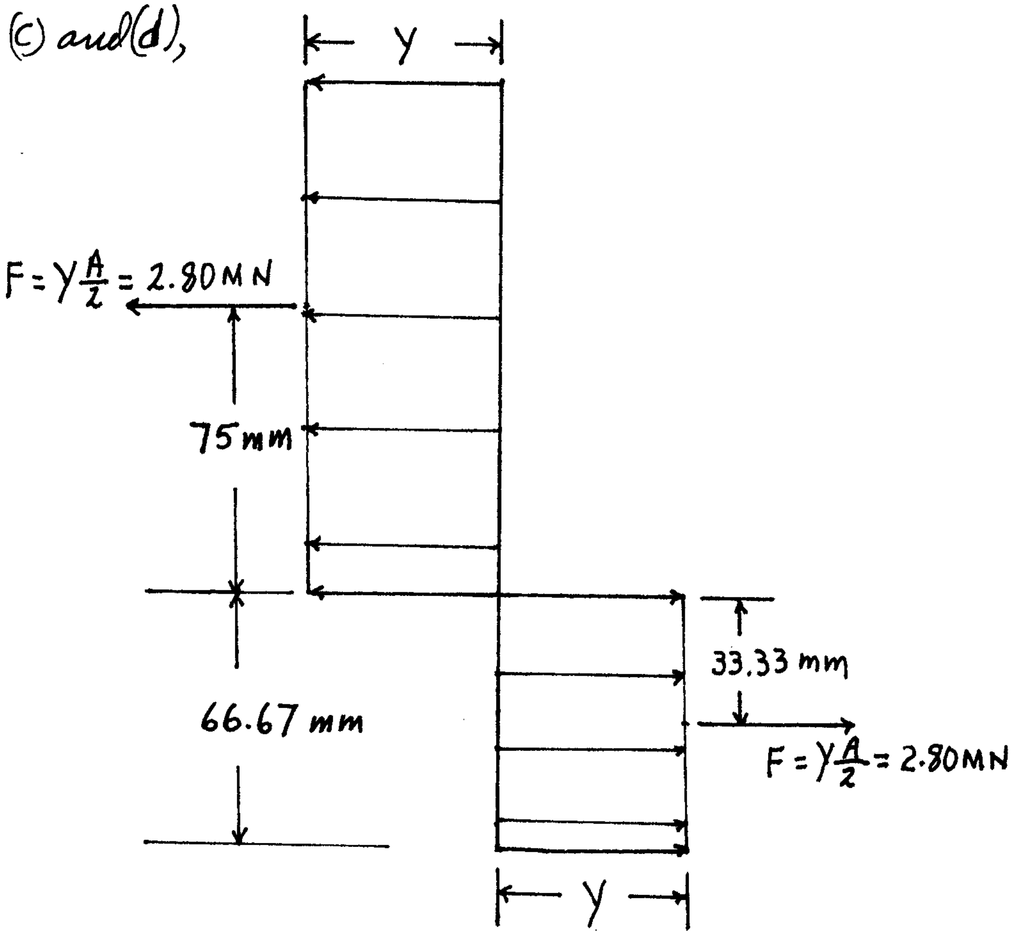
$$M_Y = \frac{Y I_x}{c} = 204.8 \text{ kN}\cdot\text{m} \qquad (c)$$

The axis N-N divides the cross-sectional area into two equal areas $A/2 = 10{,}000 \text{ mm}^2 = 0.010 \text{ m}^2$. So, the fully-plastic stress distribution is shown in Fig. b. By Fig. b, the fully-plastic moment M_P is

$$M_P = (2.80\times10^6)(0.075 + 0.03333) = 303.3 \text{ kN}\cdot\text{m} \quad (d)$$

So, by Eqs. (c) and (d),

$$\frac{M_P}{M_Y} = 1.48$$



Y
$F = Y\frac{A}{2} = 2.80 \text{ MN}$
75 mm
66.67 mm
33.33 mm
$F = Y\frac{A}{2} = 2.80 \text{ MN}$
Y

Figure b

4.23 The general stress distribution over the beam cross section is shown in Fig. a. By summation of moments about 0, we obtain, where y locates the boundary between the elastic and plastic regions of the beam,

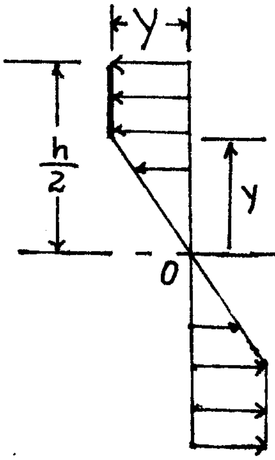


Figure a

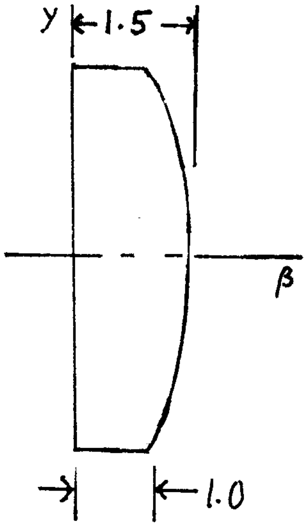


Figure b

$$M=\beta M_Y = 2\left(\frac{h}{2}-y\right)bY\frac{\left(\frac{h}{2}+y\right)}{2}+2\left(\frac{1}{2}ybY\right)\left(\frac{2}{3}y\right) \quad (a)$$

By Eq. (4.40),

$$M_Y=\frac{1}{6}bh^2Y \quad (b)$$

So, by Eqs. (a) and (b),

$$\beta = 1.5-2\left(\frac{y}{h}\right)^2 \quad (c)$$

The plot of Eq. (c) is shown in Fig. b.

4.24 (a) The moment is (see Prob. 4.23, with $\beta = 1.25$)

$$M = 1.25 M_Y \qquad (a)$$

The maximum elastic moment is

$$M_Y = \frac{Y I_x}{h/2} = \frac{1}{6} b h^2 Y \qquad (b)$$

By Eqs. (a) and (b),

$$M = \frac{5}{24} b h^2 Y \qquad (c)$$

By Fig. a, $M = \Sigma M_O = 2\left(\frac{h}{2} - e\right) bY\left(\frac{h/2 + e}{2}\right) = \left(\frac{h^2}{4} - \frac{1}{3}e^2\right) bY \quad (d)$

Equating Eqs. (c) and (d)

We find

$$e = \frac{1}{\sqrt{2}} \frac{h}{2} \qquad (e)$$

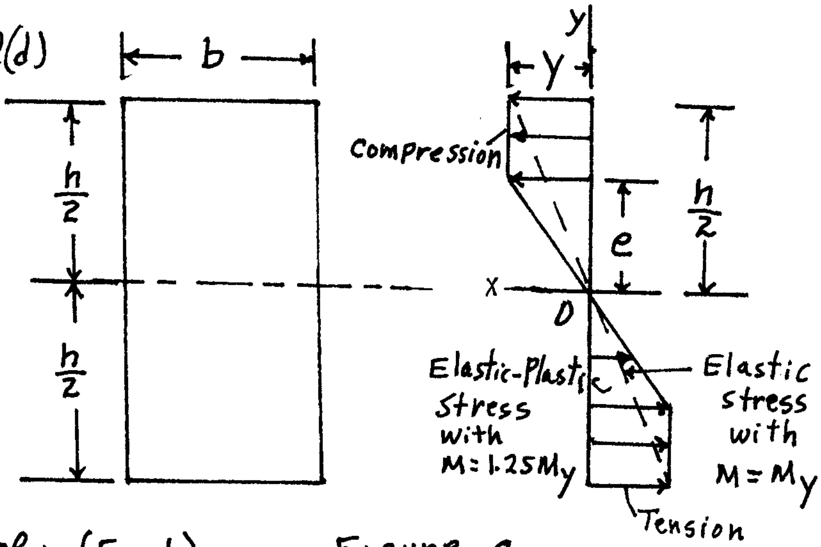


Figure a

Since plane sections remain plane, the strain ϵ is a linear function of y (Fig. b).

At $y = e$, $\epsilon = Y/E$, where E is the modulus of elasticity of the beam. The curvature of the beam is

$$\frac{1}{\rho} = \frac{\epsilon}{e} = \frac{Y}{Ee},$$ or the radius of curvature is

$$\rho = \frac{Ee}{Y}$$

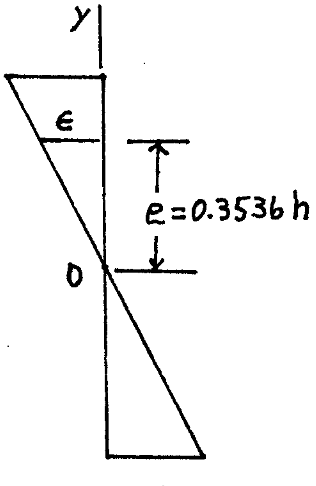


Figure b

(Cont.)

4.24 Cont. (b) With $M = 1.25 M_Y$, the elastic stress upon release is

$$\sigma_E = -\frac{My}{I_x} = -\frac{\frac{5}{4}(\frac{1}{6}bh^2Y)\,y}{\frac{1}{12}bh^3} = -\frac{5Y}{2}\left(\frac{y}{h}\right)$$

Subtracting σ_E from the elastic-plastic stress distribution, we obtain the residual stress distribution shown in Fig. c.

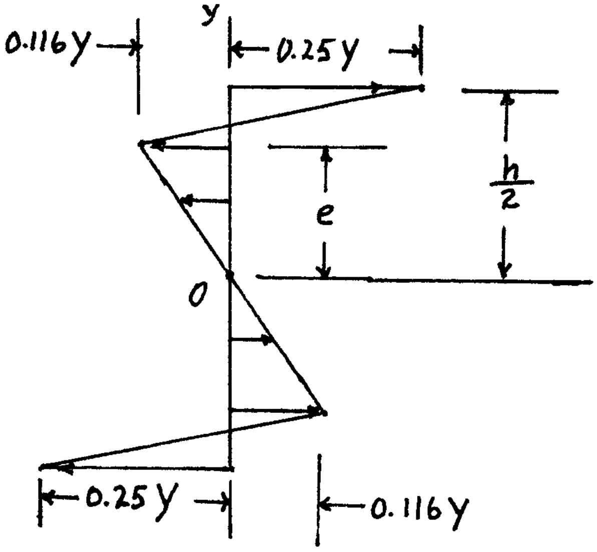


Figure c

4.25

(a) $M_Y = \frac{1}{6}bh^2Y$, $M_P = \frac{1}{4}bh^2Y$; $f = \frac{M_P}{M_Y} = \frac{6}{4} = 1.5$

(b) $M_Y = 2tbY(\frac{h}{2} - \frac{t}{2}) + 2(\frac{h}{2} - t)\frac{Y}{2}t(\frac{2}{3})(\frac{h}{2} - t)$

$\approx tbh(1 + \frac{1}{6}\frac{h}{b})Y$

$M_P = 2btY(\frac{h}{2} - \frac{t}{2}) + 2(\frac{h}{2} - t)Yt(\frac{1}{2})(\frac{h}{2} - t)$

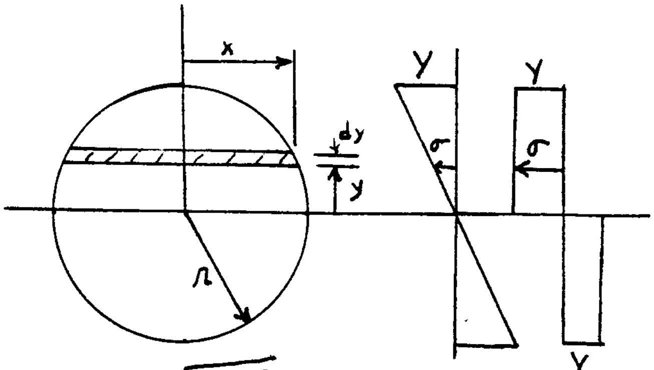
$\approx tbh(1 + \frac{1}{4}\frac{h}{b})Y$

Therefore,

$$f = \frac{M_P}{M_Y} = \frac{1 + \frac{1}{4}\frac{h}{b}}{1 + \frac{1}{6}\frac{h}{b}}$$

4.25 Continued:

(c) By Figs. (a) and (b),



$x = \sqrt{r^2-y^2}$ (a) $\sigma = \frac{Y}{r}y$ (b) $\sigma = Y$ (c)

$$dM_Y = 2[\sigma\, 2x\,dy \cdot y]$$
$$= 4\frac{Y}{r}\, y^2\sqrt{r^2-y^2}\,dy$$

or

$$M_Y = \frac{4Y}{r}\int_0^r y^2\sqrt{r^2-y^2}\,dy = \frac{\pi}{4}r^3 Y$$

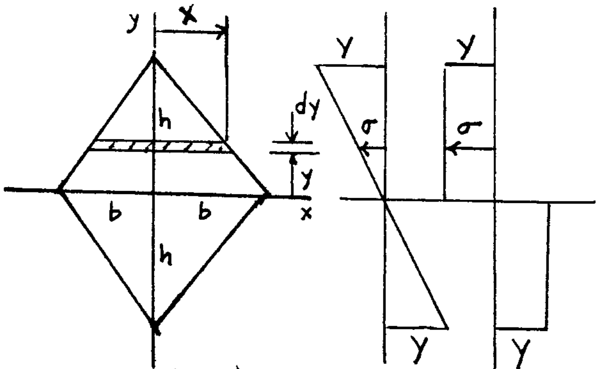
By Figs. (a) and (c),

$$dM_P = 2[\sigma\, 2x\,dy \cdot y]$$
$$= 4Y y\sqrt{r^2-y^2}\,dx$$

or

$$M_P = 4Y\int_0^r y\sqrt{r^2-y^2}\,dx = \frac{4}{3}r^3 Y \quad \therefore f = \frac{M_P}{M_Y} = \frac{16}{3\pi} = 1.70$$

(d) By Figs. (a) and (b),



$x = b(1-\frac{y}{h})$ (a) $\sigma = Y\frac{y}{h}$ (b) $\sigma = Y$ (c)

$$dM_Y = 2(2x\,dy\,\sigma \cdot y)$$
$$= 4Y\frac{b}{h}\left(y^2 - \frac{y^3}{h}\right)dy$$

or

$$M_Y = \int_0^h 4Y\frac{b}{h}\left(y^2-\frac{y^3}{h}\right)dy = \frac{bh^2Y}{3}$$

By Figs. (a) and (c),

$$dM_P = 2[2x\,dy\,\sigma \cdot y]$$
$$= 4Yb\left(y - \frac{y^2}{h}\right)dy$$

or

$$M_P = 4bY\int_0^h \left(y-\frac{y^2}{h}\right)dy = \frac{2bh^2Y}{3}$$

$$\therefore f = \frac{M_P}{M_Y} = 2.0$$

(cont.)

4.25 continued:

(e)

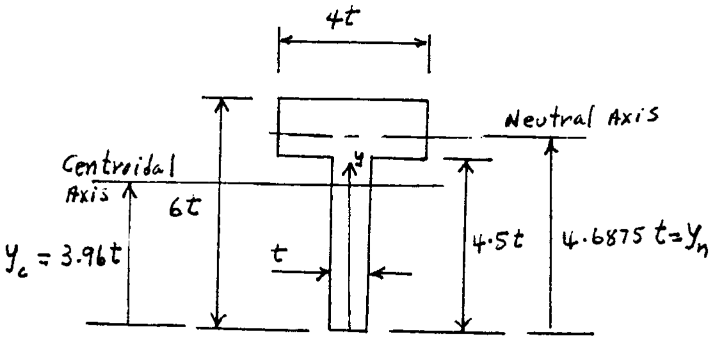
$y_c = 3.96t, \; I = 31.86t^4$

$M_y = \dfrac{Y}{y_c} I = 8.05\, Y t^3$

$y_n = 4.6875t$

$M_p = 14.48\, Y t^3$

$f = \dfrac{M_p}{M_y} = 1.80$



4.26

(a) $M_{max} = PL \;, \; I = \frac{1}{12} b h^3$

$M_y = P_Y L = \dfrac{2Y}{h} I = \dfrac{2Y}{h} \dfrac{bh^3}{12} = \dfrac{bYh^2}{6}$

$\therefore \; P_Y = \dfrac{bh^2}{6L} Y$

(b) $M_p = P_p L = \dfrac{bh^2}{4} Y$

$P_p = \dfrac{bh^2}{4L} Y$

(c) $\dfrac{P_P}{P_Y} = \dfrac{\frac{bh^2 Y}{4L}}{\frac{bh^2 Y}{6L}} = \dfrac{3}{2}$

4.27

(a) $M_B = M_P = R \dfrac{L}{2}$

$M_A = -M_p = RL - P\dfrac{L}{2}$

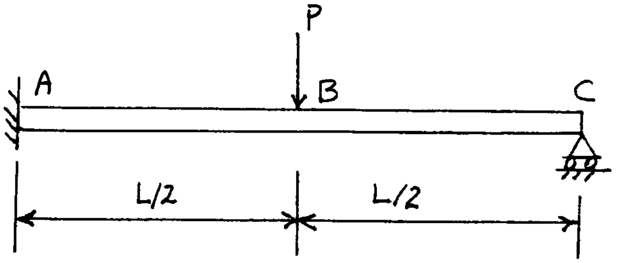
$\therefore \; 3M_p = P\dfrac{L}{2}$

or $P = \dfrac{6M_P}{L}$

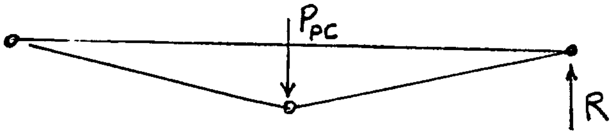
$M_P = \dfrac{bh^2}{4} Y$

$\therefore \; P_{PC} = \dfrac{6}{L} \dfrac{bh^2}{4} Y$

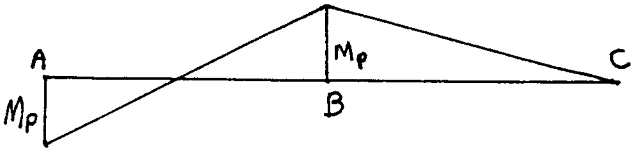
or $P_{PC} = \dfrac{3bh^2}{2L} Y$



(c)



(b)



4.28 $\sigma = \frac{P}{A} = \frac{50{,}000(4)}{\pi(20)^2} = 159.2\ \text{MPa}; \quad \tau = \frac{Tc}{J} = \frac{T(10)(32)}{\pi(20)^4} = 0.0006366T\ (\text{MPa})$

(a) Eq. (4.46) gives

$$\tau = \frac{1}{\sqrt{3}}\sqrt{Y^2 - \sigma^2} = \frac{1}{\sqrt{3}}\sqrt{330^2 - 159.2^2} = 166.9\ \text{MPa}$$

$$T = \frac{166.9}{0.0006366} = \underline{262.2\ \text{N.m}}$$

(b) $\sigma = SF\frac{P}{A} = 1.75(159.2) = 278.6\ \text{MPa}$

$$\tau = \frac{1}{\sqrt{3}}\sqrt{330^2 - 278.6^2} = 102.1 = SF\frac{Tc}{J} = 1.75(0.0006366T) = 0.001114\ T\ (\text{MPa})$$

$$T = \frac{102.1}{0.001114} = \underline{91.7\ \text{N.m}}$$

4.29

$$\sigma = SF\left(\frac{P}{A} + \frac{Mc}{I}\right) = SF\left[\frac{30{,}000}{\pi(15)^2} + \frac{150{,}000(15)(4)}{\pi(15)^4}\right] = SF(99.03)\ (\text{MPa})$$

$$\tau = SF\frac{Tc}{J} = SF\frac{250{,}006(15)(2)}{\pi(15)^4} = SF(47.16)\ (\text{MPa})$$

Eq. (4.45) gives $Y^2 = \sigma^2 + 4\tau^2$

$$SF = \sqrt{\frac{280^2}{99.03^2 + 4(47.16)^2}} = \underline{2.05}$$

4.30 $A = 2\pi Rh = 2\pi(20)(2) = 251.3\ \text{mm}^2; \quad J = AR^2 = 251.3(20)^2 = 100{,}500\ \text{mm}^4$

$$\sigma_{xx} = SF\frac{pR_i}{2h} + SF\frac{P}{A} = \frac{1.90(22.0)(19)}{2(2)} + \frac{1.90(50{,}000)}{251.3} = 576.6\ \text{MPa}$$

$$\sigma_{yy} = SF\frac{pR_i}{h} = \frac{1.90(22.0)(19)}{2} = 397.1\ \text{MPa}; \quad \tau = SF\frac{Tc}{J} = \frac{1.90T(21)}{100{,}500} = 0.000397T$$

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau^2} = \sigma_{max}$$

$$\sigma_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau^2} = \sigma_{min} \quad \text{(1st assumption)}$$

$$\sigma_3 = 0 = \sigma_{min} \quad \text{(2nd assumption)}$$

1st assumption (Eq. 4.15)

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{Y}{2} = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau^2}$$

$$\tau = \sqrt{\left(\frac{Y}{2}\right)^2 - \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2} = \sqrt{400^2 - \left(\frac{576.6 - 397.1}{2}\right)^2} = 389.8\ \text{MPa}$$

2nd assumption

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{Y}{2} = \frac{\sigma_{xx}+\sigma_{yy}}{4} + \frac{1}{2}\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau^2}$$

$$\tau = \sqrt{\left(Y - \frac{\sigma_{xx}+\sigma_{yy}}{2}\right)^2 - \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2} = \sqrt{\left(800 - \frac{576.6+397.1}{2}\right)^2 - \left(\frac{576.6-397.1}{2}\right)^2} = \underline{300.0\ \text{MPa}}$$

$$T = \frac{300.0}{0.000397} = \underline{756\ \text{N.m}}$$

4.31 $M = 30,000(250)$; $\sigma = SF\frac{Mc}{I}$

$T = 30,000(200)$; $\tau = SF\frac{Tc}{J}$

Substitue σ and τ into Eq. (4.45) to obtain

$$d_{min} = \left[\frac{32(SF)}{\pi Y}\sqrt{M^2 + T^2}\right]^{\frac{1}{3}} = \left[\frac{32(2.00)}{\pi(276)}\sqrt{7,500,000^2 + 6,000,000^2}\right]^{\frac{1}{3}} = \underline{89.2 \text{ mm}}$$

4.32 Substitute σ and τ from Prob. 4.31 into Eq. (4.46) to obtain

$$d_{min} = \left[\frac{16(SF)}{\pi Y}\sqrt{4M^2+3T^2}\right]^{\frac{1}{3}} = \left[\frac{16(2.00)}{\pi(276)}\sqrt{4(7,500,000)^2 + 3(6,000,000)^2}\right]^{\frac{1}{3}} = \underline{87.7 \text{ mm}}$$

4.33 $\Sigma M_D = 0$; $750 R_1 - 3000(450) + 6000(150) = 0$

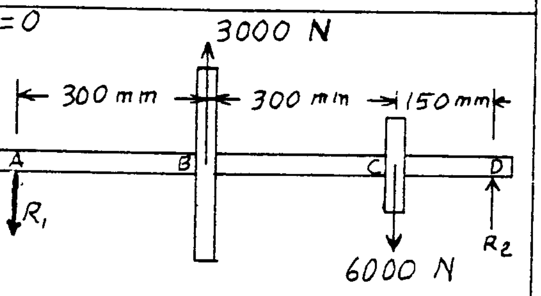
$R_1 = 600$ N ; $R_2 = 3600$ N

$M_{max} = 150 R_2 = 150(3600) = 540,000$ N.mm

$T = 3000(150) = 450,000$ N. mm

$$d_{min} = \left[\frac{32(SF)}{\pi Y}\sqrt{M^2+T^2}\right]^{\frac{1}{3}} = \left[\frac{32(1.85)}{\pi(290)}\sqrt{540,000^2+450,000^2}\right]^{\frac{1}{3}}$$

$= \underline{35.75 \text{ mm}}$



4.34 $M_{Bx} = 540,000$ N.mm ; $M_{By} = 360,000$ N.mm

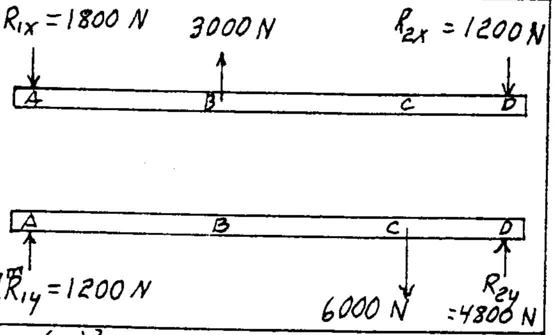
$M_{Cx} = 180,000$ N.mm ; $M_{Cy} = 720,000$ N.mm

$M_B = \sqrt{M_{Bx}^2 + M_{By}^2} = 649,000$ N. mm

$M_C = \sqrt{M_{Cx}^2 + M_{Cy}^2} = 742,200$ N. mm

$T = 450,000$ N. mm (See Prob. 4.33)

$$d_{min} = \left[\frac{32(1.85)}{\pi\, 290}\sqrt{742,200^2 + 450,000^2}\right]^{\frac{1}{3}} = \underline{38.3 \text{ mm}}$$



4.35 $\frac{I}{c} = \frac{\pi R^3}{4} = \frac{\pi(50)^3}{4} = 98,170 \text{ mm}^3$; $\frac{J}{c} = \frac{\pi R^3}{2} = \frac{\pi(50)^3}{2} = 196,350 \text{ mm}^3$

$P_x = P_y = P_z = \frac{P}{\sqrt{3}} = 0.5774 P$

$N_O = P_y$; $T_O = 400 P_x = 230.96P$; $M_{Ox} = 300 P_z - 400 P_y = -57.74P$ | $M_{Oz} = -300 P_x = -173.22P$

$M_O = \sqrt{M_{Ox}^2 + M_{Oy}^2} = P\sqrt{(-57.74)^2 + (-173.22)^2} = 182.59 P$

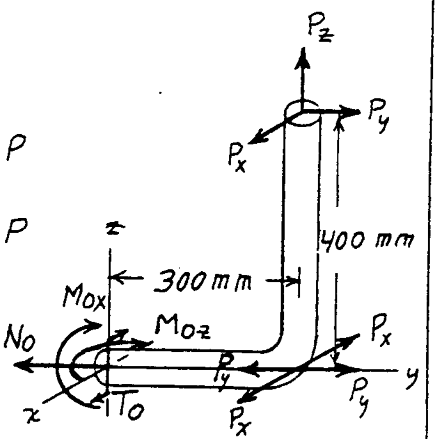
$\sigma = \frac{N_O}{A} + \frac{M_O c}{I} = \frac{0.5774P}{\pi(50)^2} + \frac{182.59 P}{98,170} = 0.001934 P$

$\tau = \frac{Tc}{J} = \frac{230.96 P}{196,350} = 0.001176 P$

$Y = \sqrt{\sigma^2 + 3\tau^2}$

$420 = P\sqrt{0.001934^2 + 3(0.001176)^2}$

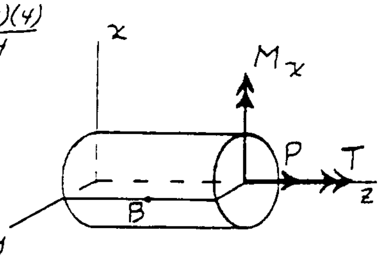
$P = \underline{149.5 \text{ kN}}$



4.36 $\sigma_B = \sigma_{zz} = SF\dfrac{P}{A} + SF\dfrac{M_x c}{I} = SF\dfrac{25{,}000}{\pi(10)^2} + SF\dfrac{50{,}000(10)(4)}{\pi(10)^4}$

$= SF(143.24)$

$\tau_B = SF\dfrac{Tc}{J} = SF\dfrac{120{,}000(10)(2)}{\pi(10)^4} = SF(76.39)$



(a) $\tau_{oct(max)} = \frac{1}{3}\sqrt{2Y^2} = \frac{SF}{3}\sqrt{2(143.24)^2 + 6(76.39)^2}$

$SF = \underline{2.05}$

(b) $\sigma_1 = SF\dfrac{143.24}{2} + SF\sqrt{\left(\dfrac{143.24}{2}\right)^2 + (76.39)^2} = SF(71.62 + 104.71)$

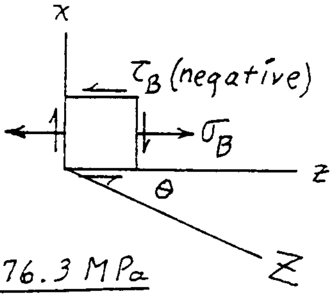
$= SF(176.33)$

$\sigma_2 = SF(71.62 - 104.71) = SF(-33.09)$

$\sigma_3 = 0$

$\tau_{max} = \dfrac{\sigma_{max} - \sigma_{min}}{2} = \dfrac{Y}{2} = \dfrac{400}{2} = \dfrac{SF(143.24) - SF(-33.09)}{2}$

$SF = \underline{1.91}$



(c) $\tan 2\theta = \dfrac{2\tau_B}{\sigma_B} = \dfrac{2(-76.39)}{143.24} = -1.0666$; $\sigma_{max} = \underline{176.3\ MPa}$

$\theta = \underline{-0.4088\ rad}$ $\sigma_{min} = \underline{-33.1\ MPa}$

4.37

(a) $\tau_{oct(max)} = \frac{1}{3}\sqrt{6\tau_Y^2} = \frac{1}{3}\sqrt{2\sigma_B^2 + 6\tau_B^2} = \frac{SF}{3}\sqrt{2(143.24)^2 + 6(76.39)^2} = \frac{1}{3}\sqrt{6(200)^2}$

$SF = \underline{1.78}$

(b) $\tau_{max} = \dfrac{\sigma_{max} - \sigma_{min}}{2} = \dfrac{\sigma_1 - \sigma_2}{2} = \tau_Y = 200 = SF\dfrac{176.33 - (-33.09)}{2}$

$SF = \underline{1.91}$

4.38

$M_x = 150(1500) = 225{,}000\ N.mm$; $M_y = 200(600) = 120{,}000\ N.mm$

$M = \sqrt{M_x^2 + M_y^2} = 255{,}000\ N.mm$

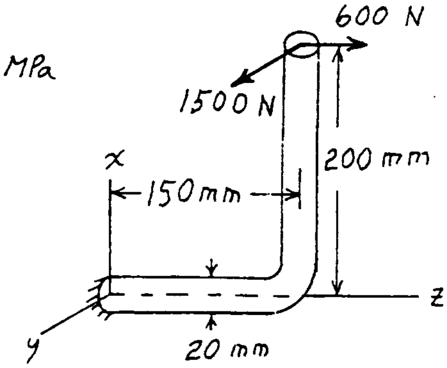
$T = 200(1500) = 300{,}000\ N.mm$

$\sigma = \dfrac{P}{A} + \dfrac{Mc}{I} = \dfrac{600}{\pi(10)^2} + \dfrac{255{,}000(10)(4)}{\pi(10)^4} = 326.6\ MPa$

$\tau = \dfrac{Tc}{J} = \dfrac{300{,}000(10)(2)}{\pi(10)^4} = 191.0\ MPa$

$\tau_{max} = \sqrt{\left(\dfrac{\sigma}{2}\right)^2 + \tau^2} = \dfrac{Y}{2} = \sqrt{\left(\dfrac{326.6}{2}\right)^2 + 191.0^2}$

$Y = \underline{502.6\ MPa}$



4.39

$$\sigma = SF\frac{P}{A} = \frac{2.00P}{\pi(25)^2} = 0.001019P;\ \tau = SF\frac{Tc}{J} = \frac{2.00(1,200,000)(25)(2)}{\pi(25)^4} = 97.8\ \text{MPa}$$

(a) $\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \tau_Y = 140 = \sqrt{\left(\frac{0.001019P}{2}\right)^2 + 97.8^2}$

$P = \underline{196.\ \ \text{kN}}$

(b) $\tau_{oct(max)} = \frac{1}{3}\sqrt{2\sigma^2 + 6\tau^2} = \frac{1}{3}\sqrt{6\tau_Y^2} = \frac{1}{3}\sqrt{6(140)^2} = \frac{1}{3}\sqrt{2(0.001019P)^2 + 6(97.8)^2}$

$P = \underline{170.3\ \text{kN}}$

4.40 $M = 200(1800+180) = 396,000\ \text{N.mm};\ T = 200(1800-180) = 324,000\ \text{N.mm}$

$$\sigma = SF\frac{Mc}{I} = \frac{2.20(396,000)(d)(64)}{2\pi d^4} = \frac{8,874,000}{d^3}\ (\text{MPa})$$

$$\tau = SF\frac{Tc}{J} = \frac{2.20(324,000)(d)(32)}{2\pi d^4} = \frac{3,630,000}{d^3}\ (\text{MPa})$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{Y}{2} = \frac{320}{2} = \frac{1}{d^3}\sqrt{\left(\frac{8,874,000}{2}\right)^2 + (3,630,000)^2}$$

$d = \underline{32.97\ \text{mm}}$

4.41 $M = \sqrt{M_x^2 + M_y^2} = \sqrt{660^2 + 480^2} = 816.1\ \text{N.m};\ A = 2\pi Rh;\ J = AR^2;\ I = \frac{J}{2}$

$$\sigma_z = \frac{pR}{2h} + \frac{P}{A} + \frac{Mc}{I} = \frac{11(38)}{2(4)} + \frac{80,000}{1005} + \frac{816,100(40)}{804,000} = 172.5\ \text{MPa}$$

$$\sigma_\theta = \frac{pR}{h} = \frac{11(38)}{4} = 104.5\ \text{MPa}$$

$$\tau = \frac{Tc}{J} = \frac{3,600,000(40)}{1,608,000} = 89.6\ \text{MPa}$$

$$\sigma_1 = \frac{\sigma_z + \sigma_\theta}{2} + \sqrt{\left(\frac{\sigma_z - \sigma_\theta}{2}\right)^2 + \tau^2} = \frac{172.5+104.5}{2} + \sqrt{\left(\frac{172.5+104.5}{2}\right)^2 + 89.6^2} = 234.3\ \text{MPa}$$

$$\sigma_2 = \frac{\sigma_z + \sigma_\theta}{2} - \sqrt{\left(\frac{\sigma_z - \sigma_\theta}{2}\right)^2 + \tau^2} = 138.5 - 95.8 = 42.7\ \text{MPa}$$

$$\sigma_3 = 0$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{234.3}{2} = \frac{Y}{2}$$

$Y = \sigma_1 = \underline{234.3\ \text{MPa}}$

4.42

$$\tau_{oct(max)} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} = \frac{1}{3}\sqrt{2Y^2} = \frac{1}{3}\sqrt{(234.3-42.7)^2 + 42.7^2 + (-234.3)^2}$$

$Y = \underline{216.6\ \text{MPa}}$

4.43 FIRST WRITE EXPRESSIONS FOR BENDING AND SHEAR STRESSES IN TERMS OF THE APPLIED LOAD AND BAR SIZE:

$$\sigma = \frac{M(d/2)}{I} = \frac{M(d/2)}{(\pi d^4/64)} = \frac{32M}{\pi d^3} \qquad (a)$$

$$\tau = \frac{T(d/2)}{J} = \frac{T(d/2)}{(\pi d^4/32)} = \frac{16T}{\pi d^3} \qquad (b)$$

SUBSTITUTE EQS. (a) & (b) INTO EQ. (4.45) TO FIND THE RELATIONSHIP AMONG DIAMETER $d = d_s$, YIELD STRESS Y, MOMENT M, AND TORQUE T FOR THE MAX. SHEAR STRESS CRITERION:

$$\left(\frac{\sigma}{2}\right)^2 + \tau^2 = \frac{Y^2}{4} \qquad (4.45)$$

$$\left(\frac{32M}{2\pi d_s^3}\right)^2 + \left(\frac{16T}{\pi d_s^3}\right)^2 = \frac{Y^2}{4}$$

SOLVE FOR d_s, THE DIAMETER AT WHICH YIELD OCCURS:

$$\underline{d_s = \sqrt[6]{\frac{1024}{\pi^2 Y^2}\left(M^2+T^2\right)}} \qquad (c)$$

NOW SUBSTITUTE EQS (a) & (b) INTO EQ. (4.46) FOR THE von MISES CRITERION:

$$\frac{2\sigma^2 + 6\tau^2}{9} = \frac{2Y^2}{9} \qquad (4.46)$$

(cont.)

4.43 (CONT.)

$$2\left(\frac{32M}{\pi d_{vm}^3}\right)^2 + 6\left(\frac{16T}{\pi d_{vm}^3}\right)^2 = 2Y^2$$

SOLVE FOR d_{vm}, THE DIAMETER AT WHICH YIELD OCCURS.

$$\underline{d_{vm} = \sqrt[6]{\frac{1024M^2 + 768T^2}{\pi^2 Y^2}}} \qquad (d)$$

TAKE THE RATIO OF EQS. (d) & (c)

$$\underline{\underline{\frac{d_{vm}}{d_s} = \sqrt[6]{\frac{M^2 + 0.75T^2}{M^2 + T^2}}}}$$

4.44 BENDING AND SHEAR STRESSES ARE (SEE SOL. 4.43)

$$\sigma = \frac{32M}{\pi d^3}, \qquad \tau = \frac{16T}{\pi d^3} \qquad (a)$$

FROM EQS (a) & (4.45), THE BAR DIAMETER ASSOCIATED WITH YIELD IS (SEE SOL. 4.43)

$$\underline{d_s = \sqrt[6]{\frac{1024}{\pi^2 Y^2}(M^2 + T^2)}} \qquad (b)$$

THE MAXIMUM PRINCIPAL STRAIN CRITERION, EQ. (4.5), CAN BE WRITTEN AS

$$\sigma_1 - \nu\sigma_3 = Y$$

(cont.)

4.44 (CONT.)

FOR THE BIAXIAL STRESS STATE (σ, τ), EQ. (4.5) CAN BE WRITTEN AS

$$\left(\frac{\sigma}{2} + R\right) - \nu\left(\frac{\sigma}{2} - R\right) = Y \qquad (c)$$

WHERE $R = \sqrt{\sigma^2/4 + \tau^2}$.

SUBSTITUTE EQS. (a) & (b) INTO EQ. (c) AND SOLVE FOR THE BAR DIAMETER $d = d_{eps}$ AT WHICH YIELD OCCURS.

$$d_{eps} = \sqrt[3]{\frac{16M(1-\nu) + 16(1+\nu)\sqrt{M^2+T^2}}{\pi Y}} \qquad (d)$$

TAKE THE RATIO OF EQS. (d) & (b):

$$\frac{d_{eps}}{d_s} = \sqrt[6]{\frac{\left[16M(1-\nu) + 16(1+\nu)\sqrt{M^2+T^2}\right]^2}{1024\,(M^2+T^2)}}$$

5.1 Let q_1, q_2, q_3 be the displacements of weights W_1, W_2, W_3, respectively (Fig. a). Then spring 1 undergoes elongation q_1, spring 2 undergoes elongation $q_2 - q_1$, and spring 3 undergoes elongation $q_3 - q_2$. Then, by Eq. (5.12), the potential energy stored in the springs is

$$U_A = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 + \frac{1}{2}k_3(q_3 - q_1)^2 \quad (a)$$

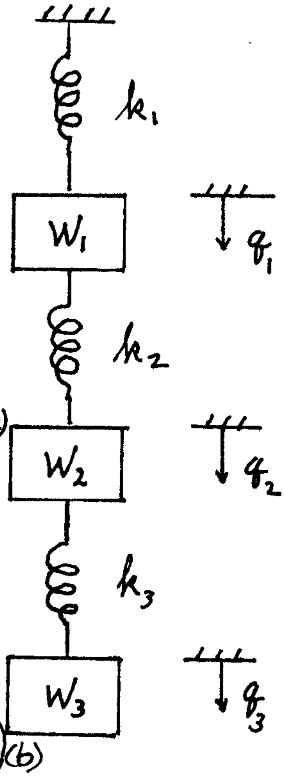


Figure a

The internal forces in springs 1, 2, and 3 are, respectively,

$$F_1 = k_1 q_1,\ F_2 = k_2(q_2 - q_1),\ F_3 = k_3(q_3 - q_2) \quad (b)$$

Also, by equilibrium,

$$F_1 = W_1 + W_2 + W_3,\ F_2 = W_2 + W_3,\ F_3 = W_3 \quad (c)$$

So, by Eqs. (a), (b), and (c),

$$U_A = \frac{1}{2k_1}(W_1 + W_2 + W_3)^2 + \frac{1}{2k_2}(W_2 + W_3)^2 + \frac{1}{2k_3}(W_3)^2 \quad (d)$$

By Eqs. (5.4) and (d), we obtain

$$q_1 = \frac{\partial U_A}{\partial W_1} = \frac{1}{k_1}(W_1 + W_2 + W_3),\ q_2 = \frac{\partial U_A}{\partial W_2} = \frac{1}{k_1}(W_1 + W_2 + W_3) + \frac{1}{k_2}(W_2 + W_3)$$

$$q_3 = \frac{\partial U_A}{\partial W_3} = \frac{1}{k_1}(W_1 + W_2 + W_3) + \frac{1}{k_2}(W_2 + W_3) + \frac{1}{k_3}W_3$$

5.2 Referring to Problem 5.1, the potential energy stored in n springs is (see also Example 5.3)

$$U_A = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 + \frac{1}{2}k_3(q_3 - q_2) + \cdots + \frac{1}{2}k_n(q_n - q_{n-1}) \quad (a)$$

$$F_1 = k_1 q_1, \quad F_2 = k_2(q_2 - q_1), \quad F_3 = k_3(q_3 - q_2), \cdots, \quad F_n = k_n(q_n - q_{n-1}) \quad (b)$$

By equilibrium,

$$\begin{aligned} F_1 &= W_1 + W_2 + W_3 + \cdots + W_n \\ F_2 &= W_2 + W_3 + W_4 + \cdots + W_n \\ F_3 &= W_3 + W_4 + W_5 + \cdots + W_m \\ &\vdots \\ F_n &= W_n \end{aligned} \qquad (c)$$

By Eqs. (a), (b), and (c), we have

$$U_A = \frac{1}{2k_1}(W_1 + W_2 + \cdots + W_n)^2 + \frac{1}{2k_2}(W_2 + W_3 + \cdots + W_n)^2 + \cdots + \frac{1}{2k_m}W_n^2 \quad (d)$$

Then, by Eqs. (5.4) and (d)

$$q_1 = \frac{\partial U_A}{\partial W_1} = \frac{1}{k_1}(W_1 + W_2 + \cdots + W_n), \quad q_2 = \overset{\frac{\partial U_A}{\partial W_2} =}{\frac{1}{k_1}}(W_1 + W_2 + \cdots + W_n) + \frac{1}{k_2}(W_2 + W_3 + \cdots + W_n)$$

$$q_3 = \frac{\partial U_A}{\partial W_3} = \frac{1}{k_1}(W_1 + W_2 + \cdots + W_n) + \frac{1}{k_2}(W_2 + W_3 + \cdots + W_n) + \frac{1}{k_3}(W_3 + W_4 + \cdots + W_n)$$

$$\vdots$$

$$q_n = \frac{\partial U_A}{\partial W_n} = \frac{1}{k_1}(W_1 + W_2 + \cdots + W_m) + \frac{1}{k_2}(W_2 + W_3 + \cdots + W_n) + \cdots$$
$$+ \frac{1}{k_{10}}(W_{10} + W_{11} + \cdots + W_n) + \cdots + \frac{1}{k_m}W_n$$

5.3 Since $F = kx^n$,

$$x = \left(\frac{F}{k}\right)^{1/n} \qquad \text{(a)}$$

where x is the elongation of the spring. By Eq. (a) and Fig. 5.1, the complementary strain energy of the spring system of Example 5.3 is

$$C_A = \int_0^{F_1} x_1\,dF + \int_0^{F_2} x_2\,dF = \int_0^{F_1}\left(\frac{F}{k_1}\right)^{1/n} dF + \int_0^{F_2}\left(\frac{F}{k_2}\right)^{1/n} dF$$

or after integrating

$$C_A = \frac{n}{n+1}\left[\frac{F_1^{(n+1)/n}}{k_1^{1/n}} + \frac{F_2^{(n+1)/n}}{k_2^{1/n}}\right] \qquad \text{(b)}$$

By equilibrium, as in Example 5.3,

$$F_1 = W_1 + W_2, \quad F_2 = W_2 \qquad \text{(c)}$$

By Eqs. (5.2) and (b), with Eqs. (c),

$$q_1 = \frac{\partial C_A}{\partial W_1} = \frac{\partial C_A}{\partial F_1}\frac{\partial F_1}{\partial W_1} + \frac{\partial C_A}{\partial F_2}\frac{\partial F_2}{\partial W_1} = \left(\frac{W_1 + W_2}{k_1}\right)^{1/n}$$

$$q_2 = \frac{\partial C_A}{\partial W_2} = \frac{\partial C_A}{\partial F_1}\frac{\partial F_1}{\partial W_2} + \frac{\partial C_A}{\partial F_2}\frac{\partial F_2}{\partial W_2} = \left(\frac{W_1 + W_2}{k_1}\right)^{1/n} + \left(\frac{W_2}{k_2}\right)^{1/n} \qquad \text{(d)}$$

For $n = 1$ and $n = 2$, Eqs. (d) reduce to the solutions of Examples 5.3 and 5.4, respectively.

5.4 The shear stress is

$$\sigma_{zy} = \frac{V_y Q}{I_x b} \qquad (a)$$

By Fig. a, the first moment Q about the x axis of the area above the ordinate y is

$$Q = b\left(\frac{h}{2} - y\right)\left[\frac{1}{2}\left(\frac{h}{2} + y\right)\right]$$

or

$$Q = \frac{bh^2}{8}\left(1 - \frac{4y^2}{h^2}\right) \qquad (b)$$

Also by Fig. a,

$$I_x = \frac{1}{12} b h^3 \qquad (c)$$

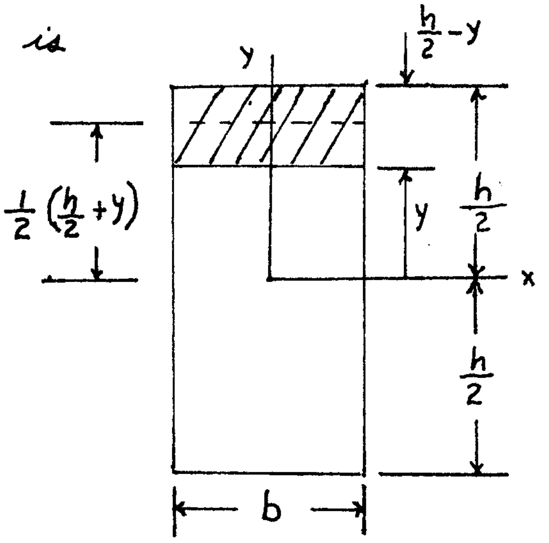


Figure a

Equations (a), (b), and (c) yield

$$\sigma_{zy} = \frac{V_y\left[\frac{bh^2}{8}\left(1 - \frac{4y^2}{h^2}\right)\right]}{\frac{1}{12} b^2 h^3} = \frac{3}{2}\frac{V_y}{A}\left(1 - \frac{4y^2}{h^2}\right); \; A = bh$$

or

$$\sigma_{zy} = \tau_{max}\left(1 - \frac{4y^2}{h^2}\right)$$

where $\tau_{max} = \frac{3}{2}\frac{V_y}{A}$ is the maximum value of σ_{zy} for $y = 0$.

5.5 By Eq. (5.6), where V = volume, the strain energy is for shear

$$U_s = \int U_0 \, dV \qquad \text{(a)}$$

By Eq. (5.7), for shear stress $\sigma_{zy} = \sigma_{yz}$,

$$U_0 = \frac{1}{2G} \sigma_{zy}^2 \qquad \text{(b)}$$

where (see Problem 5.4), the shear stress is

$$\sigma_{zy} = \frac{3}{2} \frac{V_y}{A} \left(1 - \frac{4y^2}{h^2}\right) ; \quad A = bh \qquad \text{(c)}$$

By Eqs. (a), (b), and (c), with $dV = b\,dy\,dz$, we obtain

$$U_s = \int \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{9 V_y^2}{8 G A^2} \left(1 - \frac{4y^2}{h^2}\right)^2 b\,dy\,dz \qquad \text{(d)}$$

Expansion of the integrand of Eq. (d) and integration with respect to y from $-h/2$ to $h/2$ yields

$$U_s = \int \frac{1.20 \, V_y^2}{2GA} \, dz \qquad \text{(e)}$$

Comparison of Eqs (5.14) and (e) shows that

$$k = 1.20$$

5.6 By Eqs. (5.6) and (5.7), we have

$$U_M = \int U_0 \, dV \; ; \quad V = \text{Volume} \qquad (a)$$

$$U_0 = \frac{1}{2E} \sigma_{zz}^2 \qquad (b)$$

where as given

$$\sigma_{zz} = \frac{M_x y}{I_x} \qquad (c)$$

By Eqs. (b) and (c), we obtain

$$U_0 = \frac{1}{2E} \frac{M_x^2 y^2}{I_x^2} \qquad (d)$$

Then, by Eqs. (a) and (d),

$$U_M = \int \left[\frac{1}{2E} \frac{M_x^2}{I_x^2} \int y^2 dA \right] dz \quad (e)$$

where $dV = dA\,dz$, A = beam cross-sectional area, and dz is the axial infinitesimal length of the beam.

By definition $I_x = \int y^2 dA$. Hence, Eq. (e) reduces to

$$U_M = \int \frac{M_x^2}{2EI_x} dz$$

5.7 By Eqs. (5.6) and (5.7),

$$U_T = \int U_0 dV ; \quad V = \text{Volume} \qquad \text{(a)}$$

$$U_0 = \frac{1}{2G} \sigma_{xy}^2 \qquad \text{(b)}$$

where as given,

$$\sigma_{xy} = \frac{T\rho}{J} \qquad \text{(c)}$$

By Eqs. (b) and (c),

$$U_0 = \frac{1}{2G} \frac{T^2 \rho^2}{J^2} \qquad \text{(d)}$$

Then, by Eqs. (a) and (d),

$$U_T = \int \left[\frac{1}{2G} \frac{T^2}{J^2} \int \rho^2 dA \right] dz \qquad \text{(e)}$$

where $dV = dA\,dz$, A = the cross-sectional area of the member, and dz is the axial infinitesimal length of the member.

By definition, $J = \int \rho^2 dA$. Hence, Eq. (e) reduces to

$$U_T = \int \frac{T^2}{2GJ} dz$$

5.8 For each member, the area of the cross section is $A = \pi D^2/4 = \pi (0.10)^2/4 = 7.854\times10^{-3}\ m^2$ and the axial load at the proportional limit is $N = \sigma_{PL} A = 320\times10^6 \times 7.854\times10^{-3} = 2.513\times10^6\ N$. The energy absorbed under axial loading is, by Eq. (5.8),

$$U_N = \int_0^L \frac{N^2}{2EA}\,dz = \frac{N^2 L}{2EA} \qquad (a)$$

For steel, $E = 200\,GPa$. So, by Eq. (a), with $L = 1.50\,m$,

$$U_N^{steel} = \frac{(2.513\times10^6)^2(1.50)}{2(200\times10^9)(7.854\times10^{-3})} = 3.016\ kJ$$

For aluminum, $E = 72\,GPa$. Then by Eq. (a),

$$U_N^{Aluminium} = \frac{(2.513\times10^6)^2(1.50)}{2(72\times10^9)(7.854\times10^{-3})} = 8.376\ kJ$$

Therefore, the aluminum member is capable of absorbing more energy up to the proportional limit.

Since the proportional limits, the lengths, and the cross-sectional areas of the members are identical, the aluminum member absorbs more energy since its modulus of elasticity (which occurs in the denominator of Eq. (a)) is smaller than that of steel.

5.9 By Eq. (5.8), the energy absorbed at yield is

$$U_N = \frac{N^2L}{2EA} = \frac{(YA)^2L}{2EA} = \frac{Y^2AL}{2E} \qquad (a)$$

(a) Since the member must absorb 20 kJ with a safety factor of 2.5, Eq (a) yields

$$U_N = \frac{Y^2AL}{2E} = 2.5(20\times10^3)\ \mathrm{N\cdot m} \qquad (b)$$

With $Y = 210$ MPa, $L = 2.00$ m, and $E = 82.7$ GPa, the cross-sectional area of the brass member is, by Eq. (b),

$$A = \frac{2.5(20\times10^3)(2)(82.7\times10^9)}{(210\times10^6)^2(2.00)} = 0.09376\ \mathrm{m}^2$$

(b) at the design energy, by Eq. (a)

$$U_N = \frac{N^2L}{2EA} = \frac{(\sigma A)^2L}{2EA} = 20\times10^3\ \mathrm{N\cdot m}$$

or

$$\sigma = \left[\frac{(20\times10^3)(2)(82.7\times10^9)}{(0.09376)(2.00)}\right]^{1/2} = 132.8\ \mathrm{MPa} \qquad (c)$$

(c) With $Y = 210$ MPa, Eq. (c) yields

$$\frac{Y}{\sigma} = \frac{210}{132.8} = 1.581 \neq 2.5$$

Since the energy is proportional to N^2 [Eq (a)], it is proportional to the square of stress σ. Therefore, the ratio of the stress Y at yield to the stress σ at the design energy is equal to the square root of the safety factor; that is, $\frac{Y}{\sigma} = \sqrt{2.5} = 1.581$.

5.10 Member A: The cross-sectional area is

$$A = \frac{\pi(0.020)^2}{4} = 3.1416\times10^{-4}\,m^2$$

at yield,

$$P = YA = 3.1416\times10^{-4}\,Y$$

The strain energy in the member is

$$U_N = \int_0^{0.50} \frac{N^2}{2EA}\,dz = \frac{P^2(0.50)}{2EA} = \frac{Y^2}{E}(7.854\times10^{-5}) \quad (a)$$

Member B: The cross-sectional area for the 25 mm section is

$$A_1 = \frac{\pi(0.025)^2}{4} = 4.9087\times10^{-4}\,m^2$$

and for the 50 mm section is

$$A_2 = \frac{\pi(0.050)^2}{4} = 1.9635\times10^{-3}\,m^2$$

Member B will yield first in the 25 mm section at a load

$$P = YA_1 = (4.9087\times10^{-4})\,Y$$

The strain energy in member B is

$$U_N = \int_0^{0.100} \frac{N^2}{2EA_1}\,dz + \int_{0.100}^{0.500} \frac{N^2}{2EA_2}\,dz = \frac{P^2}{2E}\left(\frac{0.100}{A_1} + \frac{0.400}{A_2}\right)$$

or

$$U_N = (4.9087\times10^{-5})\frac{Y^2}{E} \qquad (b)$$

Comparing Eqs. (a) and (b), we see that member A absorbs more energy; this is because the entire length of member A is subjected to the yield stress Y. Only the 25 mm (100 mm long) section is subjected to the yield stress Y. The stress in the 400 mm length of member B is subjected to a stress of $\sigma = \frac{P}{A_2} = 0.25Y$

5.11 Given that the cross section is a b×h rectangle,

$$A = bh, \quad I_x = \frac{1}{12} b h^3, \quad k = 1.20 \qquad (a)$$

For steel, $E = 200\ \text{GPa}$ and $G = 77.5\ \text{GPa}$

(a) The shear strain energy is [Eq. (5.14)]

$$U_S = 2 \int_0^{L/2} \frac{1.20 V_y^2}{2GA} dz \qquad (b)$$

where for the simple beam

$$V_y = \frac{P}{2}, \quad 0 \le z \le \frac{L}{2} \qquad (c)$$

Substituting G, A, and V_y into Eq (b) and integrating, we obtain

$$U_S = 1.935 \times 10^{-12} \frac{P^2 L}{bh} \qquad (d)$$

The bending energy is [Eq. (5.13)]

$$U_M = 2 \int_0^{L/2} \frac{M_x^2}{2EI_x} dz \qquad (e)$$

where

$$M_x = \frac{P}{2} z \qquad (f)$$

Substituting E, I_x, and M_x into Eq. (e) and integrating, we find

$$U_M = 6.25 \times 10^{-13} \frac{P^2 L^2}{b h^3} \qquad (g)$$

Requiring that $U_S < 0.01 U_M$, we find by Eqs. (d) and (g),

(Cont.)

5.11 Cont.

$$309.6h^2 < L^2, \text{ or } L > 17.6h$$

(b) Requiring $U_S < 0.05 U_M$, we find

$$61.92h^2 < L^2, \text{ or } L > 7.9h$$

5.12 Since the cross section is circular with diameter h,

$$A = \frac{\pi h^2}{4}, \quad I_x = \frac{\pi h^4}{64}, \quad k = 1.33 \qquad (a)$$

For steel, $E = 200$ GPa, $G = 77.5$ GPa.

The shear strain energy [Eq. (5.14)] is

$$U_S = 2\int_0^{L/2} \frac{1.33 V_y^2}{2GA} dz \qquad (b)$$

where since the beam is simply supported

$$V_y = \frac{P}{2} \qquad (c)$$

Substituting G, A, and V_y into Eq. (b) and integrating, we obtain

$$U_S = 2.054 \times 10^{-12} \frac{P^2 L}{h^2} \qquad (d)$$

The bending energy [Eq. (5.13)] is

$$U_M = 2\int_0^{L/2} \frac{M_x^2}{2EI_x} dz \qquad (e)$$

where

$$M_x = \frac{P}{2} z \qquad (f)$$

Substituting E, I_x, M_x into Eq. (e) and integrating, we find

$$U_M = 8.488 \times 10^{-12} \frac{P^2 L^3}{h^4} \qquad (g)$$

(cont.)

5.12 cont. Setting $U_S < 0.01 U_M$, we find by Eqs. (d) and (g),

$$2.054 \times 10^{-12} \frac{P^2 L}{h^2} < 8.488 \times 10^{-14} \frac{P^2 L^3}{h^4}$$

or $L/h > 4.919$. Hence, $U_S < 0.01 U_M$ for $L = 5.0 h$.

5.13 We are given that $I_x = 51.61 \times 10^6 \text{ mm}^4$, $h = 254$ mm, $b = 7.9$ mm, $A = bh = (7.9)(254) = 2006.6 \text{ mm}^2$, $E = 200$ GPa, and $G = 77.5$ GPa. By Table 5.1, $k = 1.00$. So, by Eqs. (5.13) and (5.14), we have

$$U_M = 2\int_0^{L/2} \frac{M_x^2}{2EI_x} dz, \quad U_S = 2\int_0^{L/2} \frac{kV_y^2}{2GA} dz \qquad (a)$$

where since the beam is simply supported,

$$M_x = \frac{P}{2} z, \quad V_y = \frac{P}{2} \qquad (b)$$

Substituting the given data and Eqs. (b) into Eq. (a), we find

$$U_M = 1.00917 \times 10^{-9} P^2 L^3, \quad U_S = 8.0380 \times 10^{-10} P^2 L \quad (c)$$

(a) For $U_S < 0.01 U_M$, Eqs. (c) yield

$$8.038 \times 10^{-10} P^2 L < 1.00917 \times 10^{-11} P^2 L^3$$

or

$$L > 8.295 \text{ m} = 8295 \text{ mm} = 35.1 h$$

(b) For $U_S < 0.05 U_M$, Eqs. (c) yield

$$8.038 \times 10^{-10} P^2 L < 5.046 \times 10^{-11} P^2 L^3$$

or

$$L > 3.991 \text{ m} = 3991 \text{ mm} = 15.7 h$$

5.14 By Fig. P5.14, the cross-sectional area of the members is given by member A as

$$A = \frac{\pi d^2}{4} = \frac{\pi (0.050)^2}{4} = 1.9635 \times 10^{-3}\ m^2 \qquad (a)$$

For Member A: Since member A has a solid circular cross section of diameter d = 0.050 m, the polar moment of inertia is

$$J = \frac{\pi d^4}{32} = \frac{\pi (0.050)^4}{32} = 6.1359 \times 10^{-7}\ m^4 \qquad (b)$$

By Eq. (1.4), with $\tau_{max} = 60\ MPa$,

$$T = \tau_{max} \frac{J}{(d/2)} = (60 \times 10^6) \frac{6.1359 \times 10^{-7}}{0.025} = 1472.6\ N{\cdot}m \quad (c)$$

By Eqs. (5.16), (b), and (c), with L = 500 mm = 0.500 m,

$$U_T = \int_0^{0.50} \frac{T^2}{2GJ} dz = \frac{(1472.6)^2 (0.50)}{2(27 \times 10^9)(6.1359 \times 10^{-7})} = 32.72\ N{\cdot}m \quad (d)$$

For Member B: For the solid section of length 250 mm, as for member A, with d = 0.050 m,

$$A = 1.9635 \times 10^{-3}\ m^2; \quad J = 6.1359 \times 10^{-7}\ m^4 \qquad (e)$$

The cross-sectional area of the hollow section of length 250 mm and inner diameter $d_{in} = 0.050$ m is also $A = 1.9635 \times 10^{-3}\ m^2$. To determine J for the hollow section, we need to determine the outer diameter d_{out}. Since the area is $A = 1.935 \times 10^{-3} = \frac{\pi}{4}(d_{out}^2 - d_{in}^2)$, $d_{out} = 0.07071\ m$.

Hence,

$$J = \frac{\pi}{32}(d_{out}^4 - d_{in}^4) = 1.8408 \times 10^{-6}\ m^4 \qquad (f)$$

Next, we check to see when and where the maximum shear stress first occurs.

(cont.)

5.14 cont. The maximum shear stress occurs in the solid section when $T = 1472.6$ N.m [see Eq. (c)].

For the hollow section τ_{max} occurs when

$$T = \frac{\tau_{max} J}{(d/2)} = (60\times10^{6})\left[\frac{1.8408\times10^{-6}}{(0.07071/2)}\right] = 3123.9 \text{ N.M} \quad (g)$$

Hence, τ_{max} occurs first in the solid section when $T = 1472.6$ N.m. Then, by Eqs. (e), (f), (g), and (5.16), the torsional strain energy of member B is

$$U_T = \int_0^{0.250} \frac{T^2}{2GJ} dz + \int_{0.250}^{0.500} \frac{T^2}{2GJ} dz$$

$$= \frac{(1472.6)^2 (0.250)}{2(27\times10^9)(6.1359\times10^{-7})} + \frac{(1472.6)^2 (0.500-0.250)}{2(27\times10^9)(1.8408\times10^{-6})}$$

or

$$U_T = 16.36 + 5.45 = 21.81 \text{ N.m}$$

For Member C: For member C,

$$A = 1.9635\times10^{-3} \text{ m}^2; \; J = 1.8408\times10^{-6} \text{ m}^4; \; T = 3123.9 \text{ N.m}$$

So, for member C,

$$U_T = \int_0^{0.500} \frac{T^2}{2GA} dz = \frac{(3123.9)^2 (0.500)}{2(27\times10^9)(1.8408\times10^{-6})}$$

or

$$U_T = 49.10 \text{ N.m}$$

Hence, for τ_{max}, member C is capable of absorbing the most energy.

5.15 By Fig. a,

$$M_{AB} = Px\,,\quad \frac{\partial M_{AB}}{\partial P} = x \qquad (a)$$

$$M_{BC} = P[L+R(1-\cos\theta)] \qquad (b)$$

$$\frac{\partial M_{BC}}{\partial P} = L + R(1-\cos\theta)$$

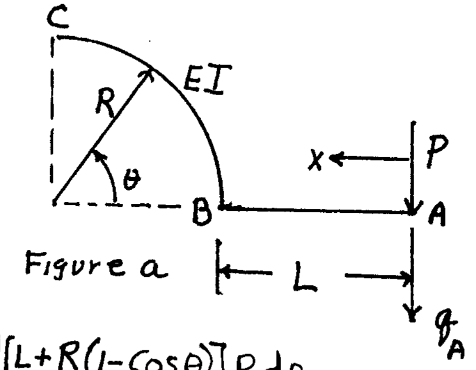


Figure a

By Eqs. (a), (b), and (5.17),

$$q_A = \frac{\partial U}{\partial P} = \int_0^L \frac{Px^2}{EI}dx + \int_0^{\frac{\pi}{2}} \frac{P}{EI}[L+R(1-\cos\theta)][L+R(1-\cos\theta)]R\,d\theta$$

$$= \frac{PL^3}{3EI} + \frac{PR}{EI}\int_0^{\frac{\pi}{2}}[(L+R)^2 - 2(L+R)R\cos\theta + R^2\cos^2\theta]d\theta$$

or

$$q_A = \frac{PL^3}{3EI} + \frac{PR}{EI}\left[\frac{\pi}{2}(L+R)^2 - 2(L+R)R + \frac{\pi}{4}R^2\right]$$

5.16 Apply a dummy moment M_A at A (Fig. a). Then, by Fig. a,

$$M_{AB} = Px + M_A \qquad (a)$$

$$\frac{\partial M_{AB}}{\partial M_A} = 1$$

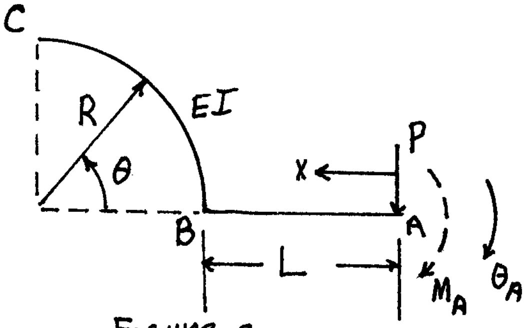


Figure a

$$M_{BC} = P[L+R(1-\cos\theta)] + M_A \qquad (b)$$

$$\frac{\partial M_{BC}}{\partial M_A} = 1$$

Then, by Eqs. (a), (b), and (5.20b), with $M_A = 0$,

$$\theta_A = \frac{\partial U}{\partial M_A} = \int_0^L \frac{Px}{EI}dx + \int_0^{\frac{\pi}{2}} \frac{P}{EI}[L+R(1-\cos\theta)]R\,d\theta$$

or

$$\theta_A = \frac{PL^2}{2EI} + \frac{PR}{EI}\left[\frac{\pi}{2}(L+R) - R\right]$$

5.17 Apply a dummy moment M_B at B (Fig. a). Then, by Fig. a,

$$M_{AB} = Px, \quad \frac{\partial M_{AB}}{\partial M_B} = 0 \qquad (a)$$

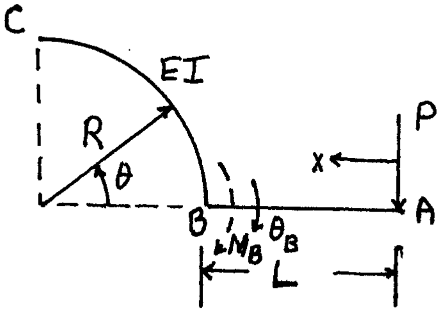


Figure a

$$M_{BC} = P[L + R(1 - \cos\theta)] + M_A$$

$$\frac{\partial M_{BC}}{\partial M_B} = 1 \qquad (b)$$

So, by Eqs. (a), (b), and (5.20b), with $M_B = 0$,

$$\theta_B = \frac{\partial U}{\partial M_B} = \int_0^L \frac{Px}{EI}(0)dx + \int_0^{\frac{\pi}{2}} \frac{P}{EI}[L + R(1 - \cos\theta)](1)Rd\theta$$

or

$$\theta_B = \frac{PR}{EI}\left[\frac{\pi}{2}(L+R) - R\right]$$

5.18 Apply a dummy load Q_B at B (Fig. a). Then, by Fig. a,

$$M_{AB} = Px, \quad \frac{\partial M_{AB}}{\partial Q_B} = 0 \qquad (a)$$

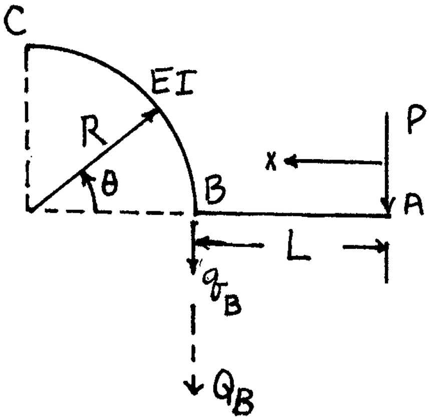


Figure a

$$M_{BC} = P[L + R(1 - \cos\theta)] + Q_B R(1 - \cos\theta)$$

$$\frac{\partial M_{BC}}{\partial Q_B} = R(1 - \cos\theta) \qquad (b)$$

Then, by Eqs. (a), (b), and (5.20a), with $Q_B = 0$,

$$q_B = \frac{\partial U}{\partial Q_B} = \int_0^L \frac{Px}{EI}(0)dx + \int_0^{\frac{\pi}{2}} \frac{P}{EI}[L + R(1 - \cos\theta)][R(1 - \cos\theta)]Rd\theta$$

$$= \frac{PR^2}{EI}\int_0^{\frac{\pi}{2}}[L(1 - \cos\theta) + R(1 - 2\cos\theta + \cos^2\theta)]\,d\theta$$

or

$$q_B = \frac{PR^2}{EI}\left[L\left(\frac{\pi}{2} - 1\right) + R\left(\frac{3\pi}{4} - 2\right)\right]$$

5.19

By Fig. a,

$$M_{AB} = 0, \quad \frac{\partial M_{AB}}{\partial Q_B} = 0 \qquad (a)$$

$$M_{BC} = PR\sin\theta$$

$$\frac{\partial M_{BC}}{\partial P} = R\sin\theta \qquad (b)$$

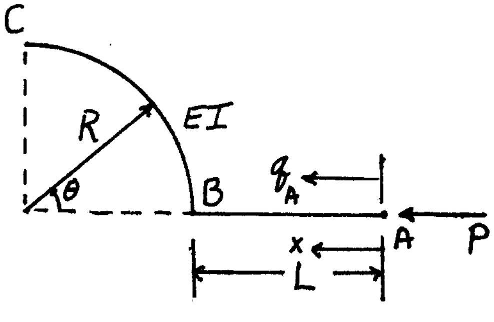


Figure a

Then, by Eqs.(a),(b), and (5.20a),

$$q_A = \frac{\partial U}{\partial P} = \int_0^L (0)(0)\,dx + \int_0^{\frac{\pi}{2}} \frac{PR^2}{EI}(\sin^2\theta)\,R\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

or

$$q_A = \frac{PR^3}{EI}\frac{\pi}{4}$$

5.20 Apply a dummy moment M_A at A (Fig. a). Then, by Fig. a,

$$M_{AB} = M_A, \quad \frac{\partial M_{AB}}{\partial M_A} = 1 \qquad (a)$$

$$M_{BC} = M_A + PR\sin\theta \qquad (b)$$

$$\frac{\partial M_{BC}}{\partial M_A} = 1$$

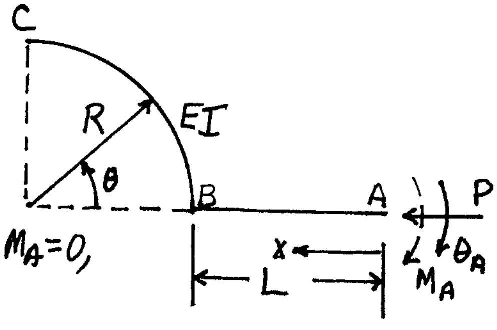


Figure a

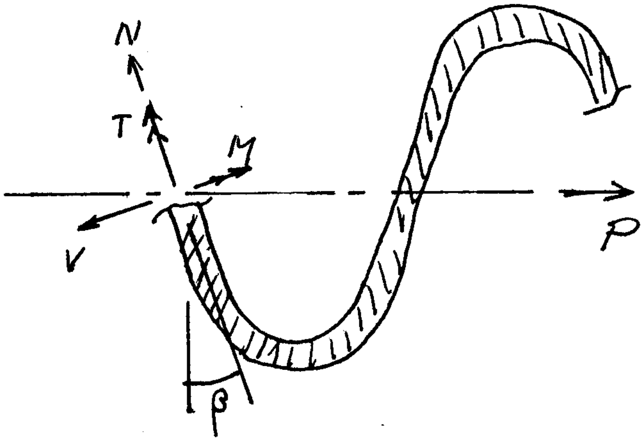
Therefore, by Eqs.(a),(b), and (5.20b), with $M_A=0$,

$$\theta_A = \frac{\partial U}{\partial M_A} = \int_0^L (0)\,dx + \int_0^{\pi/2} \frac{(PR\sin\theta)\,R\,d\theta}{EI}$$

or

$$\theta_A = \frac{PR^2}{EI}$$

5.21



THE INTERNAL ACTIONS IN THE SPRING, IN TERMS OF APPLIED LOAD P AND PITCH ANGLE β, ARE

$$N = P\sin\beta$$
$$V = P\cos\beta$$
$$M = \frac{PD}{2}\sin\beta$$
$$T = \frac{PD}{2}\cos\beta$$

FOR A UNIT LOAD ON THE SPRING, THE INTERNAL ACTIONS ARE

$$n = \sin\beta$$
$$v = \cos\beta$$
$$m = \frac{D}{2}\sin\beta$$
$$t = \frac{D}{2}\cos\beta$$

SPRING DEFLECTION IS GIVEN BY EQ. (5.20a)

$$q = \int_0^L \left(\frac{P\sin^2\beta}{EA} + \frac{P\cos^2\beta}{GA} + \frac{PD^2\sin^2\beta}{4EI} + \frac{PD^2\cos^2\beta}{4GJ} \right) ds \quad (a)$$

$L = n\pi D$ AND ALL QUANTITIES ARE INDEPENDENT OF S.

(Cont.)

5.21 (CONT.)

INTEGRATION OF EQ (a) GIVES:

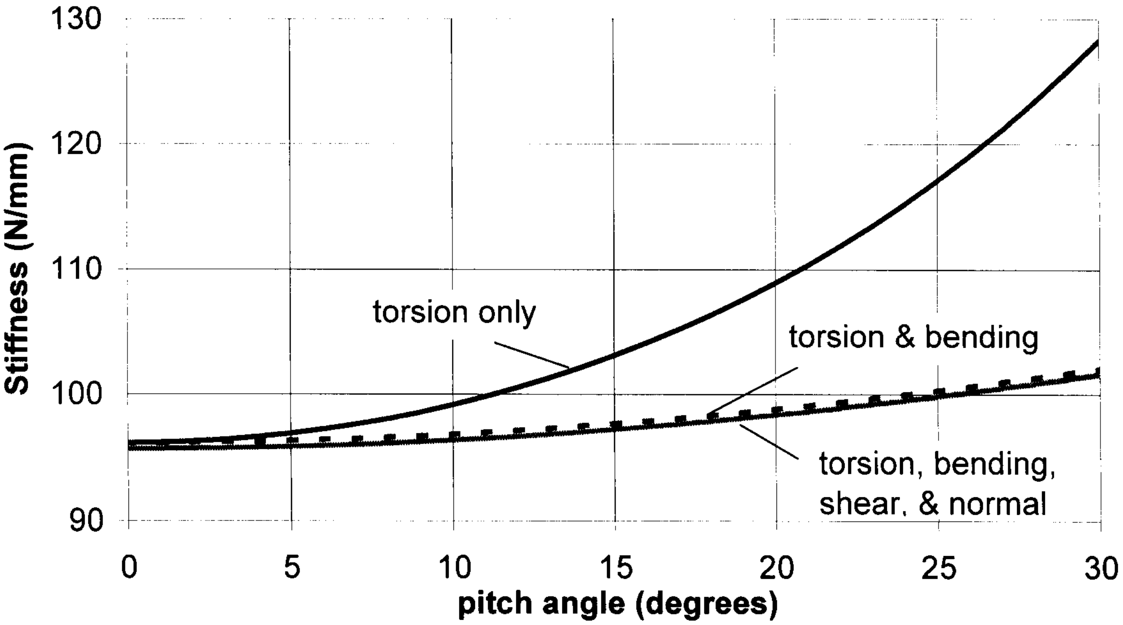
$$\delta = n\pi P D\left(\frac{\sin^2\beta}{EA} + \frac{\cos^2\beta}{GA} + \frac{D^2\sin^2\beta}{4EI} + \frac{D^2\cos^2\beta}{4GJ}\right) \qquad (b)$$

SPRING STIFFNESS IS GIVEN BY $k = P/\delta$.

SUBSTITUTION OF EQ (b) INTO THIS EQUATION, WITH A, I, AND J WRITTEN IN TERMS OF WIRE DIAMETER d, GIVES

$$k = \frac{d^2}{4nD\left[\frac{\sin^2\beta}{E}\left(1+\frac{4D^2}{d^2}\right) + \frac{\cos^2\beta}{G}\left(1+\frac{2D^2}{d^2}\right)\right]}$$

FOR $D = 100\,mm$, $d/D = 0.1$, $E = 200\,GPa$, $\nu = 0.3$ AND $n = 1$, THE FOLLOWING PLOT IS OBTAINED. STIFFNESS IS DOMINATED BY TORSION WITH BENDING HAVING SIGNIFICANT EFFECTS AS θ INCREASES. NORMAL & SHEAR EFFECTS ARE NEGLIGIBLE.



5.22 FOR A FLAT SPRING, $\beta = 0$ (SEE FIG. E5.14). NEGLECTING SHEAR & NORMAL FORCES, ONLY TORQUE CONTRIBUTES TO STRAIN ENERGY IN THE SPRING. FOR A SPIRAL SPRING (SEE FIG. P5.22), THE EFFECTIVE RADIUS R VARIES WITH Θ. ASSUMING THIS VARIATION IS LINEAR, WE HAVE

$$R = 5d\left(1 + \frac{\Theta}{2\pi}\right) \qquad (a)$$

THE TORQUE IN THE WIRE DUE TO LOAD P IS

$$T = PR \qquad (b)$$

FOR A UNIT LOAD, THE TORQUE IS

$$t = R \qquad (c)$$

THE INFINITESIMAL ARC LENGTH IS

$$ds = R\,d\Theta \qquad (d)$$

UNDER LOAD P, THE SPRING DEFLECTION IS

$$\delta = \int_0^L \frac{Tt}{GJ}\,ds \qquad (e)$$

SUBSTITUTION FROM EQS. (b), (c), & (d) INTO (e) AND ADJUSTING THE INTEGRATION LIMITS GIVES

$$\delta = \frac{1}{GJ}\int_0^{6\pi} PR^3\,d\Theta \qquad (f)$$

SUBSTITUTION FOR R FROM EQ. (a) INTO EQ. (f) GIVES

$$\delta = \frac{125Pd^3}{GJ}\int_0^{6\pi}\left(1 + \frac{3\Theta}{2\pi} + \frac{3\Theta^2}{4\pi^2} + \frac{\Theta^3}{8\pi^3}\right)d\Theta \qquad (g)$$

(CONT.)

5.22 (CONT.)

INTEGRATION OF $E_\varphi(g)$ GIVES

$$\delta = \frac{15{,}937.5\, P d^3 \pi}{G J}$$

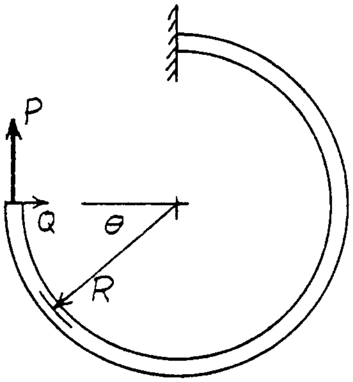
FOR $J = \frac{\pi d^4}{32}$, THE STIFFNESS OF THE SPRING IS

$$k = \frac{P}{\delta} = \frac{G(\pi d^4)}{32(15{,}937.5) d^3 \pi}$$

OR

$$\underline{\underline{k = \frac{G d}{510{,}000}}}$$

5.23 Apply an infinitesimal load Q perpendicular to P. At angle θ from free end of beam



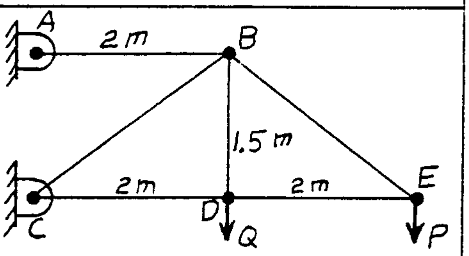
$$M = PR(1-\cos\theta) + QR\sin\theta \qquad \frac{\partial M}{\partial Q} = R\sin\theta$$

$$q_Q = \int_0^{\frac{3\pi}{2}} \frac{PR(1-\cos\theta)}{EI} R\sin\theta R\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\frac{3\pi}{2}} (\sin\theta - \sin\theta\cos\theta)\,d\theta$$

$$= \frac{PR^3}{EI}\left(-\cos\theta - \frac{\sin^2\theta}{2}\right)_0^{\frac{3\pi}{2}} = \frac{PR^3}{EI}\left(1-\frac{1}{2}\right) = \frac{6000(65^3)(12)}{2(200{,}000)(30^4)} = \underline{0.0610\text{ mm}}$$

5.24



$$N_{AB} = \frac{4}{3}(Q+2P) \qquad \frac{\partial N_{AB}}{\partial P} = \frac{8}{3}$$

$$N_{BC} = -\frac{5}{3}(Q+P) \qquad \frac{\partial N_{BC}}{\partial P} = -\frac{5}{3}$$

$$N_{BD} = Q \qquad \frac{\partial N_{BD}}{\partial P} = 0$$

$$N_{BE} = \frac{5}{3}P \qquad \frac{\partial N_{BE}}{\partial P} = \frac{5}{3}$$

$$N_{CD} = N_{DE} = -\frac{4}{3}P \qquad \frac{\partial N_{CD}}{\partial P} = \frac{\partial N_{DE}}{\partial P} = -\frac{4}{3}$$

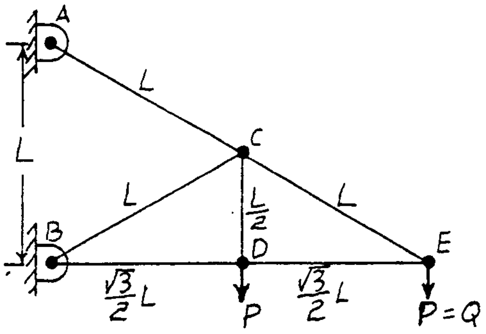
$$q_P = \frac{N_{AB}L_{AB}}{EA_{AB}}\frac{\partial N_{AB}}{\partial P} + \frac{N_{BC}L_{BC}}{EA_{BC}}\frac{\partial N_{BC}}{\partial P} + \frac{N_{BD}L_{BD}}{EA_{BD}}\frac{\partial N_{BD}}{\partial P} + \frac{N_{BE}L_{BE}}{EA_{BE}} + \frac{2N_{CD}L_{CD}}{EA_{CD}}\frac{\partial N_{CD}}{\partial P}$$

$$= \frac{4(25{,}000)(2000)(8)}{3(72{,}000)(150)(3)} + \frac{5(15{,}000)(2500)(5)}{3(72{,}000)(900)(3)} + \frac{5(10{,}000)(2500)(5)}{3(72{,}000)(150)(3)} + \frac{2(4)(10{,}000)(2000)(4)}{3(72{,}000)(900)(3)}$$

$$= 16.461 + 1.608 + 6.430 + 1.097$$

$$= \underline{25.60\text{ mm}}$$

5.25



$N_{AC} = P + 2Q \qquad \frac{\partial N_{AC}}{\partial Q} = 2$

$N_{BC} = -P \qquad \frac{\partial N_{BC}}{\partial Q} = 0$

$N_{BD} = -\sqrt{3}Q \qquad \frac{\partial N_{BD}}{\partial Q} = -\sqrt{3}$

$N_{CE} = 2Q \qquad \frac{\partial N_{CE}}{\partial Q} = 2$

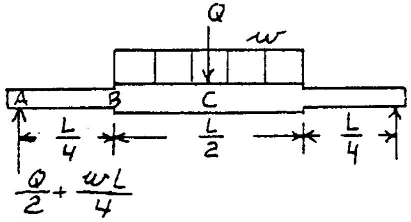
$N_{CD} = P \qquad \frac{\partial N_{CD}}{\partial Q} = 0$

$N_{DE} = -\sqrt{3}Q \qquad \frac{\partial N_{DE}}{\partial Q} = -\sqrt{3}$

$$q_Q = \frac{N_{AC}L_{AC}}{EA_{AC}}\frac{\partial N_{AC}}{\partial Q} + 0 + \frac{N_{BD}L_{BD}}{EA_{BD}}\frac{\partial N_{BD}}{\partial Q} + \frac{N_{CE}L_{CE}}{EA_{CE}}\frac{\partial N_{CE}}{\partial Q} + 0 + \frac{N_{DE}L_{DE}}{EA_{DE}}\frac{\partial N_{DE}}{\partial Q}$$

$$= \frac{(P+2P)L(2)}{AE} + \frac{(-\sqrt{3}P)(\sqrt{3}L)(-\sqrt{3})}{2AE} + \frac{2PL(2)}{AE} + \frac{(-\sqrt{3}P)(\sqrt{3}L)(-\sqrt{3})}{2AE} = \underline{15.20\frac{PL}{AE}}$$

5.26 Because of symmetry the total strain energy is twice that for the left half of the beam.



From A to B — $M = \left(\frac{Q}{2} + \frac{wL}{4}\right)z$

From B to C — $M = \left(\frac{Q}{2} + \frac{wL}{4}\right)z - \frac{w}{2}\left(z - \frac{L}{4}\right)^2$

$$q_Q = \int \frac{M}{EI}\frac{\partial M}{\partial Q}dz = 2\int_0^{L/4} \frac{wLz/4}{EI/2}\left(\frac{z}{2}\right)dz + 2\int_{L/4}^{L/2} \frac{wLz/4 - w(z-L/4)^2/2}{EI}\left(\frac{z}{2}\right)dz$$

$$= \frac{wL}{2EI}\left(\frac{z^3}{3}\right)\Bigg|_0^{L/4} + \frac{wL}{4EI}\left(\frac{z^3}{3}\right)\Bigg|_{L/4}^{L/2} - \frac{w}{2EI}\left(\frac{z^4}{4}\right)\Bigg|_{L/4}^{L/2} + \frac{wL}{4EI}\left(\frac{z^3}{3}\right)\Bigg|_{L/4}^{L/2} - \frac{wL^2}{32EI}\left(\frac{z^2}{2}\right)\Bigg|_{\frac{L}{4}}^{\frac{L}{2}}$$

$$= \frac{wL}{6EI}\left(\frac{L^3}{64}\right) + \frac{wL}{12EI}\left(\frac{L^3}{8} - \frac{L^3}{64}\right) - \frac{w}{8EI}\left(\frac{L^4}{16} - \frac{L^4}{256}\right) + \frac{wL}{12EI}\left(\frac{L^3}{8} - \frac{L^3}{64}\right) - \frac{wL^2}{64EI}\left(\frac{L^2}{4} - \frac{L^2}{16}\right)$$

$$= \frac{wL^4}{EI}\left[\frac{1}{6(64)} + \frac{7}{12(64)} - \frac{15}{8(256)} + \frac{7}{12(64)} - \frac{3}{64(16)}\right]$$

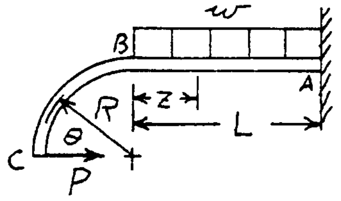
$$= \frac{wL^4}{6144EI}(16 + 56 - 45 + 56 - 18)$$

$$= \underline{\frac{65wL^4}{6144EI}}$$

5.27

From C to B —— $M = PR\sin\theta$ $\qquad \frac{\partial M}{\partial P} = R\sin\theta$

From B to A —— $M = PR + \frac{wz^2}{2}$ $\qquad \frac{\partial M}{\partial P} = R$



$$q_p = \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial P} R\,d\theta + \int_0^{L} \frac{M}{EI}\frac{\partial M}{\partial P}\,dz$$

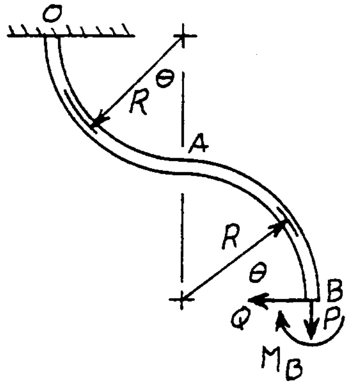
$$= \int_0^{\pi/2} \frac{PR^3}{EI}\sin^2\theta\,d\theta + \int_0^{L} \frac{PR^2}{EI}\,dz + \int_0^{L} \frac{wR}{2EI} z^2\,dz$$

$$= \underline{\frac{\pi PR^3}{4EI} + \frac{PR^2L}{EI} + \frac{wRL^3}{6EI}}$$

5.28 Let P be infinitesimal and $Q = 0$.

From B to A —— $M = M_B + PR(1-\cos\theta)$

From A to O —— $M = M_B + PR(1+\sin\theta)$



$$q_p = \int_0^{\pi/2} \frac{M_B R^2}{EI}(1-\cos\theta)\,d\theta + \int_0^{\pi/2} \frac{M_B R^2}{EI}(1+\sin\theta)\,d\theta$$

$$= \frac{M_B R^2}{EI}\left(2 - \sin\theta - \cos\theta\right)\Big|_0^{\pi/2}$$

$$= \underline{\frac{\pi R^2 M_B}{EI}}$$

5.29 Let $P = 0$ and Q be infinitesimal.

From B to A —— $M = M_B + QR\sin\theta$ $\qquad \frac{\partial M}{\partial Q} = R\sin\theta$

From A to O —— $M = M_B + QR(2-\cos\theta)$ $\qquad \frac{\partial M}{\partial Q} = R(2-\cos\theta)$

$$q_Q = \int_0^{\pi/2} \frac{M_B R^2}{EI}\sin\theta\,d\theta + \int_0^{\pi/2} \frac{M_B R^2}{EI}(2-\cos\theta)\,d\theta$$

$$= \frac{M_B R^2}{EI}\left(2\theta - \cos\theta - \sin\theta\right)\Big|_0^{\pi/2}$$

$$= \underline{\frac{\pi R^2 M_B}{EI}}$$

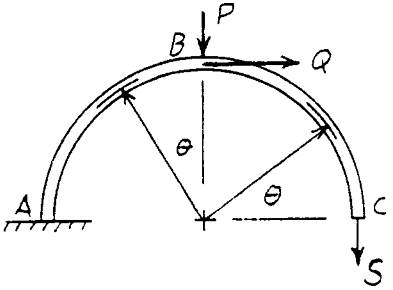
5.30 Let $P = Q = 0$.

From B to O —— $M = M_B$ $\qquad \frac{\partial M}{\partial M_B} = 1$

$$\theta_B = \int_0^{\pi/2} \frac{M_B R}{EI}\,d\theta + \int_0^{\pi/2} \frac{M_B R}{EI}\,d\theta$$

$$= \underline{\frac{\pi R M_B}{EI}}$$

5.31 Let $S=0$ and Q be an infinitesimal.



From B to A — $M = PR\sin\theta + QR(1-\cos\theta)$

$$q_P = \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial P} R\,d\theta = \int_0^{\pi/2} \frac{PR^3}{EI}\sin^2\theta\,d\theta$$

$$= \frac{\pi PR^3}{4EI}$$

$$q_Q = \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial Q} R\,d\theta = \int_0^{\pi/2} \frac{PR^3}{EI}(\sin\theta - \sin\theta\cos\theta)\,d\theta$$

$$= \underline{\frac{PR^3}{2EI}}$$

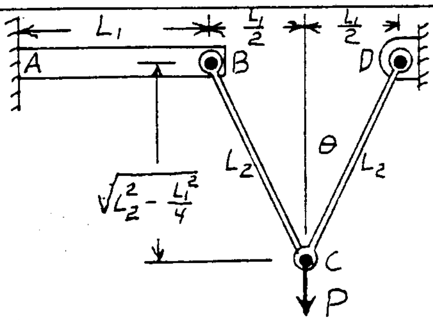
5.32 Let $Q=0$ and S be an infinitesimal.

From B to A — $M = PR\sin\theta + SR(1+\sin\theta)$ $\qquad \frac{\partial M}{\partial S} = R(1+\sin\theta)$

$$q_S = \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial S} R\,d\theta = \int_0^{\pi/2} \frac{PR^3}{EI}(\sin\theta + \sin^2\theta)\,d\theta = \frac{PR^3}{EI}\left(-\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \underline{\frac{PR^3}{EI}\left(1+\frac{\pi}{4}\right)}$$

5.33 $M_1 = \frac{P}{2}z$; $N_{CB} = N_{CD}$



$$P = 2N_{CD}\cos\theta = N_{CD}\frac{\sqrt{4L_2^2 - L_1^2}}{L_2}$$

$$\text{(a)}\ q_P = \int_0^{L_1} \frac{M_1}{E_1 I_1}\frac{\partial M_1}{\partial P}dz + 2\frac{N_{CD}L_2}{A_2E_2}\frac{\partial N_{CD}}{\partial P}$$

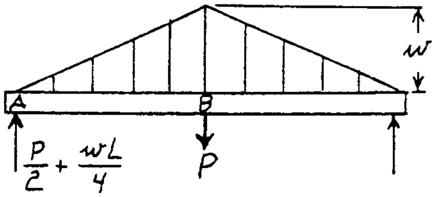
$$= \int_0^{L_1} \frac{Pz}{2E_1I_1}\left(\frac{z}{2}\right)dz + \frac{2L_2}{A_2E_2}\frac{PL_2}{\sqrt{4L_2^2-L_1^2}}\left(\frac{L_2}{\sqrt{4L_2^2-L_1^2}}\right)$$

$$= \underline{\frac{PL_1^3}{12E_1I_1} + \frac{2PL_2^3}{A_2E_2(4L_2^2-L_1^2)}}$$

$$\text{(b)}\ \frac{P(25r_1)^3(4)}{12E(\pi r_1^4)} = \frac{2P(25r_1)}{\pi r_2^2 E(3)}$$

$$\frac{r_1}{r_2} = \underline{\frac{25}{\sqrt{2}}}$$

5.34 Let P be infinitesimal.



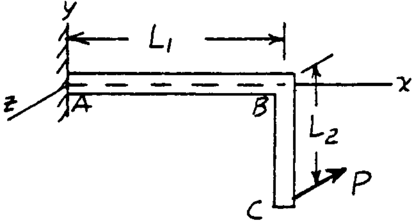
From A to B — $M=\frac{Pz}{2}+\frac{wLz}{4}-\frac{wz^3}{3L}$

$$q_P=2\int_0^{L/2}\frac{M}{EI}\frac{\partial M}{\partial P}\,dz=2\int_0^{L/2}\frac{wLz/4-wz^3/3L}{EI}\left(\frac{z}{2}\right)dz$$

$$=\frac{w}{EI}\left[\frac{Lz^3}{12}-\frac{z^5}{15L}\right]_0^{L/2}$$

$$=\underline{\frac{wL^4}{120EI}}$$

5.35 $I=\frac{\pi r^4}{4}$; $J=\frac{\pi r^4}{2}$; $G=\frac{E}{2(1+\nu)}$



From C to B — $M=Ps$ and $T=0$

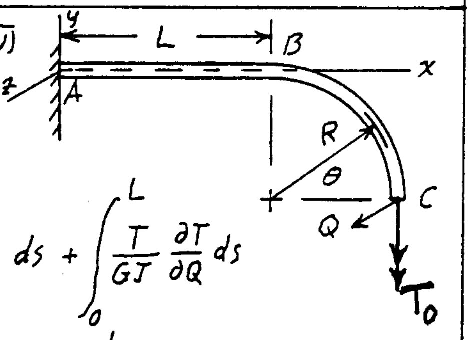
From B to A — $M=Ps$ and $T=PL_2$

$$q_P=\int_0^{L_2}\frac{M}{EI}\frac{\partial M}{\partial P}ds+\int_0^{L_1}\frac{M}{EI}\frac{\partial M}{\partial P}ds+\int_0^{L_1}\frac{T}{GJ}\frac{\partial T}{\partial P}ds$$

$$=\int_0^{L_2}\frac{Ps}{EI}(s)ds+\int_0^{L_1}\frac{Ps}{EI}(s)ds+\int_0^{L_1}\frac{PL_2}{GJ}(L_2)ds=\frac{PL_2^3}{3EI}+\frac{PL_1^3}{3EI}+\frac{PL_2^2L_1}{GJ}$$

$$=\underline{\frac{4P}{3\pi r^4E}\left[L_1^3+L_2^3+3(1+\nu)L_2^2L_1\right]}$$

5.36 Let Q be infinitesimal. $J=2I$; $G=\frac{E}{2(1+\nu)}$



From C to B — $M=T_0\sin\theta+QR\sin\theta$

$T=T_0\cos\theta-QR(1-\cos\theta)$

From B to A — $M=T_0+Q(R+s)$; $T=QR$

$$q_Q=\int_0^{\pi/2}\frac{M}{EI}\frac{\partial M}{\partial Q}R\,d\theta+\int_0^{\pi/2}\frac{T}{GJ}\frac{\partial T}{\partial Q}R\,d\theta+\int_0^{L}\frac{M}{EI}\frac{\partial M}{\partial Q}ds+\int_0^{L}\frac{T}{GJ}\frac{\partial T}{\partial Q}ds$$

$$=\int_0^{\pi/2}\frac{RT_0\sin\theta}{EI}(R\sin\theta)d\theta-\int_0^{\pi/2}\frac{RT_0\cos\theta}{GJ}\left[R(1-\cos\theta)\right]d\theta+\int_0^{L}\frac{T_0}{EI}(R+s)ds$$

$$=\frac{T_0R^2}{EI}\left(\frac{\pi}{4}\right)-\frac{T_0R^2}{GJ}\left(1-\frac{\pi}{4}\right)+\frac{T_0}{EI}\left(LR+\frac{L^2}{2}\right)$$

$$=\underline{\frac{T_0}{4EI}\left[4R^2(1+\nu)+\pi R^2\nu+4LR+2L^2+2\pi R^2\right]}$$

$$=\frac{T_0}{4EI}\left[4RL+2L^2+\pi R^2(2+\nu)-4R^2(1+\nu)\right]$$

5.37 Let $Q=0$.

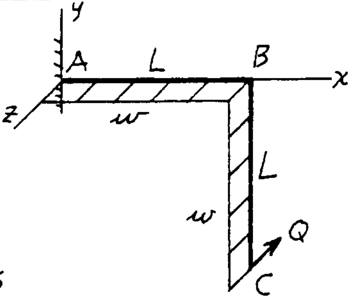
$$\theta_C = \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial T_0} R\,d\theta + \int_0^{\pi/2} \frac{T}{GJ}\frac{\partial T}{\partial T_0} R\,d\theta + \int_0^{L} \frac{M}{EI}\frac{\partial M}{\partial T_0}\,ds$$

$$= \int_0^{\pi/2} \frac{T_0 \sin\theta}{EI}(\sin\theta) R\,d\theta + \int_0^{\pi/2} \frac{T_0 \cos\theta}{GJ}(\cos\theta) R\,d\theta + \int_0^{L} \frac{T_0}{EI}(1)\,ds$$

$$= \frac{T_0 R}{EI}\left(\frac{\pi}{4}\right) + \frac{T_0 R}{GJ}\left(\frac{\pi}{4}\right) + \frac{T_0}{EI}(L)$$

$$= \underline{\frac{T_0}{4EI}\left[\pi R(2+\nu) + 4L\right]}$$

5.38 Let Q be infinitesimal. $J=2I$; $G=\frac{E}{2(1+\nu)}$



From C to B — $M = Qs + \frac{ws^2}{2}$

From B to A — $M = Qs + wLs + \frac{ws^2}{2}$

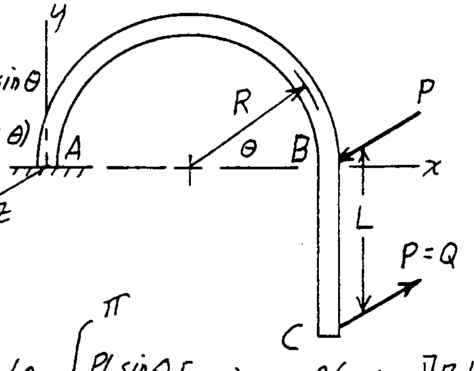
$T = QL + \frac{wL^2}{2}$

$$q_Q = \int_0^L \frac{ws^2}{2EI}(s)\,ds + \int_0^L \frac{wLs + ws^2/2}{EI}(s)\,ds + \int_0^L \frac{wL^2}{2GJ}(L)\,ds$$

$$= \frac{wL^4}{8EI} + \frac{wL^4}{3EI} + \frac{wL^4}{8EI} + \frac{wL^4}{2GJ}$$

$$= \underline{\frac{wL^4}{12EI}(13+6\nu)}$$

5.39 $J=2I$; $G=\frac{E}{2(1+\nu)}$



From C to B — $M = Qs$

From B to A — $M = Q[L\cos\theta + R\sin\theta] - PR\sin\theta$

$T = Q[L\sin\theta + R(1-\cos\theta)] - PR(1-\cos\theta)$

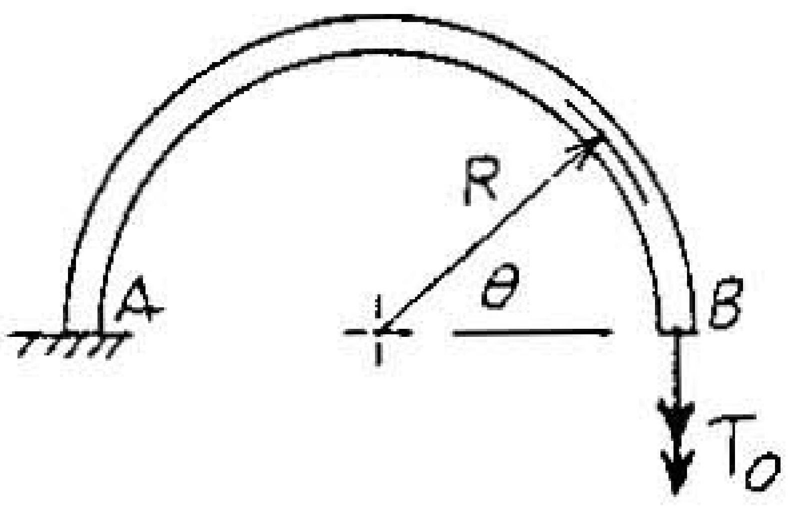
$$q_Q = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial Q}\,ds + \int_0^{\pi} \frac{M}{EI}\frac{\partial M}{\partial Q} R\,d\theta + \int_0^{\pi} \frac{T}{GJ}\frac{\partial T}{\partial Q} R\,d\theta$$

$$= \int_0^L \frac{Ps}{EI}(s)\,ds + \int_0^{\pi} \frac{PL\cos\theta}{EI}(L\cos\theta + R\sin\theta) R\,d\theta + \int_0^{\pi} \frac{PL\sin\theta}{GJ}\left[L\sin\theta + R(1-\cos\theta)\right] R\,d\theta$$

$$= \frac{PL^3}{3EI} + \frac{\pi PL^2R}{2EI} + \frac{\pi PL^2R}{2GJ} + \frac{2PLR^2}{GJ}$$

$$= \underline{\frac{P}{6EI}\left[2L^3 + 3\pi(2+\nu)L^2R + 12(1+2\nu)LR^2\right]}$$

5.40 $I = \frac{\pi r^4}{4}$; $J = \frac{\pi r^4}{2}$; $G = \frac{E}{2(1+\nu)}$



$M = T_0 \sin\theta$; $T = T_0 \cos\theta$

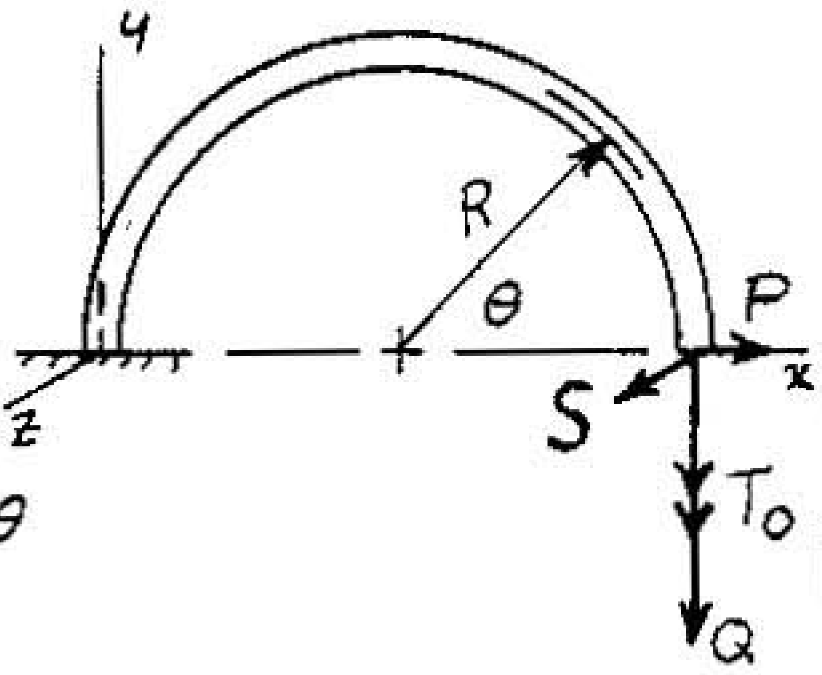
$$\theta_B = \int_0^{\pi} \frac{M}{EI}\frac{\partial M}{\partial T_0} R\,d\theta + \int_0^{\pi} \frac{T}{GJ}\frac{\partial T}{\partial T_0} R\,d\theta$$

$$= \int_0^{\pi} \frac{T_0 \sin\theta}{EI}(\sin\theta) R\,d\theta + \int_0^{\pi} \frac{T_0\cos\theta}{GJ}(\cos\theta) R\,d\theta$$

$$= \frac{\pi T_0 R}{2EI} + \frac{\pi T_0 R}{2GJ}$$

$$= \underline{\frac{2T_0 R}{Er^4}(2+\nu)}$$

5.41 Let P, Q, and S be infinitesimals.



$M = T_0 \sin\theta + SR\sin\theta$

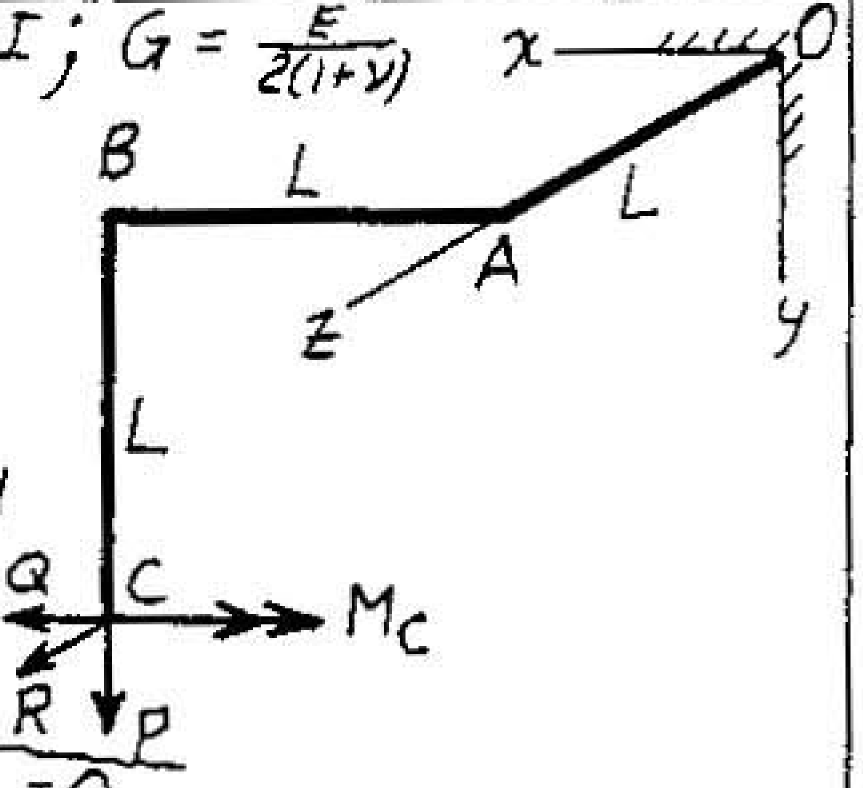
$T = T_0\cos\theta - SR(1-\cos\theta)$

$q_P = q_Q = 0$ since M and T do not contain loads P and Q.

$$q_S = \int_0^{\pi} \frac{T_0 \sin\theta}{EI}(R\sin\theta)R\,d\theta + \int_0^{\pi}\frac{T_0\cos\theta}{GJ}(-R+R\cos\theta)R\,d\theta$$

$$= \underline{\frac{T_0R^2}{EI}\left(\frac{\pi}{2}\right) + \frac{T_0R^2}{GJ}\left(\frac{\pi}{2}\right) = \frac{2T_0R^2}{Er^4}(2+\nu)}$$

5.42 Let P, Q, and R be infinitesimals. $J = 2I$; $G = \frac{E}{2(1+\nu)}$



From C to B

$M_z = -Qs$; $M_x = -M_C + Rs$; $M_y = T = 0$

From B to A

$M_z = Ps - QL$; $M_x = T = -M_C + RL$; $M_y = -Rs$

From A to O

$M_z = T = PL - QL$; $M_x = -M_C - Ps + RL$; $M_y = Qs - RL$

$$u = q_Q = \int_0^{L(C-B)} \frac{M_x}{EI}\frac{\partial M_x}{\partial Q}ds + \int_0^{L(B-A)}\frac{T}{GJ}\frac{\partial T}{\partial Q}ds + \int_0^{L(A-O)}\frac{M_x}{EI}\frac{\partial M_x}{\partial Q}ds = 0$$

$$v = q_P = \int_0^{L(C-B)} \frac{M_x}{EI}\frac{\partial M_x}{\partial P}ds + \int_0^{L(B-A)}\frac{T}{GJ}\frac{\partial T}{\partial P}ds + \int_0^{L(A-O)}\frac{M_x}{EI}\frac{\partial M_x}{\partial P}ds = \int_0^{L(A-O)}\frac{-M_C}{EI}(-s)\,ds = \frac{M_C L^2}{2EI}$$

$$w = q_R = \int_0^{L(C-B)} \frac{M_x}{EI}\frac{\partial M_y}{\partial R}ds + \int_0^{L(B-A)}\frac{T}{GJ}\frac{\partial T}{\partial R}ds + \int_0^{L(A-O)}\frac{M_x}{EI}\frac{\partial M_x}{\partial R}ds = \int_0^{L(B-A)}\frac{-M_C}{GJ}(L)\,ds + \int_0^{L(A-O)}\frac{-M_C}{EI}(L)\,ds$$

$$= \underline{-\frac{M_C L^2}{2EI}(2+\nu)}$$

5.43 Let subscript L denote upper half of the bar and subcript U denote lower half. The force P is transmitted to the free end of the bar; thus, the force P acts throoghout the bar. Hence, $P = \sigma_L A_L = \sigma_U A_U$, where σ denotes stress and A denotes cross sectional area. Similarly, for the spring $P = k x_n$, where $k = 200$ MN/m $= 200$ kN/mm, and x_n is the displacement of the top of the spring relative to the bottom of the spring. Since $A_U > A_L$, $\sigma_L = \frac{Y}{SF} = \frac{330}{1.8} = 1.833$ MPa is the design stress and $P_d = \sigma_L A_L$ is the design load $P_d = \frac{330}{1.8}(1600\pi) = 921.533$ kN ($A_L = \pi d_L^2/4 = \pi 80^2/4 = 1600\pi$ mm^2 and $A_U = \pi d_U^2/4 = \pi 120^2/4 = 3600\pi$ mm^2). The elongation of the upper half of the bar is $x_U = PL_U/(EA_U)$ and the elongation of the lower half of the bar is $x_L = PL_L/(EA_L)$. Hence, the displacement of the top of the spring is

$$x = x_n + x_U + x_L = \frac{P_d}{k} + \frac{P_d}{E}\left[\frac{L_U}{A_U} + \frac{L_L}{A_L}\right]$$

$$= \frac{921.533}{200} + \frac{921.533}{72}\left[\frac{1000}{1600\pi} + \frac{1000}{3600\pi}\right] = 4.608 + 3.678 \text{ mm}$$

or $\underline{x = 8.286 \text{ mm}}$.

Alternatively, by Eqs. (5.8) and (5.17), with $U_{spring} = \frac{1}{2} k x_n^2 = \frac{1}{2}\frac{P^2}{k}$ and $N = P$ for the upper and lower parts of the bar,

$$x = \frac{\partial}{\partial P}\left[\frac{P^2}{2k} + \frac{1}{E}\left(\frac{P^2 L_U}{2A_U} + \frac{P^2 L_L}{2A_L}\right)\right]$$

$$= \frac{P_d}{k} + \frac{1}{E}\left(\frac{P_d L_U}{A_u} + \frac{P_d L_L}{A_L}\right) = 4.608 + 3.678 = 8.286 \text{ mm}.$$

5.44 By Fig. (b). $\Sigma M_A = 0 \rightarrow BD = 1.625Q$

$\therefore$ The traction in bar BD is

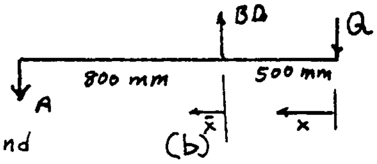
$N = 1.625Q$; $\frac{\partial N}{\partial Q} = 1.625$ (a)

The moment in BC is $M = Qx$ and

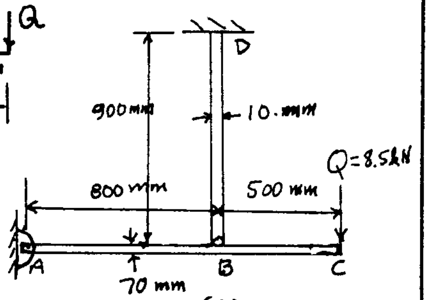
$\frac{\partial M}{\partial Q} = x$; $0 \le x \le 500$. (b)

The moment in section AB is $M = 500Q - 0.625Q\bar{x}$

and $\frac{\partial M}{\partial Q} = 500 - 0.625\bar{x}$; $0 \le \bar{x} \le 800$ (c)



(b)



(a)

(Cont.)

5.44 Continued: Therefore, the vertical deflection at C is given by

$$\delta_C = \frac{N L_{BD}}{E_{BD} A_{BD}} \frac{\partial N}{\partial Q} + \int_0^{500} \frac{Qx}{E_{AC} I_{AC}} x\,dx + \int_0^{800} \frac{500Q - 0.625Q\bar{x}}{E_{AC} I_{AC}} (500 - 0.625\bar{x})\,d\bar{x} \quad (d)$$

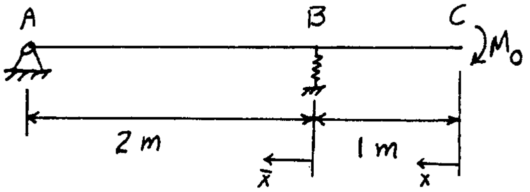
Thus, by Eqs. (a), (b), (c), (d), we have

$$\delta_C = \frac{(1.625)(8.5)(900)(1.625)}{(72)\pi(5)^2} + \frac{(8.5)(500)^3(12)}{(200)(3)(50)(70)^3} + \frac{(8.5)\left[(500)^2(800) - 2(500)(0.625)(800)^2/2 + (0.625)^2(800)^3/3\right](12)}{(200)(50)(70)^3}$$

or

$$\delta_C = 3.572 + 1.239 + 1.983 = 6.793 \text{ mm}$$

5.45 $\Sigma M_A = 0 \rightarrow B = \frac{M_0}{2}$ (a)



The moment in section BC is

$$M = M_0 \; ; \; \frac{\partial M}{\partial M_0} = 1 \, , \quad 0 \le x \le 1 \quad (b)$$

The moment in section AB is

$$M = M_0 - \frac{M_0}{2}\bar{x} \; ; \; \frac{\partial M}{\partial M_0} = 1 - \frac{\bar{x}}{2} \, , \quad 0 \le \bar{x} \le 2 \quad (c)$$

$$I = \frac{\pi}{64}(180^4 - 150^4) = 26.679 \times 10^6 \text{ mm}^4$$

Therefore, the rotation at C is given by

$$\theta_C = \int_0^1 \frac{M_0}{EI} \cdot 1 \cdot dx + \int_0^2 \frac{M_0 (1 - \frac{\bar{x}}{2})^2}{EI} dx + \frac{B}{k} \frac{\partial B}{\partial M_0} \quad (d)$$

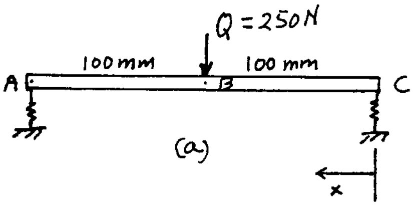
By Eqs. (a), (b), (c) and (d), we find (with $E = 200$ GPa and $I = 26.679 \times 10^6 \text{ mm}^4$)

$$\theta_C = \frac{10^6}{(200)(26.679 \times 10^6)} \left[40 + \frac{2}{3}(40)\right] + \frac{40}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2 \times 10^3}\right)$$

or

$$\theta_C = 0.0125 + 0.005 = 0.0175 \text{ rad.}$$

5.46 $I = \frac{1}{12} b h^3 = \frac{1}{12}(10)(10)^3 = 833.33 \text{ mm}^4$



Reactions at A and C are $R_A = R_C = \frac{Q}{2} = 125$ N. (a)

The moment in section BC is

$$M = \frac{Q}{2} x \; ; \; \frac{\partial M}{\partial Q} = \frac{x}{2} \, , \quad 0 \le x \le 100 \quad (b)$$

The vertical deflection at B is

$$\delta_B = 2 \int_0^{100} \frac{(\frac{Q}{2}x)(\frac{x}{2})dx}{EI} + \frac{(\frac{Q}{2})(\frac{1}{2})(2)}{k} = \frac{250(100)^3(2)}{(4)(3)(83,000)(833.33)} + \frac{(2)(250)}{(4)(30)}$$

$$= 0.602 + 4.167$$

or

$$\delta_B = 4.769 \text{ mm.}$$

5.47 Similar to Prob. 5.46, $\Sigma M_A = 0 \rightarrow R_c = \frac{3}{4}Q,\ R_A = \frac{1}{4}Q.$

$\therefore\ \frac{\partial R_c}{\partial Q} = \frac{3}{4},\ \frac{\partial R_A}{\partial Q} = \frac{1}{4}.$ The moment in section BC is

$M = \frac{3}{4}Qx,\ \frac{\partial M}{\partial Q} = \frac{3}{4}x,\ 0 \le x \le 50$, and the moment in section AB

is $M = \frac{1}{4}Q\bar{x},\ \frac{\partial M}{\partial Q} = \frac{1}{4}\bar{x},\ 0 \le \bar{x} \le 150$. Hence, the deflection at

B is:

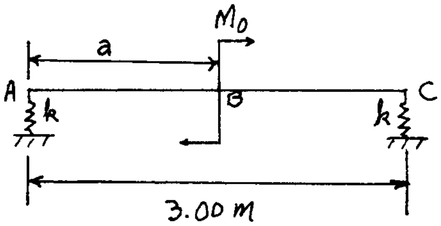
$$\delta_b = \frac{1}{(83{,}000)(833.33)}\left[\frac{(9)(250)}{(16)(3)}(50)^3 + \frac{(250)}{(16)(3)}(150)^3\right] + \frac{1}{30}\left[\frac{9}{16}(250) + \frac{1}{16}(250)\right]$$

$$= 0.339 + 5.208 = 5.547 \text{ mm.}$$

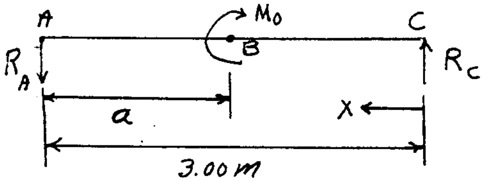
5.48 $\Sigma F = -R_A + R_c = 0;\ R_A = R_c$

$\Sigma M_A = M_0 - 3R_c = 0 \quad R_A = R_c = \frac{M_0}{3}$

$\therefore\ \frac{\partial R_A}{\partial M_0} = \frac{\partial R_c}{\partial M_0} = \frac{1}{3}.$



(a)



(b)

$I = \frac{1}{12}(80)(100)^3 - \frac{1}{12}(60)(80)^3 = 4.107 \times 10^6 \text{ mm}^4$

The moment in section BC is

$M_{BC} = \frac{M_0}{3}x;\ \frac{\partial M_{BC}}{\partial M_0} = \frac{1}{3}x,\ 0 \le x \le 3-a$

The moment in section AB is

$M_{AB} = M_0(\frac{x}{3} - 1);\ \frac{\partial M_{AB}}{\partial M_0} = \frac{x}{3} - 1,\ 3-a \le x \le 3$

For $a = 1.5$ m $= 1500$ mm,

$$\theta_B = 2\int_0^{1.5} \frac{M_0 x}{EI(3)} \cdot \frac{x}{3}\,dx + \frac{2\left(\frac{M_0}{3}\right)\left(\frac{1}{3}\right)}{k}$$

or

$$\theta_B = \frac{(2)(15)(1.5)^3}{(3)(3)(72)(4.107)(3)} + \frac{2(15)}{9(300)} = 0.01268 + 0.01111$$

$\therefore\ \theta_B = 0.02379$ rad

5.49 As in Prob. 5.48, $R_A = R_c = \frac{M_0}{3};\ \frac{\partial R_A}{\partial M_0} = \frac{\partial R_c}{\partial M_0} = \frac{1}{3}$

For $a = 2.5$, with the results of Prob. 5.48, the moment M_{BC} is

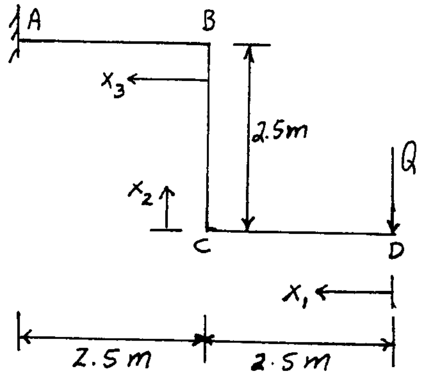
$M_{BC} = \frac{M_0}{3}x;\ \frac{\partial M_{BC}}{\partial M_0} = \frac{1}{3}x,\ 0 \le x \le 0.5$, and the moment M_{AB} is

$M_{AB} = M_0(\frac{x}{3} - 1);\ \frac{\partial M_{AB}}{\partial M_0} = \frac{x}{3} - 1,\ 0.5 \le x \le 3.0.$ Hence,

$$\theta_B = \int_0^{0.5} \frac{\left(\frac{M_0}{3}\right)x \cdot \frac{x}{3}}{EI}dx + \int_{0.5}^{3} \frac{M_0(\frac{x}{3}-1)^2}{EI}dx + \frac{2\left(\frac{M_0}{3}\right)\left(\frac{1}{3}\right)}{k} = \left.\frac{M_0 x^3}{27EI}\right|_0^{0.5} + \left.\frac{M_0}{EI}\left(\frac{x^3}{27} - \frac{x^2}{3} + x\right)\right|_{0.5}^{3.0} + \frac{2M_0}{9k}$$

$$= 0.0002348 + 0.02936 + 0.01111 = 0.04070 \text{ rad.}$$

5.50 (a) In section BC, the max. stress is



$$\sigma_{BC(max)} = \frac{Q}{A} + \frac{2.5Qc}{I_x} = \frac{Y}{2}$$

$$\therefore\ Q_{BC} = \frac{Y}{2}\ \frac{1}{\frac{1}{A} + \frac{1}{I_x/c}} = \frac{250\times10^3\ N/m^2}{2\left[\frac{10^3}{18.19} + \frac{2.5\times10^3}{2.696}\right]\frac{1}{m^2}}$$

or

$$Q_{BC} = 127.25\ kN$$

In section AB, the max stress is

$$\sigma_{AB(max)} = \frac{5.0Qc}{I_x} = \frac{Y}{2}$$

$$\frac{I_x}{c} = \frac{695\times10^6\ mm^4}{515.6/2\ mm} = 2.696\times10^{-3}\ m^3$$

$$A = Area = 18190\ mm^2 = 18.190\times10^{-3}\ m^2$$

$$\therefore\ Q_{AB} = \frac{Y}{2}\frac{I_x}{(5c)} = \frac{(250\times10^3 N/m^2)(2.696\times10^{-3} m^3)}{10\ m}$$

$$= 67.4\ kN$$

Thus, the max. allowable load is $Q = 67.4\ kN$

(b) For section CD, $M = Qx_1$, $\frac{\partial M}{\partial Q} = x_1$; $V = Q$, $\frac{\partial V}{\partial Q} = 1$; $0 \le x_1 \le 2.5m$

For section BC, $M = 2.5Q$, $\frac{\partial M}{\partial Q} = 2.5$; $N = Q$, $\frac{\partial N}{\partial Q} = 1$; $0 \le x_2 \le 2.5m$

For section AB, $M = (2.5+x_3)Q$, $\frac{\partial M}{\partial Q} = 2.5+x_3$; $V = Q$, $\frac{\partial V}{\partial Q} = 1$; $0 \le x_3 \le 2.5m$

$$\delta_D = \frac{1}{EI}\left[\int_0^{2.5} Q x_1^2 dx_1 + \int_0^{2.5}(2.5Q)2.5dx_2 + \int_0^{2.5}(2.5+x_3)Q(2.5+x_3)dx_3\right] + \int_0^{2.5}\frac{Q}{EA}\cdot 1\cdot dx_2$$

$$+ \frac{k}{GA}\int_0^{2.5} Q\cdot 1\cdot dx_1 + \frac{k}{GA}\int_0^{2.5} Q\cdot 1\cdot dx_3 \ ;\ k = 1.0 \text{ (see Table 5.1)}$$

$$= \frac{1}{(200\times10^3)(695)}\left\{\frac{67.4\times10^3}{3}(2.5)^3 + (67.4\times10^3)\left[(2.5)^3 + (2.5)^3 + (2.5)^3 + \frac{(2.5)^3}{3}\right]\right\}$$

$$+ \frac{(67.4\times10^3)(2.5)}{(200\times10^6)(18.19)} + \frac{1}{(77.5\times10^6)(18.19)}\left[(67.4\times10^3)(2.5) + (67.4\times10^3)(2.5)\right]$$

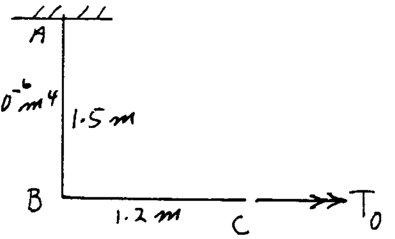
$$= \underbrace{(0.02778)}_{Bending} + \underbrace{(.0463\times10^{-3})}_{Axial} + (0.2391\times10^{-3})_{shear}$$

$$= 0.028065\ m = 28.065\ mm$$

(c) Neglecting axial energy, error is $\left(\frac{0.0463\times10^{-3}}{0.028065}\right)\times100 = 0.165\%$

Neglecting shear energy, error is $\left(\frac{0.2391\times10^{-3}}{0.028065}\right)\times100 = 0.852\%$

5.51 For circular bar, $J = \frac{\pi d^4}{32} = \frac{\pi (60)^4}{32} = 1.272 \times 10^6\ mm^4 = 1.272 \times 10^{-6}\ m^4$



For rectangular bar $I_x = \frac{1}{12} bh^3 = \frac{1}{12}(30)(70)^3$

$= 857500\ mm^4 = 0.8575 \times 10^{-6}\ m^4$

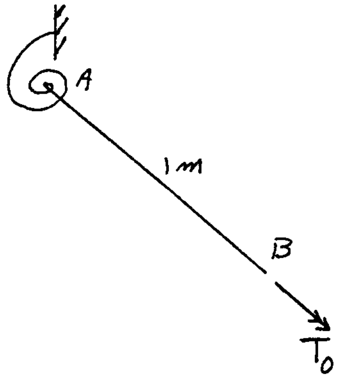
For BC: $T = T_0,\ \frac{\partial T}{\partial T_0} = 1$

For AB: $M = T_0\ \ \frac{\partial M}{\partial T_0} = 1$

$$\therefore \theta_C = \int_0^{1.5} \frac{T_0}{EI_x} dx + \int_0^{1.2} \frac{T_0}{GJ} = \frac{1.5 T_0}{EI_x} + \frac{1.2 T_0}{GJ} = \frac{(1.5)(2.50 \times 10^3)}{(200 \times 10^3)(0.8575)} + \frac{(1.2)(2.50 \times 10^3)}{(77.5 \times 10^3)(1.272)}$$

$= 0.02187 + 0.03043 = 0.0523$ rad.

5.52 $J = \frac{\pi (80)^4}{32} = 4.0212 \times 10^6\ mm^4 = 4.0212 \times 10^{-6}\ m^4$



Shaft: $T_{sh} = T_0,\ \frac{\partial T_{sh}}{\partial T_0} = 1$

Spring: $T_{sp} = T_0\ ,\ \frac{\partial T_{sp}}{\partial T_0} = 1$

$$U = \int_0^1 \frac{T_{sh}^2}{2GJ} dx + \frac{1}{2}\frac{T^2}{k};\ \therefore \theta_B = \frac{\partial U}{\partial T_0} = \int_0^1 \frac{T_0}{GJ}dx + \frac{T_0}{k}$$

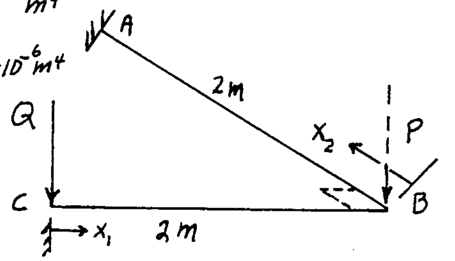
$$\theta_B = \frac{T_0 \cdot 1}{GJ} + \frac{T_0}{k} = \frac{(4.0 \times 10^3)(1)}{(27 \times 10^3)(4.0212)} + \frac{(4.0 \times 10^3)}{200 \times 10^3} = 0.03684 + 0.0200$$

or $\theta_B = 0.05684$ rad

5.53 $J_{AB} = \frac{\pi (180)^4}{32} = 103.06 \times 10^6\ mm^4 = 103.06 \times 10^{-6}\ m^4$

$I_{BC} = \frac{1}{12}(100)(180)^3 - \frac{1}{12}(60)(140)^3 = 34.88 \times 10^6\ mm^4 = 34.88 \times 10^{-6}\ m^4$

$I_{AB} = \frac{1}{2} J_{AB} = 51.53 \times 10^6\ mm^4 = 51.53 \times 10^{-6}\ m^4$



(a)

The moment in section BC is

$M_{BC} = Qx_1,\ \frac{\partial M_{BC}}{\partial Q} = x_1,\ 0 \le x_1 \le 2$

The moment in section AB is

$M_{AB} = Qx_2,\ \frac{\partial M_{AB}}{\partial Q} = x_2,\ \ 0 \le x_2 \le 2$

The twisting moment in section AB is

$T_{AB} = 2Q,\ \frac{\partial T_{AB}}{\partial Q} = 2\ ;\ \ 0 \le x_2 \le 2$

Therefore, the vertical deflection at C is

$$\delta_C = \int_0^2 \frac{Qx_1^2}{E_{BC} I_{BC}} dx_1 + \int_0^2 \frac{Qx_2^2}{E_{AB} I_{AB}} dx_2 + \int_0^2 \frac{(2Q)2}{G_{AB} J_{AB}} dx_2 = \frac{(16)(8)}{(3)(72)(34.88} + \frac{(16)(8)}{(3)(72)(51.53)} + \frac{(16)(8)}{(27)(103.06} = 0.0745 m$$

$= 74.50$ mm

(Cont.)

5.53 Continued: To compute the vertical deflection at B, we apply the force P at B (Fig. a). Then as above:

The moment in section BC is

$$M_{BC} = Qx_1 \,, \quad \frac{\partial M_{BC}}{\partial P} = 0 \quad 0 \le x_1 \le 2$$

The moment in section AB is

$$M_{AB} = (Q+P)x_2 \,, \quad \frac{\partial M_{AB}}{\partial P} = x_2 \,, \quad 0 \le x_2 \le 2$$

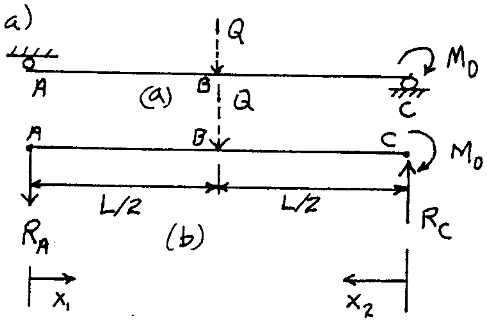
and the twisting moment is

$$T_{AB} = 2Q \,, \quad \frac{\partial T}{\partial P} = 0 \; ; \quad 0 \le x_2 \le 2 .$$

Therefore, the vertical deflection at B is

$$\delta_B = \frac{\partial U}{\partial P} = \int_0^2 \frac{(Q+P)x_2 \cdot x_2}{E_{AB} I_{AB}} \bigg|_{P=0} dx_2 = \frac{Q x_2^3}{3 E_{AB} I_{AB}} = \frac{(16)(8)}{(3)(72)(51.53)} = 0.0115 \text{ m} = 11.5 \text{ mm}.$$

5.54 Introduce the dummy load Q at B. (Fig. a)



Then, by Fig. (b), $\Sigma M_A = 0 \Rightarrow R_C = \frac{M_0}{L} + \frac{Q}{2}$, $R_A = \frac{M_0}{L} - \frac{Q}{2}$. Therefore, the bending moment in section AB is

$$M_{AB} = \left(-\frac{M_0}{L} + \frac{Q}{2}\right)x_1 \,, \quad \frac{\partial M_{AB}}{\partial Q} = \frac{x_1}{2} \; ; \quad 0 \le x_1 \le \frac{L}{2}$$

and in section BC

$$M_{BC} = -M_0 + \left(\frac{M_0}{L} + \frac{Q}{2}\right)x_2 \,, \quad \frac{\partial M_{BC}}{\partial Q} = \frac{x_2}{2}$$

Hence,

$$\delta_B = \frac{1}{EI}\left[\int_0^{L/2} \left(-\frac{M_0}{L} + \frac{Q}{2}\right)x_1 \cdot \frac{x_1}{2} \bigg|_{Q=0} dx_1 + \int_0^{L/2} \left(-M_0 + \left(\frac{M_0}{L} + \frac{Q}{2}\right)x_2\right) \frac{x_2}{2} \bigg|_{Q=0} dx_2 \right]$$

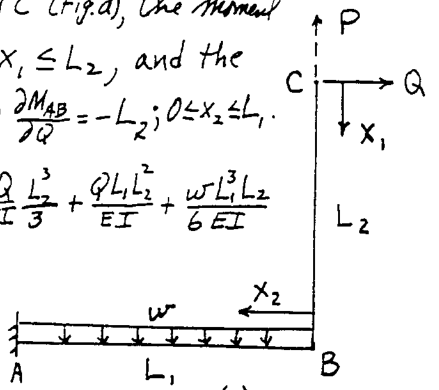
$$= \frac{1}{EI}\left[-\frac{M_0}{2L}\left(\frac{L}{2}\right)^3\left(\frac{1}{3}\right) - \frac{M_0}{2}\left(\frac{L}{2}\right)^2\left(\frac{1}{2}\right) + \frac{M_0}{2L}\left(\frac{L}{2}\right)^3\left(\frac{1}{3}\right) \right]$$

or

$$\delta_B = -\frac{M_0 L^2}{16EI}$$

5.55 (a) For the horizontal displacement of point C (Fig. a), the moment in section BC is $M_{BC} = QX_1$, $\frac{\partial M_{BC}}{\partial Q} = X_1$; $0 \le X_1 \le L_2$, and the moment in section AB is $M_{AB} = -(QL_2 + \frac{w x_2^2}{2})$, $\frac{\partial M_{AB}}{\partial Q} = -L_2$; $0 \le x_2 \le L_1$.

$$\therefore \delta_Q = \int_0^{L_2} \frac{(QX_1)(X_1)dX_1}{EI} + \int_0^{L_1} \frac{(QL_2 + \frac{wX_2^2}{2})(L_2)\,dX_2}{EI} = \frac{Q}{EI}\frac{L_2^3}{3} + \frac{QL_1L_2^2}{EI} + \frac{wL_1^3L_2}{6EI}$$



(a)

(b) For the vertical displacement of point C, we add the dummy load P. Then,

$M_{BC} = QX_1$, $\frac{\partial M_{BC}}{\partial P} = 0$; $0 \le X_1 \le L_2$, and

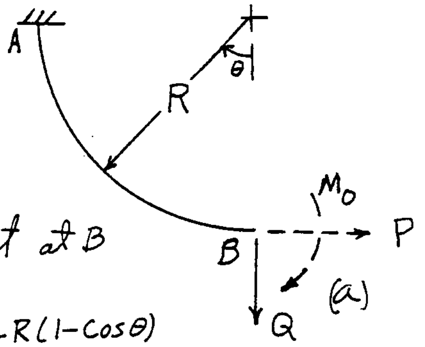
$M_{AB} = -(QL_2 + \frac{w X_2^2}{2}) + PX_2$, $\frac{\partial M_{AB}}{\partial P} = X_2$; $0 \le X_2 \le L_1$.

$$\therefore \delta_P = \int_0^{L_1} \left[\frac{-(QL_2 + \frac{wX_2^2}{2}) + PX_2}{EI}\right]\Bigg|_{P=0} X_2\,dX_2 = -\frac{QL_1^2}{2EI}\left(L_2 + \frac{wL_1^2}{4Q}\right)$$

5.56 (a) To determine the vertical displacement at B, note that the moment in the beam (due to Q) is

$M_Q = QR\sin\theta$, $\frac{\partial M_Q}{\partial Q} = R\sin\theta$; $0 \le \theta \le \frac{\pi}{2}$

$$\therefore \delta_Q = \frac{1}{EI}\int_0^{\pi/2} (QR\sin\theta)(R\sin\theta)\,Rd\theta = \frac{\pi QR^3}{4EI}$$



(a)

(b) Similarly, for the horizontal displacement at B (after introducing the dummy load P),

$M_{P+Q} = QR\sin\theta - PR(1-\cos\theta)$; $\frac{\partial M_{P+Q}}{\partial P} = -R(1-\cos\theta)$

$$\therefore \delta_P = \frac{1}{EI}\int_0^{\pi/2} [QR\sin\theta - PR(1-\cos\theta)][-R(1-\cos\theta)]\Big|_{P=0} Rd\theta = -\frac{QR^3}{2EI}$$

5.57 To determine the rotation of the section at B (Fig. a, Prob. 5.34), we add the dummy moment M_0 at B. Then, the moment in the beam is

$M_{AB} = QR\sin\theta + M_0$, $\frac{\partial M_{AB}}{\partial M_0} = 1$; $0 \le \theta \le \frac{\pi}{2}$

$$\therefore \theta_B = \frac{1}{EI}\int_0^{\pi/2} (QR\sin\theta + M_0)\Big|_{M_0=0} 1 \cdot Rd\theta = \frac{QR^2}{EI}$$

$\boxed{5.58}$ Since L is large compared to the cross-sectional dimensions, neglect shear. Therefore,

$$U = \int \frac{M^2}{2EI}\,ds \qquad (a)$$

The reaction R at C is determined by the condition

$$\frac{\partial U}{\partial R} = \int \frac{M}{EI}\frac{\partial M}{\partial R}\,ds = 0 \qquad (b)$$

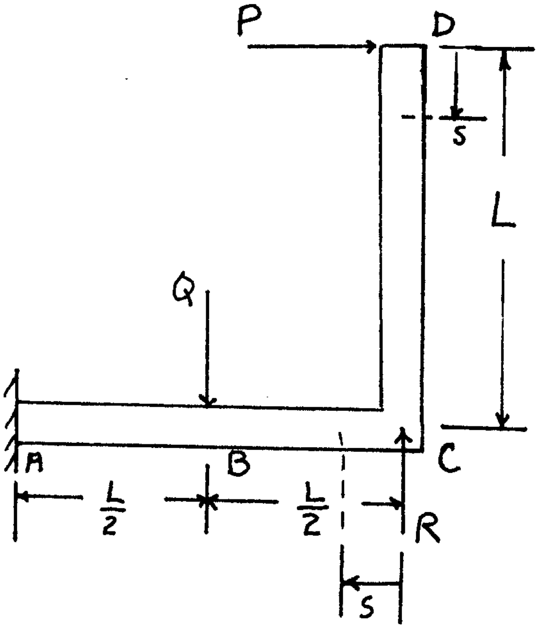


Figure a

where (Fig. a),

$$M_{DC} = -Ps, \; \frac{\partial M_{DC}}{\partial R} = 0; \; 0 \le s \le L$$

$$M_{BC} = Rs - PL, \; \frac{\partial M_{BC}}{\partial R} = s; \; 0 \le s \le \frac{L}{2} \qquad (c)$$

$$M_{AB} = Rs - Q\left(s - \frac{L}{2}\right) - PL; \; \frac{\partial M_{AB}}{\partial R} = s; \; \frac{L}{2} \le s \le L$$

Then, by Eqs. (a), (b), and (c),

$$\int_0^{L/2} \frac{M_{BC}}{EI}\,s\,ds + \int_{L/2}^{L} \frac{M_{AB}}{EI}\,s\,ds = 0$$

or

$$\int_0^{L/2} (Rs - PL)\,s\,ds + \int_{L/2}^{L} \left[Rs - Q\left(s - \frac{L}{2}\right) - PL\right] s\,ds = 0 \qquad (d)$$

Integration of Eq. (d) yields

$$R = \frac{3}{2}P + \frac{5}{16}Q$$

For $Q = P$,

$$R = 1.8125P \qquad (e)$$

The displacement δ_B of B is determined by the condition [see Eq. (a)]

$$\delta_B = \frac{\partial U}{\partial Q} = \int \frac{M}{EI}\frac{\partial M}{\partial Q}\,ds; \text{ positive downward} \qquad (f)$$

(cont.)

5.58 cont. Since Q occurs only in M_{AB} [see Eqs. (c)], $M = M_{AB}$ and $\frac{\partial M}{\partial Q} = \frac{\partial M_{AB}}{\partial Q} = \frac{L}{2} - s$. So, by Eq. (f) and Fig. a,

$$\delta_B = \frac{1}{EI}\int_{L/2}^{L}\left[Rs + Q\left(\tfrac{L}{2}-s\right) - PL\right]\left(\tfrac{L}{2}-s\right)ds$$

Integration yields

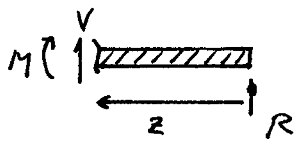
$$\delta_B = \frac{L^3}{48EI}\left(-5R + 2Q + 6P\right) \qquad (g)$$

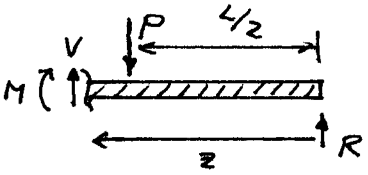
Then, by Eqs. (e) and (g), with $Q = P$,

$$\delta_B = -0.02214\,\frac{PL^3}{EI} \quad \text{(upward)}$$

5.59

(a) SELECT THE RIGHT REACTION AS THE REDUNDANT. APPLY THE REACTION AS A LOAD TO THE STATICALLY DETERMINATE BEAM. THE BENDING MOMENT IN THE TWO SECTIONS OF THE BEAM IS:

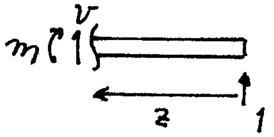




$M = Rz \quad (0 \le z < L/2)$ $\qquad$ $M = Rz - P(z - L/2) \quad (L/2 < z < L)$

FOR A UNIT LOAD APPLIED AT THE RIGHT END OF THE BEAM, THE MOMENT IS

$m = 1(z)$



(cont.)

5.59 (CONT.)

(a) (CONT.) SINCE THE DISPLACEMENT AT THE RIGHT END IS ZERO

$$0 = \int_0^{L/2} \frac{Rz}{EI}(z)\,dz + \int_{L/2}^{L} \frac{Rz - P(z - L/2)}{EI}(z)\,dz$$

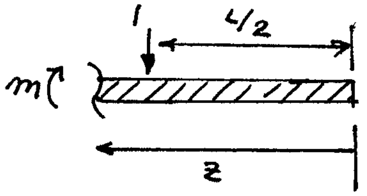
AFTER INTEGRATION, WE SOLVE FOR R TO GET

$$\underline{\underline{R = \tfrac{5}{16}P}}$$

(b) TO FIND THE DEFLECTION AT MIDSPAN, APPLY A UNIT LOAD (DOWNWARD) AT $z = L/2$. FOR $0 \le z < L/2$, $m = 0$ SO WE CONSIDER ONLY $L/2 < z \le L$, WHERE

$M = Rz - P(z - L/2)$ (FROM PART (a)) AND

$m = -(z - L/2)$



THE DEFLECTION AT THE LOAD P IS δ_P, GIVEN BY

$$\delta_P = \frac{1}{EI}\int_{L/2}^{L} \left[Rz - P(z - L/2)\right]\left[-(z - L/2)\right]dz$$

$$\delta_P = \frac{PL^3}{24EI} - \frac{5RL^3}{48EI}$$

$$\underline{\underline{\delta_P = \frac{7PL^3}{768EI}}}$$

(Cont.)

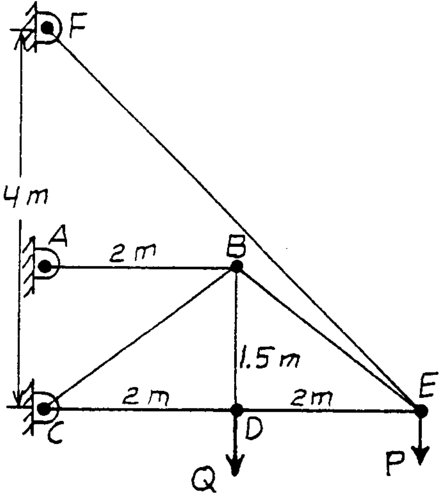
5.59 (CONT.)

(C) FOR SUPPORT SETTLEMENT OF $\frac{PL^3}{32EI}$ (WHICH IS IN THE OPPOSITE DIRECTION OF THE UNIT LOAD AT THE RIGHT END IN PART a), WE WRITE:

$$\delta_R = \frac{-PL^3}{32EI} = \int_0^{L/2} \frac{Rz}{EI}(z)\,dz + \int_{L/2}^{L} \frac{Rz - P(z - L/2)}{EI}(z)\,dz$$

$$-\frac{PL^3}{32EI} = \frac{RL^3}{3} - \frac{5PL^3}{48} \Rightarrow \underline{\underline{R = \frac{7P}{32}}}$$

5.60 Let member EF be redundant.



$$N_{AB} = \frac{4}{3}(Q+2P) - \frac{4\sqrt{2}}{3}N_{EF}$$

$$N_{BC} = -\frac{5}{3}(Q+P) + \frac{5\sqrt{2}}{6}N_{EF}$$

$$N_{BD} = Q$$

$$N_{BE} = \frac{5}{3}P - \frac{5\sqrt{2}}{6}N_{EF}$$

$$N_{CD} = N_{DE} = -\frac{4}{3}P + \frac{\sqrt{2}}{6}N_{EF}$$

$$\frac{\partial U}{\partial N_{EF}} = 0$$

$$= \frac{N_{AB}L_{AB}}{EA_{AB}}\frac{\partial N_{AB}}{\partial N_{EF}} + \frac{N_{BC}L_{BC}}{EA_{BC}}\frac{\partial N_{BC}}{\partial N_{EF}} + \frac{N_{BD}L_{BD}}{EA_{BD}}\frac{\partial N_{BD}}{\partial N_{EF}} + \frac{N_{BE}L_{BE}}{EA_{BE}}\frac{\partial N_{BE}}{\partial N_{EF}} + 2\frac{N_{CD}L_{CD}}{EA_{CD}}\frac{\partial N_{CD}}{\partial N_{EF}} + \frac{N_{EF}L_{EF}}{EA_{EF}}\frac{\partial N_{EF}}{\partial N_{EF}}$$

$$= \left[\frac{4}{3}(25{,}000) - \frac{4\sqrt{2}}{3}N_{EF}\right]\left(\frac{2000}{150}\right)\left(-\frac{4\sqrt{2}}{3}\right) + \left[-\frac{5}{3}(15{,}000) + \frac{5\sqrt{2}}{6}N_{EF}\right]\left(\frac{2500}{900}\right)\left(\frac{5\sqrt{2}}{6}\right) + 0$$

$$+ \left[\frac{5}{3}(10{,}000) - \frac{5\sqrt{2}}{6}N_{EF}\right]\left(\frac{2500}{150}\right)\left(-\frac{5\sqrt{2}}{6}\right) + 2\left[-\frac{4}{3}(10{,}000) + \frac{\sqrt{2}}{6}N_{EF}\right]\left(\frac{2000}{900}\right)\left(\frac{\sqrt{2}}{6}\right) + N_{EF}\frac{4000\sqrt{2}}{150}$$

$$112.37\,N_{EF} = 1{,}261{,}225$$

$N_{EF} = \underline{11{,}224\text{ N}}$

$N_{AB} = \underline{12{,}169\text{ N}}$

$N_{BC} = \underline{-11{,}772\text{ N}}$

$N_{BD} = \underline{5{,}000\text{ N}}$

$N_{BE} = \underline{3{,}439\text{ N}}$

$N_{CD} = N_{DE} = \underline{-10{,}688\text{ N}}$

$$q_P = \frac{N_{AB}L_{AB}}{EA_{AB}}\frac{\partial N_{AB}}{\partial P} + \frac{N_{BC}L_{BC}}{EA_{BC}}\frac{\partial N_{BC}}{\partial P} + \frac{N_{BD}L_{BD}}{EA_{BD}}\frac{\partial N_{BD}}{\partial P} + \frac{N_{BE}L_{BE}}{EA_{BE}}\frac{\partial N_{BE}}{\partial P} + 2\frac{N_{CD}L_{CD}}{EA_{CD}}\frac{\partial N_{CD}}{\partial P}$$

$$= \frac{12{,}169(2000)}{72{,}000(150)}\left(\frac{8}{3}\right) - \frac{11{,}772(2500)}{72{,}000(900)}\left(-\frac{5}{3}\right) + 0 + \frac{3439(2500)}{72{,}000(150)}\left(\frac{5}{3}\right) + \frac{2(-10{,}688)(2000)}{72{,}000(900)}\left(-\frac{4}{3}\right)$$

$$= 6.0094 + 0.7569 + 1.3268 + 0.8797$$

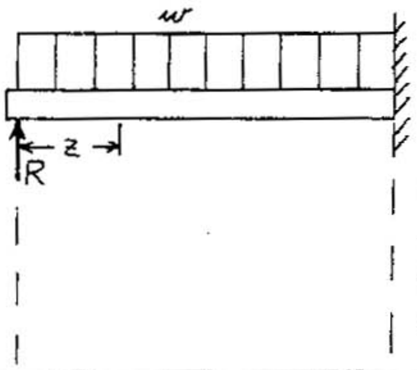
$$= \underline{8.973\text{ mm}}$$

5.61 $M = Rz - \frac{wz^2}{2}$

$$q_R = 0 = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R}\,dz = \int_0^L \frac{Rz - \frac{wz^2}{2}}{EI}(z)\,dz$$

$$0 = \frac{RL^3}{3} - \frac{wL^4}{8}$$

$$\underline{R = \frac{3}{8}wL}$$



5.62 $V = R - wz = \frac{3}{8}wL - wz$

(a) $|M_{max}| = \frac{1}{8}wL^2 = \frac{\sigma I}{c}$; $I = \frac{\pi D^4}{64} = 125{,}700\ \text{mm}^4$

$$w = \frac{8\sigma I}{cL^2} = \frac{8(140)(125{,}700)}{20(2000^2)} = 1.76\ \text{N/mm}$$

(b) $q_R = -5 = \frac{RL^3}{3EI} - \frac{wL^4}{8EI}$

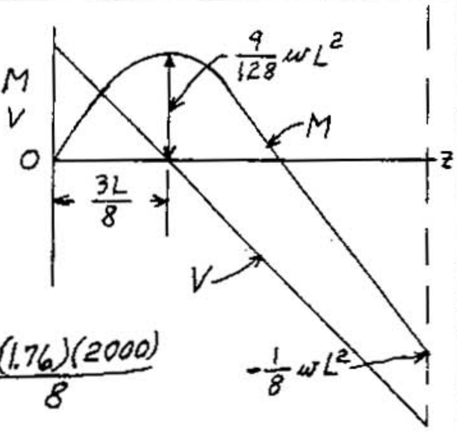
$$R = -\frac{15EI}{L^3} + \frac{3wL}{8} = -\frac{15(200{,}000)(125{,}700)}{2000^3} + \frac{3(1.76)(2000)}{8}$$

$$= 1273\ \text{N}$$

$$|M_{max}| = \frac{wL^2}{2} - RL = \frac{1.76(2000^2)}{2} - 1273(2000) = 974{,}000\ \text{N.mm}$$

$$\sigma = \frac{M_{max}c}{I} = \frac{974{,}000(20)}{125{,}700} = 155\ \text{MPa}$$

Percentage increase in stress $= \frac{155-140}{140}(100) = \underline{10.7\ \%}$



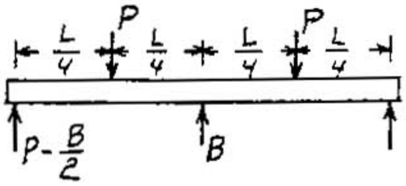
5.63 Consider half of beam.

$$q_B = 0 = 2\int_0^{L/4} \frac{M}{EI}\frac{\partial M}{\partial B}\,dz + 2\int_{L/4}^{L/2} \frac{M}{EI}\frac{\partial M}{\partial B}\,dz$$

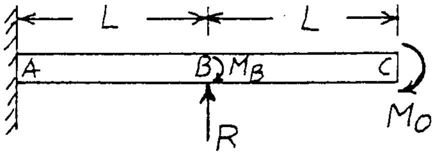
$$0 = 2\int_0^{\frac{L}{4}} \frac{Pz - \frac{Bz}{2}}{EI}\left(-\frac{z}{2}\right)dz + 2\int_{L/4}^{L/2} \frac{Pz - \frac{Bz}{2} - P\left(z - \frac{L}{4}\right)}{EI}\left(-\frac{z}{2}\right)dz$$

$$= \left(-\frac{Pz^3}{6} + \frac{Bz^3}{12}\right)_0^{\frac{L}{4}} + \left(-\frac{Pz^3}{6} + \frac{Bz^3}{6} + \frac{Pz^3}{6} - \frac{PLz^2}{16}\right)_{L/4}^{\frac{L}{2}}$$

$$\underline{B = \frac{11}{8}P}$$



5.64 Let $M_B = 0$



From B to A — $M = Rz - M_0$

$$q_R = 0 = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R}\,dz = \int_0^L \frac{Rz - M_0}{EI}(z)\,dz$$

$$0 = \frac{RL^3}{3} - \frac{M_0L^2}{2}$$

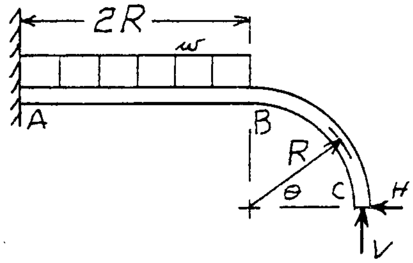
$$\underline{R = \frac{3M_0}{2L}}$$

Let M_B be infinitesimal. From B to A — $M = M_B + M_0 - Rz$

$$\theta_B = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_B}\,dz = \int_0^L \frac{M_0 - Rz}{EI}(1)\,dz = \frac{M_0L}{EI} - \frac{RL^2}{2EI}$$

$$= \underline{\frac{M_0L}{4EI}}$$

5.65



From C to B — $M = HR\sin\theta - VR(1-\cos\theta)$

From B to A — $M = HR + \frac{wz^2}{2} - V(R+z)$

$$q_H = 0 = \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial H}R\,d\theta + \int_0^{2R} \frac{M}{EI}\frac{\partial M}{\partial H}\,dz$$

$$= \int_0^{\pi/2} \frac{HR\sin\theta - VR(1-\cos\theta)}{EI}(R\sin\theta)R\,d\theta + \int_0^{2R} \frac{HR + \frac{wz^2}{2} - V(R+z)}{EI}(R)\,dz$$

$$= \frac{\pi HR^3}{4} - VR^3\left(1-\frac{1}{2}\right) + 2HR^3 + \frac{8wR^4}{6} - V(2R^3 + 2R^3)$$

$$0 = H\left(\frac{\pi}{4}+2\right) - 4.5V + \frac{4}{3}wR$$

$$q_V = 0 = \int_0^{\pi/2} \frac{HR\sin\theta - VR(1-\cos\theta)}{EI}[-R(1-\cos\theta)]R\,d\theta + \int_0^{2R} \frac{HR + \frac{wz^2}{2} - V(R+z)}{EI}[-(R+z)]\,dz$$

$$= -HR^3 + \frac{HR^3}{2} + \frac{\pi VR^3}{2} - 2VR^3 + \frac{\pi VR^3}{4} - 2HR^3 - 2HR^3 - \frac{8wR^4}{6} - \frac{16wR^4}{8} + 2VR^3 + 4VR^3 + \frac{8VR^3}{3}$$

$$0 = -4.5H + V\left(\frac{3\pi}{4} + \frac{20}{3}\right) - \frac{10wR}{3}$$

$V = \underline{0.6727\ wR}$

$H = \underline{0.6080\ wR}$

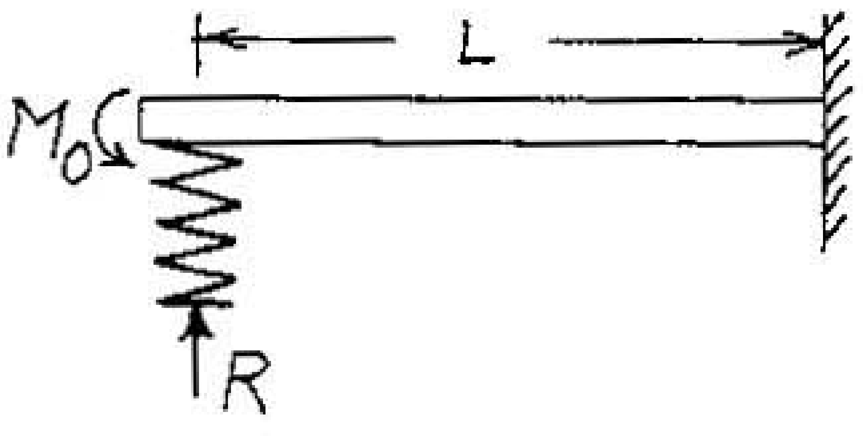
5.66 $U_{spring} = \frac{1}{2}Re = \frac{R^2}{2K}$ since $e = \frac{R}{K}$

For beam, $M = Rz - M_0$

$$q_R = 0 = \frac{\partial U_{spring}}{\partial R} + \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R}\,dz$$

$$0 = \frac{R}{K} + \int_0^L \frac{Rz - M_0}{EI}(z)\,dz = \frac{R}{K} + \frac{RL^3}{3EI} - \frac{M_0L^2}{2EI}$$

$$R = \underline{\frac{3M_0KL^2}{6EI + 2KL^3}}$$



5.67 Consider half of member due to symmetry.

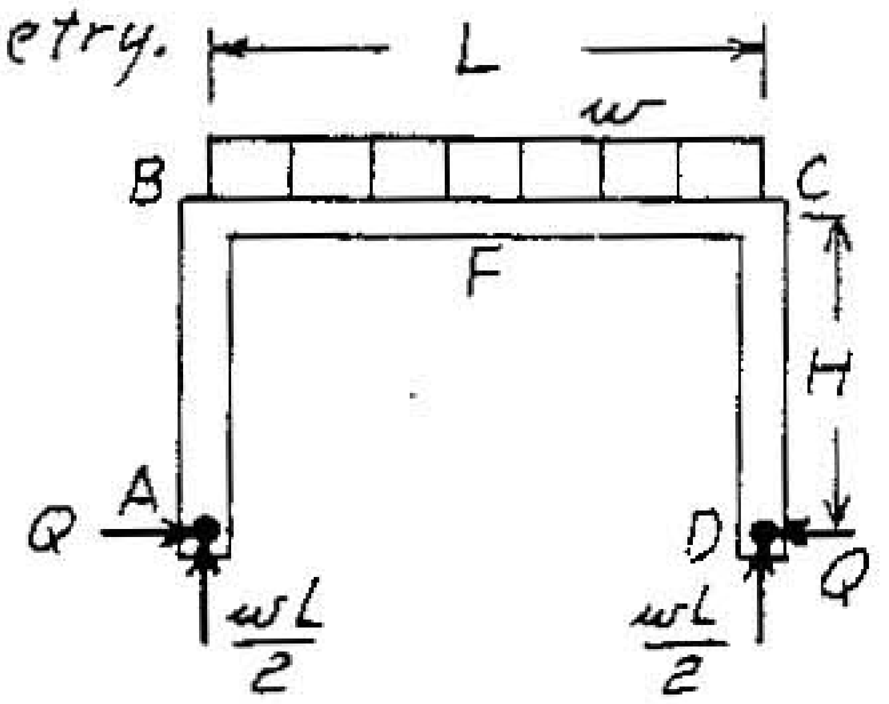
From D to C — $M = Qs$

From C to F — $M = QH - \frac{wL}{2}s + \frac{ws^2}{2}$

$$q_Q = 0 = \int_0^H \frac{Qs}{EI_1}(s)\,ds + \int_0^{L/2} \frac{QH - \frac{wL}{2}s + \frac{ws^2}{2}}{EI_2}(H)\,ds$$

$$= \frac{QH^3}{3I_1} + \frac{QH^2L}{2I_2} - \frac{wHL^3}{16I_2} + \frac{wHL^3}{48I_2}$$

$$Q = \underline{\frac{wL^3I_1}{8H^2I_2 + 12HLI_1}}$$



5.68 Consider quarter of member due to symmetry.

From C to B — $M = M_C$

From B to A — $M = M_C - \frac{PR}{2}(1-\cos\theta)$

$$\theta_C = 0 = \int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial M_C}\,ds + \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial M_C}\,R\,d\theta$$

$$0 = \int_0^{L/2} \frac{M_C}{EI}(1)\,ds + \int_0^{\pi/2} \frac{M_C - \frac{PR}{2}(1-\cos\theta)}{EI}(1)\,R\,d\theta$$

$$0 = M_C\frac{L}{2} + M_CR\frac{\pi}{2} - \frac{\pi PR^2}{4} + \frac{PR^2}{2}$$

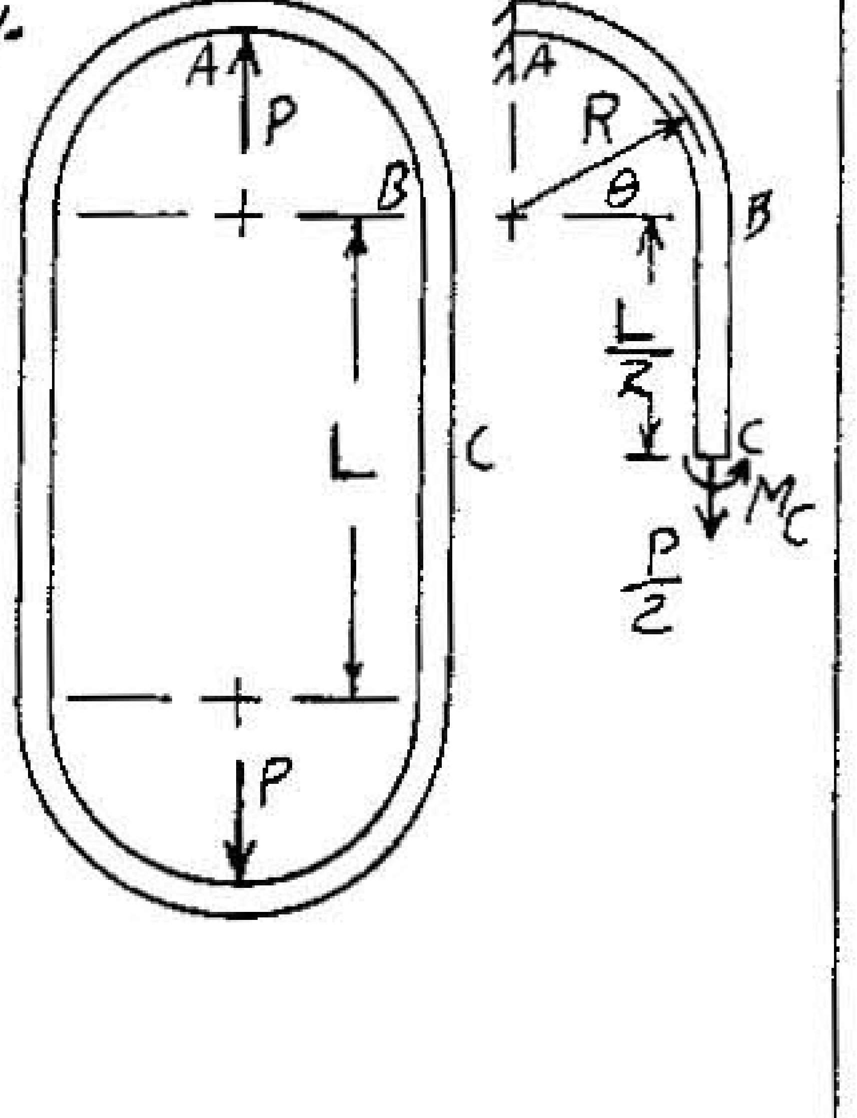
$$M_C = \underline{\frac{PR^2(\pi - 2)}{2(\pi R + L)}}$$

Check the moment at A.

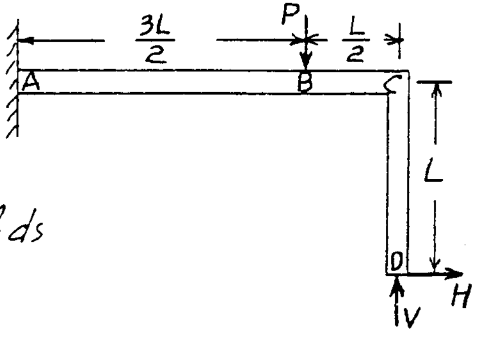
$$M_A = M_C - \frac{PR}{2} = -\frac{PR(L+2R)}{2(\pi R + L)}$$

The moment with largest absolute magnitude is $|M_A|$

$$M_{max} = \underline{\frac{PR(L+2R)}{2(\pi R + L)}}$$



5.69 Let H be infinitesimal.



From D to C — $M = Hs$

From C to B — $M = HL + Vs$

From B to A — $M = HL + V(s+\frac{L}{2}) - Ps$

$$q_V = 0 = \int_0^{L} \frac{M}{EI}\frac{\partial M}{\partial V}\,ds + \int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial V}\,ds + \int_0^{3L/2} \frac{M}{EI}\frac{\partial M}{\partial V}\,ds$$

$$0 = \int_0^{L/2} \frac{Vs}{EI}(s)\,ds + \int_0^{3L/2} \frac{V(s+\frac{L}{2}) - Ps}{EI}\left(s+\frac{L}{2}\right)ds$$

$$0 = \frac{VL^3}{24} + \frac{27VL^3}{24} + \frac{9VL^3}{8} + \frac{3VL^3}{8} - \frac{27PL^3}{24} - \frac{9PL^3}{16}$$

$$V = \frac{81P}{128} = \underline{0.6328P}$$

$$q_H = \int_0^{L/2} \frac{Vs}{EI}(L)\,ds + \int_0^{3L/2} \frac{V(s+\frac{L}{2}) - Ps}{EI}(L)\,ds = \frac{VL^3}{8EI} + \frac{9VL^3}{8EI} + \frac{3VL^3}{4EI} - \frac{9PL^3}{8EI}$$

$$= \frac{2VL^3}{EI} - \frac{9PL^3}{8EI} = \frac{2L^3}{EI}\,\frac{81P}{128} - \frac{9PL^3}{8EI} = \underline{\frac{9PL^3}{64EI}}$$

5.70 Let H be finite.

$$q_V = 0 = \int_0^{L} \frac{M}{EI}\frac{\partial M}{\partial V}\,ds + \int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial V}\,ds + \int_0^{3L/2} \frac{M}{EI}\frac{\partial M}{\partial V}\,ds$$

$$0 = \int_0^{L/2} \frac{HL+Vs}{EI}(s)\,ds + \int_0^{3L/2} \frac{HL+V(s+\frac{L}{2}) - Ps}{EI}\left(s+\frac{L}{2}\right)ds$$

$$0 = \frac{HL^3}{8} + \frac{VL^3}{24} + \frac{9HL^3}{8} + \frac{3HL^3}{4} + \frac{27VL^3}{24} + \frac{9VL^3}{8} + \frac{3VL^3}{8} - \frac{27PL^3}{24} - \frac{9PL^3}{16}$$

$$0 = 2H + \frac{8}{3}V - \frac{27}{16}P$$

$$q_H = 0 = \int_0^{L} \frac{Hs}{EI}(s)\,ds + \int_0^{L/2} \frac{HL+Vs}{EI}(L)\,ds + \int_0^{3L/2} \frac{HL+V(s+\frac{L}{2}) - Ps}{EI}(L)\,ds$$

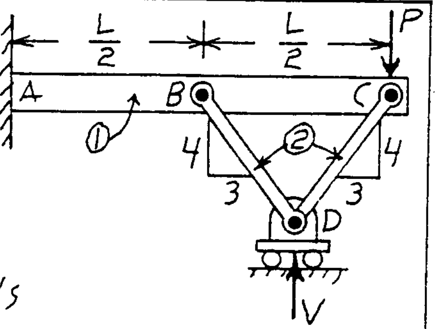
$$0 = \frac{HL^3}{3} + \frac{HL^3}{2} + \frac{VL^3}{8} + \frac{3HL^3}{2} + \frac{9VL^3}{8} + \frac{3VL^3}{4} - \frac{9PL^3}{8}$$

$$0 = \frac{7}{3}H + 2V - \frac{9}{8}P$$

$$H = \underline{-0.1688\,P}$$

$$V = \underline{0.7594\,P}$$

$\boxed{5.71}$ From free body diagram of pin at D,

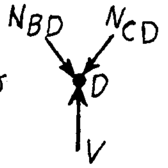


$N_{BD} = N_{CD} = N_2 = \frac{V}{1.6}$; $L_2 = \frac{5}{12}L$

From C to B —— $M = (P - \frac{V}{2})s$

From B to A —— $M = P(s+\frac{L}{2}) - V(s+\frac{L}{4})$

$$q_V = 0 = 2\frac{N_2 L_2}{A_2 E_2}\frac{\partial N_2}{\partial V} + \int_0^{L/2}\frac{M}{E_1 I_1}\frac{\partial M}{\partial V}ds + \int_0^{L/2}\frac{M}{E_1 I_1}\frac{\partial M}{\partial V}ds$$



$$0 = \frac{2(5)VL}{1.6(12)A_2E_2}\left(\frac{1}{1.6}\right) + \int_0^{L/2}\frac{Ps - \frac{Vs}{2}}{E_1I_1}\left(-\frac{s}{2}\right)ds + \int_0^{L/2}\frac{Ps + \frac{PL}{2} - Vs - \frac{VL}{4}}{E_1I_1}\left[-(s+\frac{L}{4})\right]ds$$

$$0 = \frac{10VLE_1I_1}{30.72A_2E_2} - \frac{PL^3}{48} + \frac{VL^3}{96} - \frac{PL^3}{24} - \frac{PL^3}{32} - \frac{PL^3}{16} - \frac{PL^3}{16} + \frac{VL^3}{24} + \frac{VL^3}{32} + \frac{VL^3}{32} + \frac{VL^3}{32}$$

$$0 = \frac{10VE_1I_1}{30.72L^2A_2E_2} + \frac{7V}{48} - \frac{7P}{32}$$

$$0 = \frac{10(200{,}000)(30)(40^3)V}{12(30.72)(600^2)(25^2)(10{,}000)} + \frac{7V}{48} - \frac{7P}{32}$$

$V = \underline{1.454\,P}$

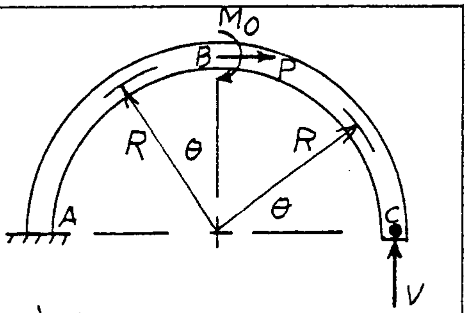
$M_{max} = M_B = \frac{0.273PL}{2} = \frac{0.273(9000)(600)}{2} = 737{,}100$ N.mm

$\sigma_{bending} = \frac{M_{max}c}{I_1} = \frac{737{,}100(20)(12)}{30(40^3)} = \underline{92.1\text{ MPa}}$

$N_2 = \frac{V}{1.6} = \frac{1.454(9000)}{1.6} = 8179$ N

$\sigma_{wood} = \frac{N_2}{A_2} = \frac{8179}{25^2} = \underline{13.1\text{ MPa}}$

5.72 Let P be infinitesimal.



From C to B — $M = VR(1-\cos\theta)$

From B to A — $M = VR(1+\sin\theta) - M_0 - PR(1-\cos\theta)$

$$q_V = 0 = \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial V} R\,d\theta + \int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial V} R\,d\theta$$

$$= \int_0^{\pi/2} \frac{VR(1-\cos\theta)}{EI}[R(1-\cos\theta)]R\,d\theta + \int_0^{\pi/2} \frac{VR(1+\sin\theta)-M_0}{EI}[R(1+\sin\theta)]R\,d\theta$$

$$0 = \frac{\pi VR^3}{2} - 2VR^3 + \frac{\pi VR^3}{4} + \frac{\pi VR^3}{4} + 2VR^3 + \frac{\pi VR^3}{4} - \frac{\pi M_0 R^2}{2} - M_0R^2$$

$$V = \frac{M_0(\pi+2)}{3\pi R} = \underline{0.5455\frac{M_0}{R}}$$

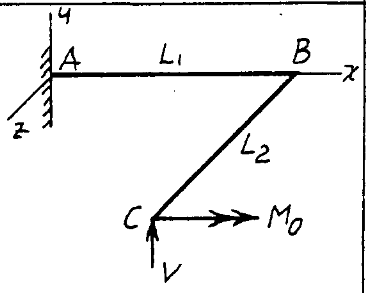
$$q_P = \int_0^{\pi/2} \frac{VR(1+\sin\theta)-M_0}{EI}[-R(1-\cos\theta)]R\,d\theta$$

$$= -\frac{VR^3}{EI}\left(\theta - \cos\theta - \sin\theta - \frac{\sin^2\theta}{2}\right)_0^{\pi/2} + \frac{M_0R^2}{EI}(\theta - \sin\theta)_0^{\pi/2}$$

$$= -\frac{VR^3}{EI}\left(\frac{\pi}{2} + 1 - 1 - \frac{1}{2}\right) + \frac{M_0R^2}{EI}\left(\frac{\pi}{2} - 1\right)$$

$$= \underline{\frac{M_0R^2}{6\pi EI}(2\pi^2 - 7\pi + 2)}$$

5.73 $J = 2I$; $G = \frac{E}{2(1+\nu)}$



From C to B — $M = Vs - M_0$

From B to A — $M = Vs$ $\quad T = VL_2 - M_0$

$$q_V = 0 = \int_0^{L_2} \frac{M}{EI}\frac{\partial M}{\partial V}ds + \int_0^{L_1} \frac{M}{EI}\frac{\partial M}{\partial V}ds + \int_0^{L_1} \frac{T}{GJ}\frac{\partial T}{\partial V}ds$$

$$= \int_0^{L_2} \frac{Vs - M_0}{EI}(s)ds + \int_0^{L_1} \frac{Vs}{EI}(s)ds + \int_0^{L_1} \frac{VL_2 - M_0}{GJ}(L_2)ds$$

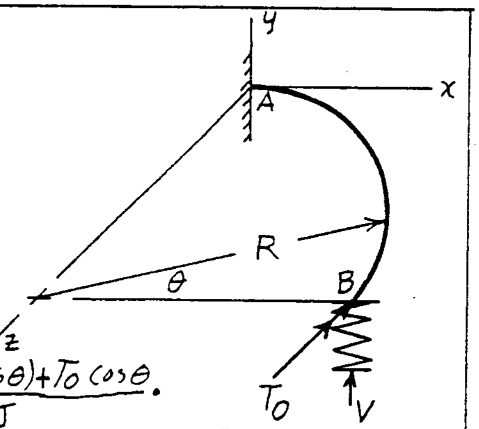
$$= \frac{VL_2^3}{3EI} - \frac{M_0L_2^2}{2EI} + \frac{VL_1^3}{3EI} + \frac{VL_1L_2^2}{GJ} - \frac{M_0L_1L_2}{GJ}$$

$$V = \underline{\frac{M_0[3L_2^2 + 6L_1L_2(1+\nu)]}{2L_1^3 + 2L_2^3 + 6L_1L_2(1+\nu)}}$$

5.74 $U_{spring} = \frac{1}{2}Ve = \frac{V^2}{2K}$; $J = 2I$; $G = \frac{E}{2(1+\nu)}$

$M = VR\sin\theta - T_0\sin\theta$; $\frac{\partial M}{\partial V} = R\sin\theta$

$T = VR(1-\cos\theta) + T_0\cos\theta$; $\frac{\partial T}{\partial V} = R(1-\cos\theta)$



$$q_V = 0 = \frac{\partial}{\partial V}\left(\frac{V^2}{2K}\right) + \int_0^{\pi/2}\frac{M}{EI}\frac{\partial M}{\partial V}R\,d\theta + \int_0^{\pi/2}\frac{T}{GJ}\frac{\partial T}{\partial V}R\,d\theta$$

$$0 = \frac{V}{K} + \int_0^{\pi/2}\frac{VR\sin\theta - T_0\quad\theta}{EI}(R\sin\theta)R\,d\theta + \int_0^{\pi/2}\frac{VR(1-\cos\theta)+T_0\cos\theta}{GJ}\cdot[R(1-\cos\theta)]R\,d\theta$$

$$= \frac{V}{K} + \frac{\pi VR^3}{4EI} - \frac{T_0R^2}{2EI} + \frac{\pi VR^3}{2GJ} - \frac{2VR^3}{GJ} + \frac{\pi VR^3}{4GJ} - \frac{T_0R^2}{GJ} + \frac{\pi T_0R^2}{4GJ}$$

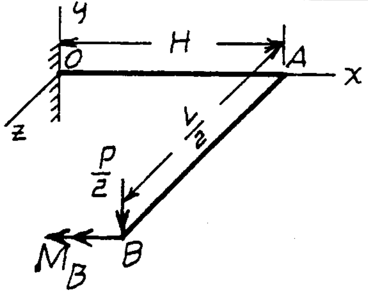
$$\underline{V = \frac{T_0[\pi(2+\nu) - 4(1+\nu)]}{\frac{4EI}{KR^2} + \pi R(4+3\nu) - 8R(1+\nu)}}$$

5.75 Due to symmetry, consider half of the member. $J = 2I$; $G = \frac{E}{2(1+\nu)}$

From B to A — $M = M_B - \frac{Ps}{2}$

From A to O — $M = \frac{Ps}{2}$

$T = M_B - \frac{PL}{4}$



$$\theta_B = 0 = \int_0^{L/2}\frac{M}{EI}\frac{\partial M}{\partial M_B}ds + \int_0^{H}\frac{M}{EI}\frac{\partial M}{\partial M_B}ds + \int_0^{H}\frac{T}{GJ}\frac{\partial T}{\partial M_B}ds$$

$$= \int_0^{L/2}\frac{M_B - \frac{Ps}{2}}{EI}(1)\,ds + 0 + \int_0^{H}\frac{M_B - \frac{PL}{4}}{GJ}(1)\,ds$$

$$= \frac{M_BL}{2EI} - \frac{PL^2}{16EI} + \frac{M_BH}{GJ} - \frac{PLH}{4GJ}$$

$$\underline{M_B = \frac{P[L^2 + 4LH(1+\nu)]}{8L + 16H(1+\nu)}}$$

$$M_{Oz} = M_O = \underline{\frac{PH}{2}}$$

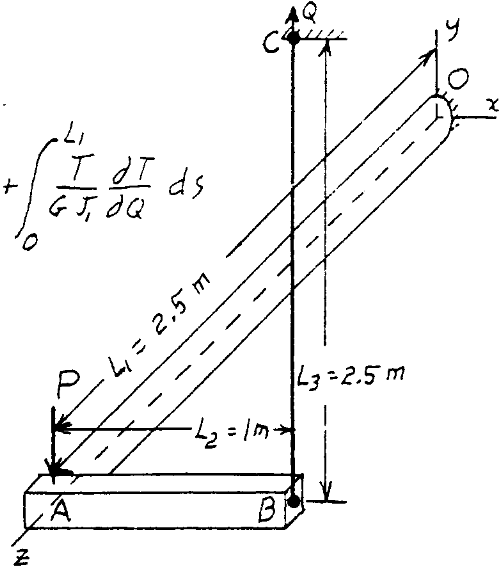
$$M_{Ox} = T_O = -M_B + \frac{PL}{4}$$

$$= \underline{\frac{PL^2}{8L + 16H(1+\nu)}}$$

5.76 $J_1 = 2I_1$; $G = \frac{E}{2(1+\nu)}$; $N_3 = Q$

From B to A —— $M = Qs$

From A to O —— $M = (Q-P)s$; $T = QL_2$



$$q_Q = 0 = \frac{\partial}{\partial Q}\left(\frac{N_3^2 L_3}{2EA_3}\right) + \int_0^{L_2} \frac{M}{EI_2}\frac{\partial M}{\partial Q} ds + \int_0^{L_1} \frac{M}{EI_1}\frac{\partial M}{\partial Q} ds + \int_0^{L_1} \frac{T}{GJ_1}\frac{\partial T}{\partial Q} ds$$

$$0 = \frac{QL_3}{EA_3} + \int_0^{L_2} \frac{Qs}{EI_2}(s) ds + \int_0^{L_1} \frac{(Q-P)s}{EI_1}(s) ds + \int_0^{L_1} \frac{QL_2}{GJ_1}(L_2) ds$$

$$= \frac{QL_3}{EA_3} + \frac{QL_2^3}{3EI_2} + \frac{QL_1^3}{3EI_1} - \frac{PL_1^3}{3EI_1} + \frac{2(1+\nu)QL_1L_2^2}{2EI_1}$$

$$P = Q\left[1 + \frac{L_2^3 I_1}{L_1^3 I_2} + \frac{3L_2^2(1+\nu)}{L_1^2} + \frac{3I_1L_3}{A_3L_1^3}\right]$$

$$= Q\left[1 + \frac{(1000^3)(\pi)(100^4)(12)}{64(2500^3)(50)(75^3)} + \frac{3(1000^2)(1+0.29)}{(2500^2)} + \frac{3\pi(100^4)(2500)(4)}{64(\pi)(7.5^2)(2500^3)}\right] = 1.8513\,Q$$

$Q = 0.5402\,P$

Assume member No. 1 yields first.

$$\sigma_1 = SF\frac{Mc_1}{I_1} = \frac{2.00(0.4598P)(2500)(50)(64)}{\pi(100^4)} = 0.02342\,P$$

$$\tau_1 = SF\frac{Tc_1}{J_1} = \frac{2.00(0.5402P)(1000)(50)(32)}{\pi(100^4)} = 0.00550\,P$$

$$\tau_{1(max)} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{Y}{2} = \frac{\sigma_1}{4} + \frac{1}{2}\sqrt{\left(\frac{\sigma_1}{2}\right)^2 + \tau_1^2} = 0.012325\,P = \frac{420}{2}$$

$P = 17.04$ kN

Assume member No 2 yields first.

$$\sigma_2 = SF\frac{Mc_2}{I_2} = \frac{2.00(0.5402P)(1000)(37.5)(12)}{50(75^3)} = 0.02305P = Y = 420$$

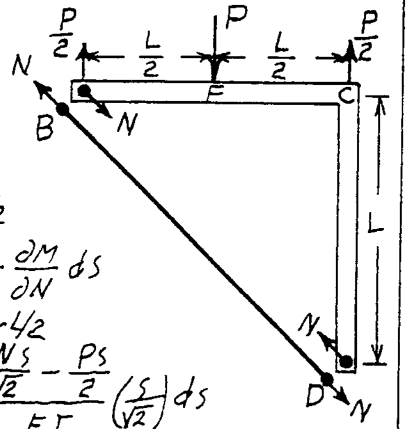
$P = 18.22$ kN

Assume member No 3 yields first.

$$\sigma_3 = SF\frac{Q}{A_3} = \frac{2.00(0.5402P)(4)}{\pi(7.5^2)} = 0.02446\,P = Y = 420;\quad P = 17.17 \text{ kN}$$

Member No. 1 yields first and $P = \underline{17.04 \text{ kN}}$

5.77



From D to C —— $M = \frac{1}{\sqrt{2}} Ns$

From C to F —— $M = \frac{1}{\sqrt{2}} N(L-s) - \frac{1}{2} Ps$

From F to B —— $M = \frac{1}{\sqrt{2}} Ns - \frac{1}{2} Ps$

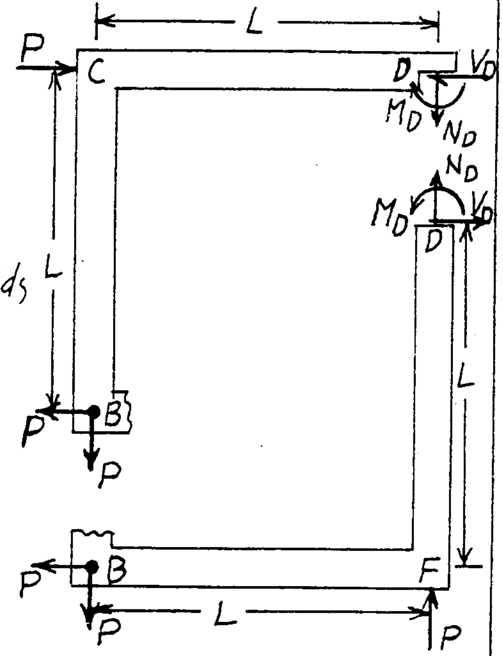
$$\frac{\partial U}{\partial N} = 0 = \frac{\partial}{\partial N}\left(\frac{N^2 L_{BD}}{2A_{BD}E}\right) + \int_0^L \frac{M}{EI}\frac{\partial M}{\partial N} ds + \int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial N} ds + \int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial N} ds$$

$$0 = \frac{NL\sqrt{2}}{AE} + \int_0^L \frac{Ns}{\sqrt{2}EI}\left(\frac{s}{\sqrt{2}}\right) ds + \int_0^{L/2} \frac{\frac{1}{\sqrt{2}}N(L-s) - \frac{1}{2}Ps}{EI}\left(\frac{L-s}{\sqrt{2}}\right) ds + \int_0^{L/2} \frac{\frac{Ns}{\sqrt{2}} - \frac{Ps}{2}}{EI}\left(\frac{s}{\sqrt{2}}\right) ds$$

$$= \frac{\sqrt{2}NL}{AE} + \frac{NL^3}{6EI} + \frac{NL^3}{4EI} - \frac{NL^3}{8EI} + \frac{NL^3}{48EI} - \frac{PL^3}{16\sqrt{2}EI} + \frac{PL^3}{48\sqrt{2}EI} + \frac{NL^3}{48EI} - \frac{PL^3}{48\sqrt{2}EI}$$

$$N = \underline{\frac{3PAL^2}{96I + 16\sqrt{2}AL^2}}$$

5.78



From D to C —— $M = M_D + N_D s$

From C to B —— $M = M_D + N_D L + Ps - V_D s$

From D to F —— $M = M_D - V_D s$

From F to B —— $M = M_D - V_D L + N_D s + Ps$

$$EI\frac{\partial U}{\partial N_D} = 0 = \int_0^L (M_D + N_D s)(s)\, ds + \int_0^L (M_D + N_D L + Ps - V_D s)(L)\, ds$$

$$+ \int_0^L (M_D - V_D L + N_D s + Ps)(s)\, ds$$

$$0 = 12M_D + 10N_D L - 6V_D L + 5PL$$

$$EI\frac{\partial U}{\partial V_D} = 0 = \int_0^L (M_D + N_D L + Ps - V_D s)(-s)\, ds$$

$$+ \int_0^L (M_D - V_D s)(-s)\, ds + \int_0^L (M_D - V_D L + N_D s + Ps)(-L)\, ds$$

$$0 = 12M_D + 6N_D L - 10V_D L + 5PL$$

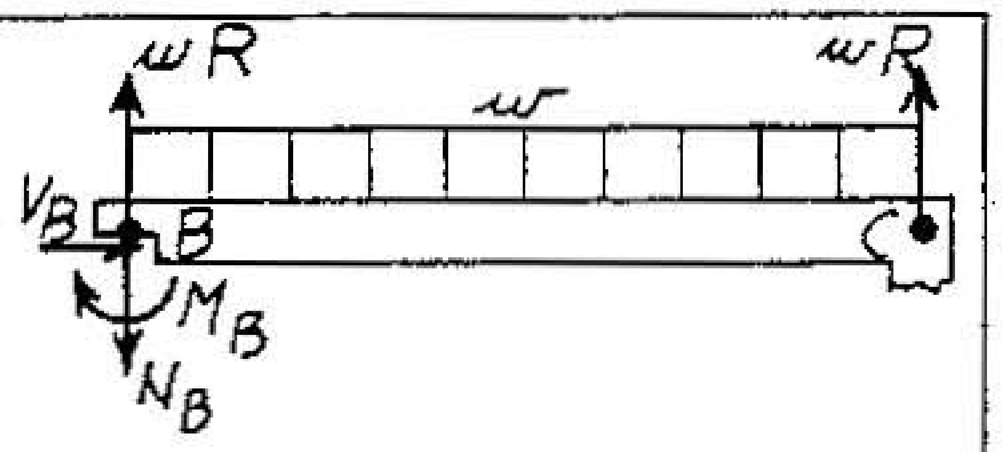
$$EI\frac{\partial U}{\partial M_D} = 0 = \int_0^L (M_D + N_D s)(1)\, ds + \int_0^L (M_D + N_D L + Ps - V_D s)(1)\, ds + \int_0^L (M_D - V_D s)(1)\, ds$$

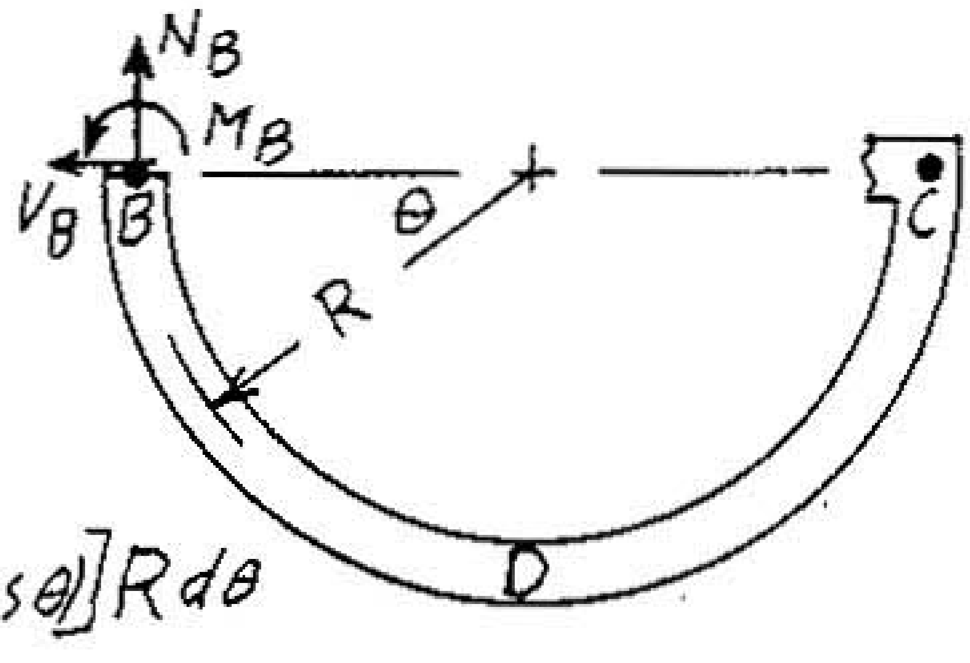
$$+ \int_0^L (M_D - V_D L + N_D s + Ps)(1)\, ds$$

$$0 = 4M_D + 2N_D L - 2V_D L + PL$$

$N_D = \underline{-\frac{1}{2}P}$; $V_D = \underline{\frac{1}{2}P}$; $M_D = \underline{\frac{1}{4}PL}$

5.79





From B to C

$$M = M_B + wRs - N_B s - \frac{ws^2}{2}$$

From B to D to C

$$M = M_B + V_B R \sin\theta - N_B R(1-\cos\theta)$$

$$\frac{\partial U}{\partial N_B} = 0 = \int_0^{2R} \frac{M_B + wRs - N_B s - \frac{ws^2}{2}}{EI}(-s)\,ds + \int_0^{\pi} \frac{M_B + V_B R\sin\theta - N_B R(1-\cos\theta)}{EI}\left[-R(1-\cos\theta)\right]R\,d\theta$$

$$0 = M_B(2+\pi) - \frac{N_B R}{6}(16+9\pi) + 2V_B R + \frac{2}{3}wR^2$$

$$\frac{\partial U}{\partial V_B} = 0 = \int_0^{\pi} \frac{M_B + V_B R\sin\theta - N_B R(1-\cos\theta)}{EI}(R\sin\theta)R\,d\theta$$

$$0 = 2M_B - 2N_B R + \frac{1}{2}V_B R\pi$$

$$\frac{\partial U}{\partial M_B} = 0 = \int_0^{2R} \frac{M_B + wRs - N_B s - \frac{ws^2}{2}}{EI}(1)\,ds + \int_0^{\pi} \frac{M_B + V_B R\sin\theta - N_B R(1-\cos\theta)}{EI}(1)R\,d\theta$$

$$0 = M_B(2+\pi) - N_B R(2+\pi) + 2V_B R + \frac{2}{3}wR^2$$

$$N_B = \underline{0}$$

$$V_B = \frac{8wR}{3(\pi^2+2\pi-8)} = \underline{0.3271\ wR}$$

$$M_B = -\frac{2\pi wR^2}{3(\pi^2+2\pi-8)} = \underline{-0.2569\ wR^2}$$

5.80 This problem is similar to Example 5.18, with 3 springs. Because of symmetry, there are only two unknown spring forces $A = C$, and B. Let the load carried by spring B be the redundant. Then by equilibrium $A = C = \frac{1}{2}(wL - B)$, and $\frac{\partial A}{\partial B} = \frac{\partial C}{\partial B} = -\frac{1}{2}$; $\frac{\partial B}{\partial B} = 1$.

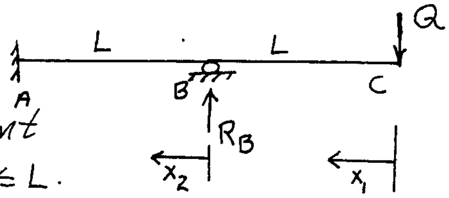
The bending moment in section AB is $M_{AB} = \frac{1}{2}(wL - B)x - \frac{wx^2}{2}$, $0 < x < \frac{L}{2}$, and $\frac{\partial M_{AB}}{\partial B} = -\frac{x}{2}$. Therefore, using symmetry, we have

$$\frac{\partial U}{\partial B} = \frac{2}{EI}\int_0^{L/2}\left[\frac{1}{2}(wL - B)x - \frac{wx^2}{2}\right]\left(-\frac{x}{2}\right)dx + 2\left[\left(\frac{wL-B}{2k}\right)\left(-\frac{1}{2}\right)\right] + \frac{B(1)}{k} = 0$$

Integrating and solving for B, we find.

$$B = \frac{\frac{wL}{2k} + \frac{5wL^4}{384EI}}{\frac{3}{2k} + \frac{L^3}{48EI}}, \text{ and } A = C = \frac{1}{2}(wL - B) = \frac{\frac{wL}{k} + \frac{wL^4}{128EI}}{\frac{3}{k} + \frac{L^3}{24EI}}$$

5.81 Let the force at the roller be the redundant.



In section BC, the bending moment is $M_{BC} = Qx_1$, $\frac{\partial M_{BC}}{\partial R_B} = 0$; $0 \le x_1 \le L$.

Likewise, the moment in section AB is

$$M_{AB} = QL + (Q - R_B)x_2, \quad \frac{\partial M_{AB}}{\partial R_B} = -x_2; \quad 0 \le x_2 \le L$$

Hence, $\frac{\partial U}{\partial R_B} = \frac{1}{EI}\int_0^L [QL + (Q - R_B)x_2](-x_2)\,dx_2 = 0$

$$\therefore \quad -\frac{QL(L^2)}{2} - \frac{(Q - R_B)L^3}{3} = 0 \quad \text{or} \quad 5Q - 2R_B = 0; \; R_B = \frac{5Q}{2}.$$

5.82 Let the spring force P be the redundant. Then the moment at any section distant x from the spring is $M = Px - \frac{wx^2}{2}$, $0 < x < L$, and $\frac{\partial M}{\partial P} = x$.

Hence, $\frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^L \left(Px - \frac{wx^2}{2}\right)(x)\,dx + \frac{P(1)}{k} = 0$

or

$$\frac{1}{EI}\left(\frac{PL^3}{3} - \frac{wL^4}{8}\right) + \frac{P}{k} = 0 \quad \therefore \quad P = \frac{\frac{wL^4}{8EI}}{\frac{1}{k} + \frac{L^3}{3EI}}$$

5.83 Let the vertical reaction P at the roller support be the redundant. Then, the moment at any section θ is

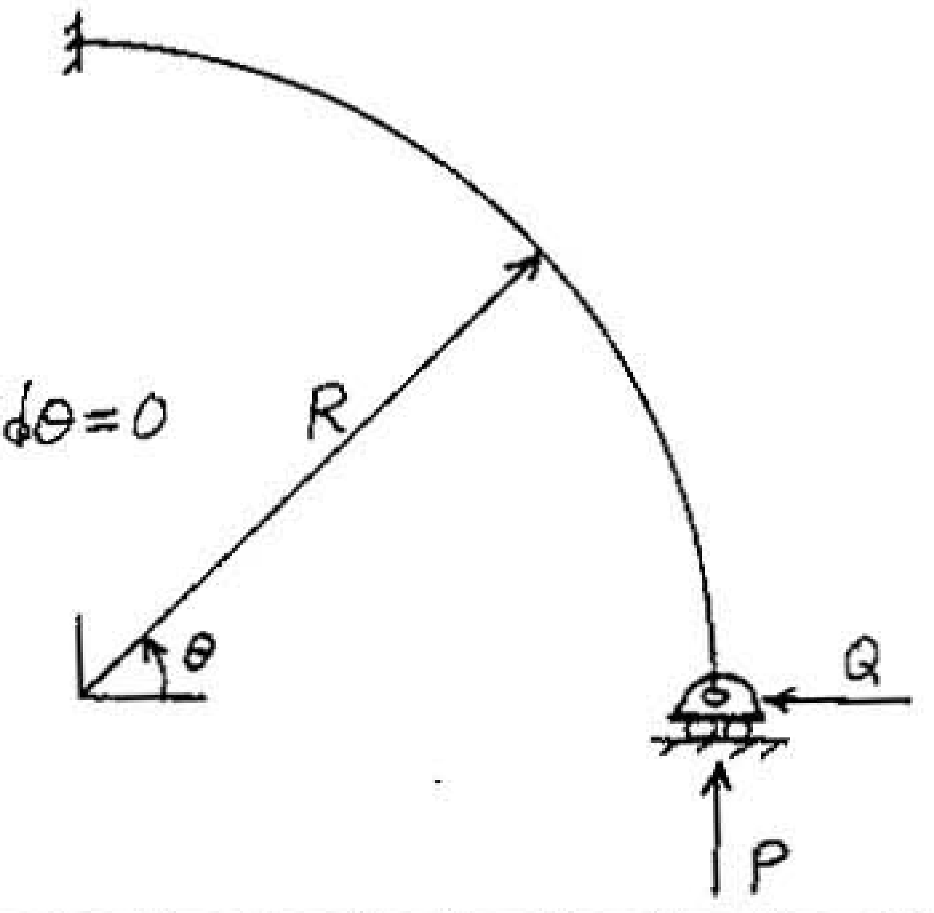
$M = PR(1-\cos\theta) - QR\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}$, and

$\frac{\partial M}{\partial P} = R(1-\cos\theta)$. Hence,

$$EI\frac{\partial U}{\partial P} = \int_0^{\pi/2} [PR(1-\cos\theta) - QR\sin\theta][R(1-\cos\theta)]R\,d\theta = 0$$

$$\therefore \quad PR^3\left(\frac{3\pi}{4}-2\right) = \frac{1}{2}Q$$

or $$P = \frac{2Q}{3\pi-8}$$



5.84 Let the reaction at B, R_B, be redundant. Then the reactions at A and C, respectively, are $R_A = \frac{wL}{2} - \frac{L_2}{L}R_B$ and $R_C = \frac{wL}{2} - \frac{L_1}{L}R_B$, where $L = L_1 + L_2$. The bending moment in section AB is $M_{AB} = R_A x_1 - \frac{wx_1^2}{2}$, $0 \le x_1 \le L_1$, and $\frac{\partial M_{AB}}{\partial R_B} = -\frac{L_2}{L}x_1$.

Similarly, the moment in section BC is

$$M_{BC} = R_C x_2 - \frac{wx_2^2}{2}, \quad 0 \le x_2 \le L_2, \text{ and } \frac{\partial M_{BC}}{\partial R_B} = -\frac{L_1}{L}x_2$$

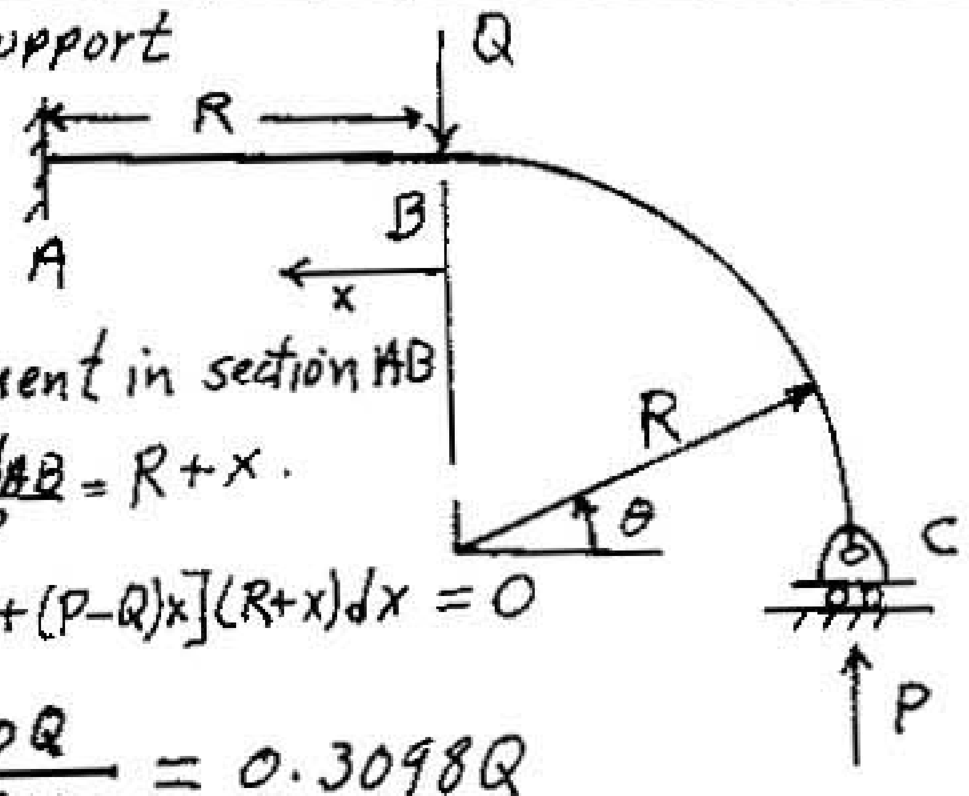
Therefore,

$$\frac{\partial U}{\partial R_B} = \frac{1}{EI}\left[\int_0^{L_1}\left(R_A x_1 - \frac{wx_1^2}{2}\right)\left(-\frac{L_2}{L}x_1\right)dx_1 + \int_0^{L_2}\left(R_C x_2 - \frac{wx_2^2}{2}\right)\left(-\frac{L_1}{L}x_2\right)dx_2\right] = 0$$

$$\therefore \quad \left(\frac{1}{3}R_A L_1^3 - \frac{wL_1^4}{8}\right)L_2 + \left(\frac{1}{3}R_C L_2^3 - \frac{wL_2^4}{8}\right)L_1 = 0$$

and $$R_B = \frac{w(L_1^3 + L_2^3 + 4L_1^2L_2 + 4L_1L_2^2)}{8L_1L_2} = \frac{wL}{8L_1L_2}(L_1^2 + 3L_1L_2 + L_2^2)$$

5.85 Let the reaction P at the roller support be the redundant. Then, the moment in section BC is



$M_{BC} = PR(1-\cos\theta), \quad 0 \le \theta \le \frac{\pi}{2}$, and $\frac{\partial M_{BC}}{\partial P} = R(1-\cos\theta)$. Similarly, the moment in section AB is $M_{AB} = PR + (P-Q)x$, $0 \le x \le R$, and $\frac{\partial M_{AB}}{\partial P} = R + x$.

$$\therefore \frac{\partial U}{\partial P} = \frac{1}{EI}\left[\int_0^{\pi/2} PR^2(1-\cos\theta)^2 R\,d\theta + \int_0^R [PR + (P-Q)x](R+x)\,dx\right] = 0$$

$$\therefore PR^3\left(\frac{3\pi}{4}+\frac{1}{3}\right) - \frac{5QR^3}{6} = 0 \quad \text{or} \quad P = \frac{10Q}{9\pi+4} = 0.3098Q$$

5.86 From Prob. 5.81, we have $M_{BC} = Qx_1$, $\frac{\partial M_{BC}}{\partial Q} = x_1$; $0 \le x_1 \le L$.

Also $M_{AB} = QL + (Q - R_B)x_2 = QL - \frac{3}{2}Qx_2$, $\frac{\partial M_{AB}}{\partial Q} = L - \frac{3}{2}x_2$; $0 \le x_2 \le L$.

$$\therefore \ \delta_C = \frac{1}{EI}\int_0^L (Qx_1)(x_1)\,dx_1 + \frac{1}{EI}\int_0^L \left[Q\left(L - \tfrac{3}{2}x_2\right)\right]\left(L - \tfrac{3}{2}x_2\right)dx_2$$

or

$$\delta_C = \frac{QL^3}{3EI} + \frac{QL^3}{4EI} = \frac{7QL^3}{12EI}$$

5.87 From Prob. 5.83, we have $M = PR(1-\cos\theta) - QR\sin\theta = \frac{2QR(1-\cos\theta)}{3\pi - 8} - QR\sin\theta$

$$\therefore \ \frac{\partial M}{\partial Q} = \frac{2R(1-\cos\theta)}{3\pi - 8} - R\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}. \text{ Therefore,}$$

$$\delta_Q = \frac{1}{EI}\int_0^{\pi/2} Q\left[\frac{2R(1-\cos\theta)}{3\pi-8} - R\sin\theta\right]\left[\frac{2R(1-\cos\theta)}{3\pi-8} - R\sin\theta\right]R\,d\theta$$

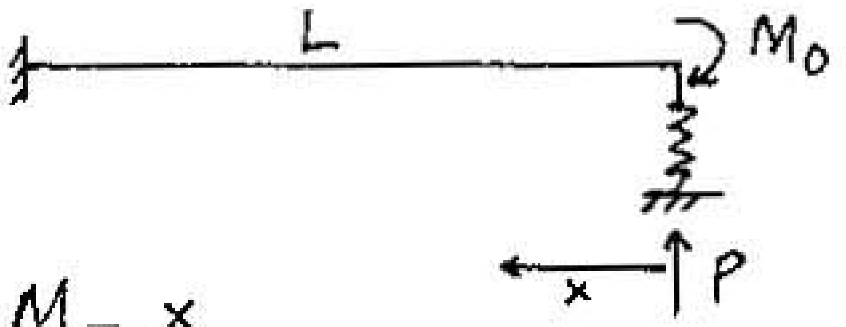
$$= \frac{QR^3}{EI}\left[\frac{\pi}{4} - \frac{1}{3\pi-8}\right] = \frac{QR^3(3\pi^2 - 8\pi - 4)}{4EI(3\pi - 8)} = 0.0835\frac{QR^3}{EI}$$

5.88 By Prob. 5.85, $M_{AB} = P(R+x) - Qx$, $0 \le x \le R$. We need consider only the strain energy of section AB (see Example 5.16). Thus, with $\frac{\partial M_{AB}}{\partial Q} = -x$,

$$\delta_Q = \int_0^R \frac{[P(R+x) - Qx]}{EI}(-x)\,dx = -\frac{5PR^3}{6EI} + \frac{QR^3}{3EI}$$

Since $P = 0.3098Q$ (See Prob. 5.85), $\delta_Q = 0.0752\frac{QR^3}{EI}$

5.89 Let the force in the spring be redundant. Then, the bending moment is

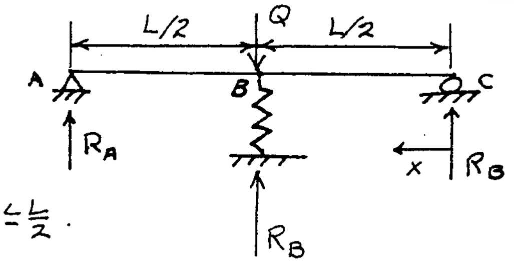


$$M = Px - M_0, \quad 0 \le x \le L, \text{ and } \frac{\partial M}{\partial P} = x$$

$$\therefore \ \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^L (Px - M_0)(x)\,dx + \frac{P \cdot 1}{k} = \frac{PL^3}{3EI} - \frac{M_0L^2}{2EI} + \frac{P}{k} = 0$$

$$\therefore \ P = \frac{3M_0kL^2}{2kL^3 + 6EI}$$

5.90 Let R_B be the redondant.



Then, by Equilibrium, $R_A = R_C = (Q-R_B)/2$.

The moment in section BC is

$$M_{BC} = \frac{Q-R_B}{2}x, \quad \frac{\partial M_{BC}}{\partial R_B} = -\frac{1}{2}, \quad 0 \le x \le \frac{L}{2}.$$

Therefore, with symmetry, we have

$$\frac{\partial U}{\partial R_B} = \frac{2}{EI}\int_0^{L/2}\left[\left(\frac{Q-R_B}{2}\right)x\right]\left(-\frac{x}{2}\right)dx + \frac{R_B \cdot 1}{k} = 0$$

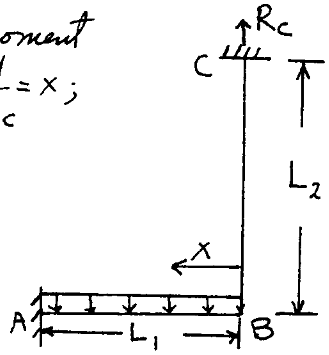
or $\frac{2}{EI}\left[\left(\frac{Q-R_B}{2}\right)\left(-\frac{L^3}{48}\right)\right] + \frac{R_B}{k} = 0$. Thus, $R_B = \frac{QkL^3}{48EI + kL^3}$.

$$\therefore \quad R_B = \frac{(30{,}000)(10^6)(5^3)}{48(200\times10^9)(24.0\times10^{-6}) + (10^6)(5^3)} = 10551\text{ N} = 10.551\text{ kN}$$

$$M_{max} = M_{BC}\left(\frac{L}{2}\right) = \frac{Q-R_B}{2}\,\frac{L}{2} = \frac{30000-10551}{4}(5) = 24311\text{ N·m} = 24.31\text{ kN·m}$$

$$\therefore \sigma_{max} = \frac{M_{max}\,c}{I_x} = \frac{(24{,}311)(0.2032/2)}{240\times10^{-6}} = 102.92\text{ MPa}$$

5.91 Let R_c be the redondant. Then the moment in the beam AB is $M_{AB} = R_c x - \frac{wx^2}{2}$, $\frac{\partial M}{\partial R_c} = x$; $0 \le x \le L_1$. Hence,



$$\frac{\partial U}{\partial R_c} = \frac{1}{E_1 I}\int_0^{L_1}\left(R_c x - \frac{wx^2}{2}\right)(x)dx + \frac{R_c \cdot L_2}{E_2 A_2} = 0$$

∴ After integration, we have (solving for R_c)

$$R_c = \frac{\dfrac{wL_1^4}{8E_1I}}{\dfrac{L_1^3}{3E_1I} + \dfrac{L_2}{E_2A_2}} = \frac{3E_2A_2wL_1^4}{8E_2A_2L_1^3 + 24E_1IL_2}$$

5.92 $I = \frac{1}{12}(30)(90)^3 = 1.8225\times10^6\text{ mm}^4 = 1.8225\times10^{-6}\text{ m}^4$; Area $= \left(\frac{5}{2}\right)^2\pi = 19.635\text{ mm}^2$
$= 19.635\times10^{-6}\text{ m}^2$
$= A_2$

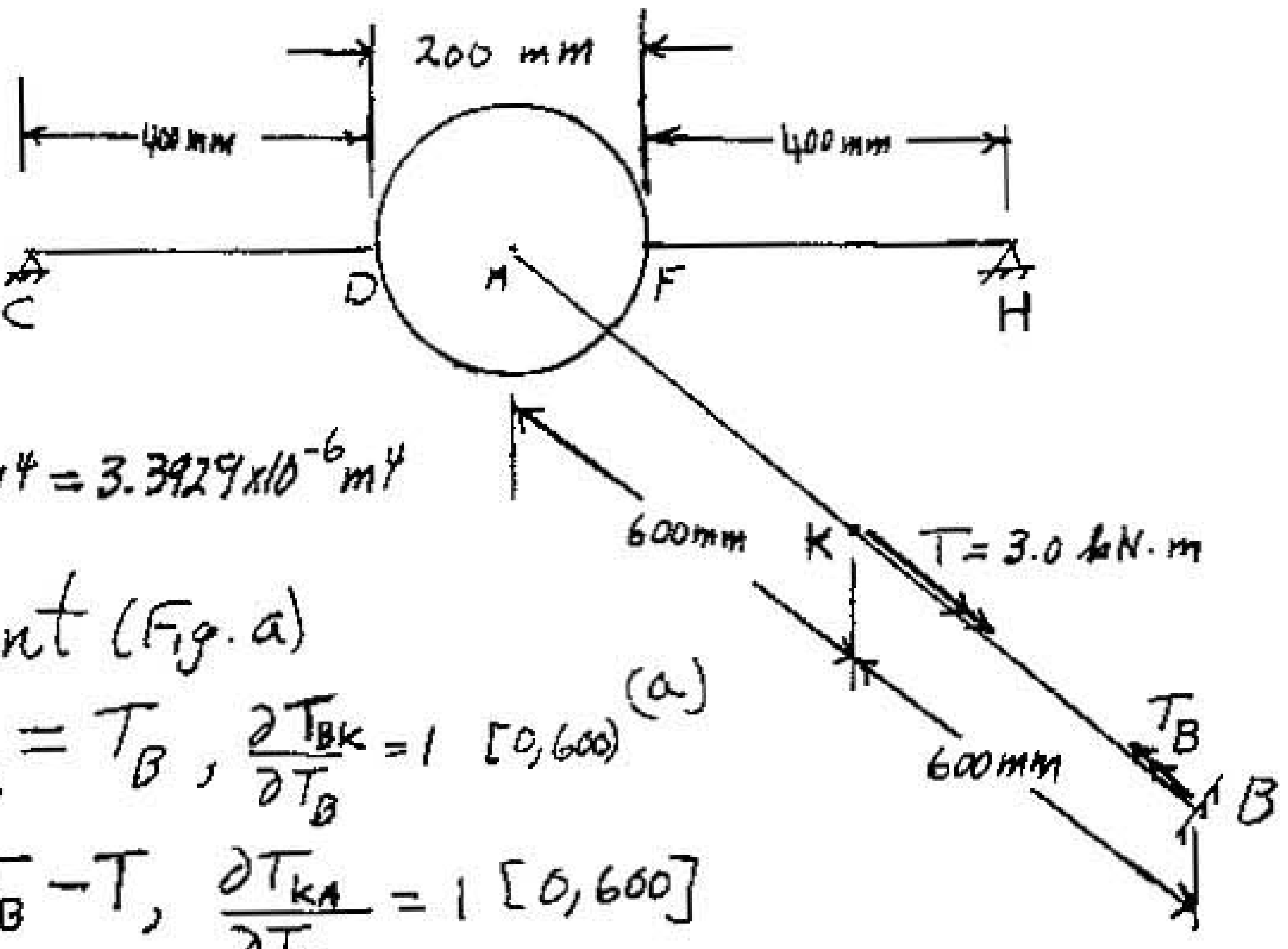
With the result of Prob. 5.91, we have

$$R_c = \frac{(3)(72\times10^9)(19.635\times10^{-6})(2)^4 w}{8(72\times10^9)(19.635\times10^{-6})(2)^3 + 24(200\times10^9)(1.8225\times10^{-6})(4)}$$

$= 0.5408\,w$ (N), with w = N/m

5.93

By Fig. P5.93,



(a)

$$I_{CD} = I_{FH} = \frac{1}{12}(40)(40)^3$$
$$= 0.2133 \times 10^6 \text{ mm}^4$$
$$= 0.2133 \times 10^{-6} \text{ m}^4$$

$$J_{AB} = \frac{\pi (60)^4}{3} = 3.3929 \times 10^6 \text{ mm}^4 = 3.3929 \times 10^{-6} \text{ m}^4$$

Let T_B be the redundant (Fig. a)

Then in Section BK, $T_{BK} = T_B$, $\frac{\partial T_{BK}}{\partial T_B} = 1$ [0,600)

IN section KA, $T_{KA} = T_B - T$, $\frac{\partial T_{KA}}{\partial T_B} = 1$ [0,600]

In sections CD (FH),

$$M_{CD} = \frac{T_B - T}{1000} x, \quad \frac{\partial M_{CD}}{\partial T_B} = \frac{x}{1000} \quad [0,400]$$

$$\therefore \frac{\partial U}{\partial T_B} = \frac{2}{EI}\int_0^{400} \left(\frac{T_B - T}{1000}\right)(x)\left(\frac{x}{1000}\right)dx + \frac{1}{GJ}\int_0^{600} T_B \cdot 1\, dx + \frac{1}{GJ}\int_0^{600} (T_B - T)\cdot 1\, dx = 0$$

or

$$\frac{128(T_B - T)}{3EI} + \frac{600\, T_B}{GJ} + \frac{600(T_B - T)}{GJ} = 0$$

or

$$\left(\frac{1200}{GJ} + \frac{128}{3EI}\right)T_B = \left(\frac{128}{3EI} + \frac{600}{GJ}\right)T$$

$$5.5638\, T_B = 3.2820\, T \rightarrow T_B = 0.5899 T = 1.7697 \text{ kN·m}$$

(a) $T_B - T = -1.2303$ kN·m

$$\therefore \sigma_{max} = \frac{(1.2303)(0.4)(20)}{0.2133} = 46.704 \text{ MPa in CD (or FH)}$$

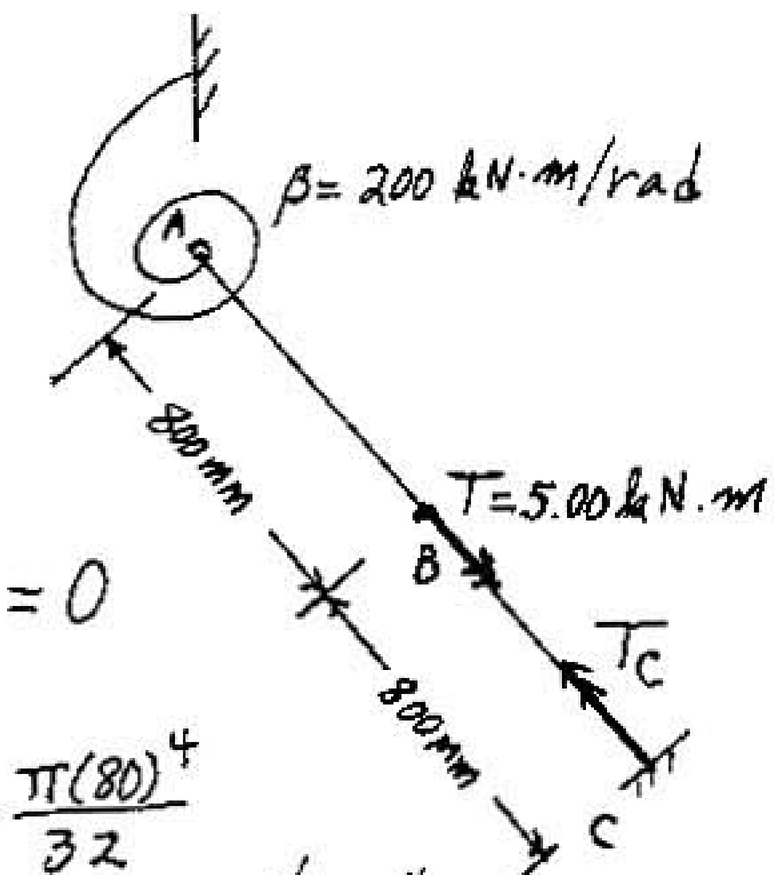
(b) $$\tau_{max} = \frac{T_{max}\, r}{J} = \frac{(1.7697)(30)}{3.3929} = 15.648 \text{ MPa}$$

5.94 Let T_C be the redundant.

Then

$$T_{BC} = T_C \; ; \; \frac{\partial T_{BC}}{\partial T_C} = 1, \; [0, .800] \text{ m}$$

$$T_{AB} = T_C - T \; ; \; \frac{\partial T_{AB}}{\partial T_C} = 1, \; [0, .800] \text{ m}$$



$$\frac{\partial U}{\partial T_C} = \frac{1}{GJ}\left[\int_0^{.800} T_C\,dx + \int_0^{.800} (T_C - T)\cdot 1 \cdot dx\right] + \frac{(T_C - T)\cdot 1}{\beta} = 0$$

$$\therefore \left(\frac{1.600}{GJ} + \frac{1}{\beta}\right) T_C = \left(\frac{0.800}{GJ} + \frac{1}{\beta}\right) T \; ; \quad J = \frac{\pi (80)^4}{32}$$

$$= 4.021 \times 10^6 \text{ mm}^4$$

$$= 4.021 \times 10^{-6} \text{ m}^4$$

$$\therefore \; 1.0134 \times 10^{-5}\, T_C = 7.5672 \times 10^{-6}\, T$$

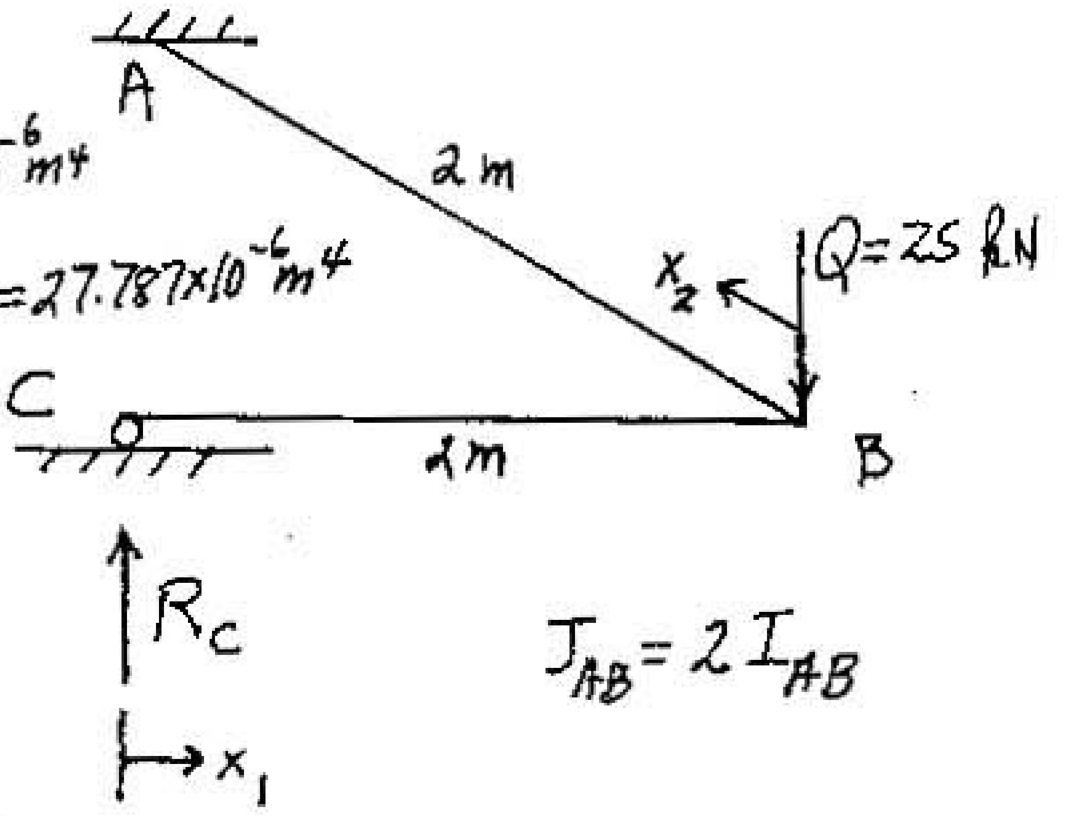
or $\quad T_C = 0.7467\, T$

$$\tau_{max} = \frac{T_{max}\, r}{J} = \frac{(0.7467)(5)(80/2)}{4.021} = 37.14 \text{ MPa}$$

5.95 Let R_C be the Redondant.

$$I_{AB} = \frac{\pi (180)^4}{64} \doteq 51.530 \times 10^6 \text{ mm}^4 = 51.530 \times 10^{-6} \text{ m}^4$$

$$I_{BC} = \frac{1}{12}(100)(200)^3 - \frac{1}{12}(80)(180)^3 = 27.787 \times 10^6 \text{ mm}^4 = 27.787 \times 10^{-6} \text{ m}^4$$



$J_{AB} = 2 I_{AB}$

For Section BC,

$$M_{BC} = R_C x_1, \; 0 \le x_1 \le 2 \; ; \; \frac{\partial M_{BC}}{\partial R_C} = x_1$$

For section AB,

$$M_{AB} = (R_C - Q)\, x_2, \; 0 \le x_2 \le 2 \; ; \; \frac{\partial M_{AB}}{\partial R_C} = x_2$$

$$T_{AB} = 2R_C, \; 0 \le x_2 \le 2 \; ; \; \frac{\partial T_{AB}}{\partial R_C} = 2$$

$$\therefore \frac{\partial U}{\partial R_C} = \frac{1}{ET}\int_0^2 (R_C x_1)\cdot x_1\, dx_1 + \frac{1}{EI_{AB}}\int_0^2 [(R_C - Q)x_2]\cdot x_2\, dx_2 + \frac{1}{GJ_{AB}}\int_0^2 (2R_C)\cdot 2 \cdot dx_2 = 0$$

$$= \frac{R(2)^3}{(200 \times 10^3)(27.787)(3)} + \frac{(R_C - Q)(2)^3}{(200 \times 10^3)(51.53)(3)} + \frac{R_C (2)^3}{(77.5 \times 10^3)(2 \times 51.53)} = 0$$

Hence,

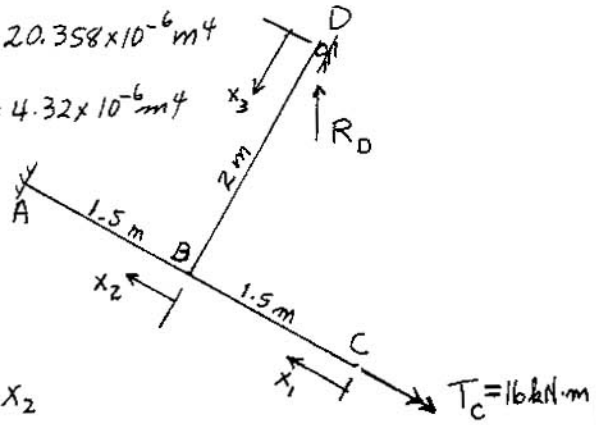
$$1.7401\, R_C = 0.2587\, Q$$

$$R_C = 0.1487\, Q = 3.717 \text{ kN}$$

5.96 Let R_D be redundant.

$$J_{AC} = \frac{\pi(120)^4}{32} = 20.358\times10^6\,\text{mm}^4 = 20.358\times10^{-6}\,\text{m}^4$$

$$I_{BD} = \frac{1}{12}(30)(120)^3 = 4.320\times10^6\,\text{mm}^4 = 4.32\times10^{-6}\,\text{m}^4$$



For section BD,

$$M_{BD} = R_D x_3,\ 0 \le x_3 \le 2;\ \frac{\partial M_{BD}}{\partial R_D} = x_3$$

For section AB,

$$M_{AB} = R_D x_2,\ 0 \le x_2 \le 1.5;\ \frac{\partial M_{AB}}{\partial R_D} = x_2$$

$$T_{AB} = T_C + 2R_D,\ 0 \le x_2 \le 1.5;\ \frac{\partial T_{AB}}{\partial R_D} = 2$$

For section BC,

$$T_{BC} = T_C,\ 0 \le x_1 \le 1.5;\ \frac{\partial T_{BC}}{\partial R_D} = 0$$

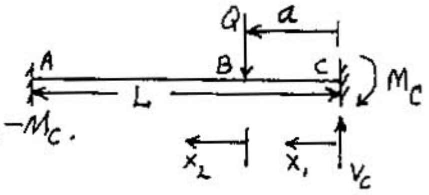
$$\therefore\ \frac{\partial U}{\partial R_D} = \frac{1}{EI_{BD}}\int_0^2 (R_D x_3)\cdot x_3\cdot dx_3 + \frac{1}{EI_{AB}}\int_0^{1.5}(R_D x_2)\cdot x_2\cdot dx_2 + \frac{1}{GJ_{AC}}\int_0^{1.5}(T_C+2R_D)\cdot 2\,dx_2 = 0$$

$$= \frac{R_D(2)^3(10)^3}{3(200)(4.32)} + \frac{R_D(1.5)^3(10)^3}{(3)(200)(20.358/2)} + \frac{2(T_C+2R_D)(1.5)(10)^3}{(77.5)(20.358)} = 0$$

$$\therefore\ 7.4419\,R_D = -1.9014\,T_C$$

or $\quad R_D = -0.2555\,\text{kN} = -255.5\,\text{N}$ (downward)

5.97 Let V_c, M_c be the redundants.



For section BC, $M_{BC} = V_c x_1 - M_C$, $0 \le x_1 \le a$; $\frac{\partial M_{BC}}{\partial V_0} = x_1$, $\frac{\partial M_{BC}}{\partial M_C} = -1$. For section AB, $M_{AB} = V_C(a+x_2) - Qx_2 - M_C$.

$0 \le x_2 \le L-a$; $\frac{\partial M_{AB}}{\partial V_C} = x_2 + a$, $\frac{\partial M_{AB}}{\partial M_C} = -1$.

$$\therefore\ \frac{\partial U}{\partial V_c} = \int_0^a (V_c x_1 - M_c)\cdot x_1\cdot dx_1 + \int_0^{L-a}[V_c(a+x_2) - Qx_2 - M_c](a+x_2) = 0 \qquad (a)$$

$$\frac{\partial U}{\partial M_c} = \int_0^a (V_c x_1 - M_c)(-1)dx_1 + \int_0^{L-a}[V_c(a+x_2) - Qx_2 - M_c](-1)dx_2 = 0 \qquad (b)$$

Eqs. (a) and (b) yield

$$\left.\begin{aligned} 2M_C L - V_C L^2 &= -Q(L-a)^2 \\ 3M_C L^2 - 2V_C L^3 &= -Q(L-a)^2(2L+a) \end{aligned}\right\} \quad (c)$$

The solution of Eqs. (c) is $M_C = Q(aL^2 - 2a^2L + a^3)/L^2$, $V_C = Q(L^3 - 3a^2L + 2a^3)/L^3$

5.98 $I = \frac{\pi(150)}{64} = 24.85\times10^{6}\ mm^4 = 24.85\times10^{-6}\ m^4$

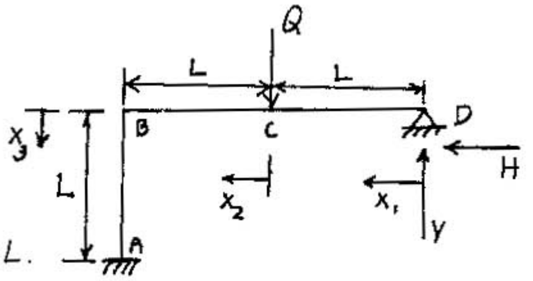
By the results of Prob. 5.97, with $L = 2\ m$ and $a = L/3 = \frac{2}{3} m$,

$$V_C = Q\left[2^3 - 3\left(\tfrac{2}{3}\right)^2(2) + 2\left(\tfrac{2}{3}\right)^3\right]/2^3 = 0.7407Q$$

$$M_C = Q\left[\left(\tfrac{2}{3}\right)(2)^2 - 2\left(\tfrac{2}{3}\right)^2(2) + \left(\tfrac{2}{3}\right)^3\right]/2^2 = 0.2963Q$$

$$M_{max} = M_c = 0.2963Q \therefore \sigma_{max} = 100\ MPa = \frac{(0.2963Q)(75)}{24.85} \rightarrow Q = 111.82 MPa$$

5.99 Let H and V be the redundants.



Then for section CD,

$M_{CD} = Vx_1,\ 0 \le x_1 \le L;\ \frac{\partial M_{CD}}{\partial V} = x_1$

$\frac{\partial M_{CD}}{\partial H} = 0.$

For section BC, $M_{BC} = V(L+x_2) - Qx_2,\ 0 \le x_2 \le L.$

$\frac{\partial M_{BC}}{\partial V} = L + x_2,\ \frac{\partial M_{BC}}{\partial H} = 0.$

For section AB, $M_{AB} = 2VL - QL + Hx_3,\ 0 \le x_3 \le L.\ \frac{\partial M_{AB}}{\partial V} = 2L,\ \frac{\partial M_{AB}}{\partial H} = x_3.$

$$\therefore \frac{\partial U}{\partial V} = \int_0^L (Vx_1)x_1\,dx_1 + \int_0^L [V(L+x_2) - Qx_2](L+x_2)dx_2 + \int_0^L (2VL - QL + Hx_3)(2L)dx_3 = 0$$

$$\therefore \frac{VL^3}{3} + V\left(\frac{7L^3}{3}\right) - \frac{QL^3}{2} - \frac{QL^3}{3} + 4VL^3 - 2QL^3 + HL^3 = 0 \qquad (a)$$

$$\frac{\partial U}{\partial H} = \int_0^L (2VL - QL + Hx_3)x_3\,dx_3 = 0$$

$$\therefore \quad VL^3 - \frac{QL^3}{2} + \frac{HL^3}{3} = 0 \quad (b)$$

The solution of Eqs. (a) and (b) is

$$V = \frac{4}{11}Q,\quad H = \frac{9}{22}Q.$$

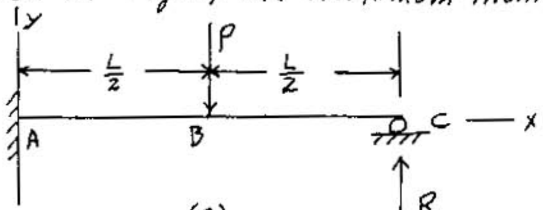
5.100 The maximum moment is $\frac{4}{11}QL$ at C. $\therefore 120 = \frac{4}{11}\frac{Q(800)(40)}{3.413\times10^6}$ since $I = \frac{1}{12}(80)^4 = 3.413\times10^6\ mm^4$. $\therefore Q = 35{,}197\ N = 35.197\ kN.$

From Prob. 5.99, $M_{BC} = V(L+x_2) - Qx_2$, and $M_{AB} = 2VL - QL + Hx_3$.

$$\therefore \frac{\partial U}{\partial Q} = \frac{1}{EI}\int_0^L [V(L+x_2) - Qx_2](-x_2)dx_2 + \frac{1}{EI}\int_0^L (2VL - QL + Hx_3)(-L)\,dx_3 = \delta_Q$$

or $$\frac{L^3}{6EI}(-17V + 8Q - 3H) = \frac{QL^3}{6EI}\left(-17\times\tfrac{4}{11} + 8 - 3\times\tfrac{9}{22}\right) = 0.09848\frac{QL^3}{EI} = 2.60\,mm = \delta_Q$$

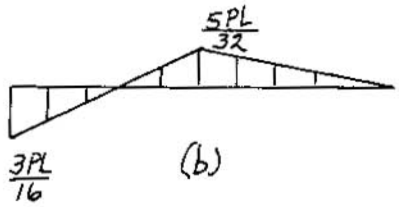
5.101 From Example 5.15, we have $R = \frac{5P}{16}$ (Fig. a). Hence, the moment diagram for the beam is shown in Fig. b; the maximum moment is $M_{max} = \frac{3PL}{16}$ at the wall. Therefore yield occurs first at the wall.



(a)

(a) $\therefore Y = \frac{M_{max}\frac{h}{2}}{\frac{1}{12}bh^3} = \frac{3P_YL}{16}\frac{6}{bh^2} = \frac{9P_YL}{8bh^2}$

or $P_Y = \frac{8bh^2}{9L}Y$



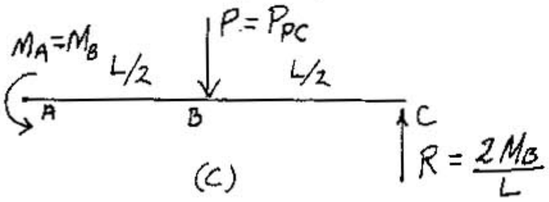
(b)

(b) For a fully plastic hinge at the wall, the moment is equal to $\frac{1}{4}bh^2Y$. $\therefore \frac{3PL}{16} = \frac{1}{4}bh^2Y$; or $P = \frac{4}{3}\frac{bh^2}{L}Y$

(c) For a plastic hinge at B, $M_B = \frac{1}{4}bh^2Y = R\frac{L}{2}$. Then, $R = \frac{2M_B}{L}$. Also, there is a plastic hinge at the wall A, where the moment is $M_A = \frac{1}{4}bh^2Y = M_B$ (Fig. c)

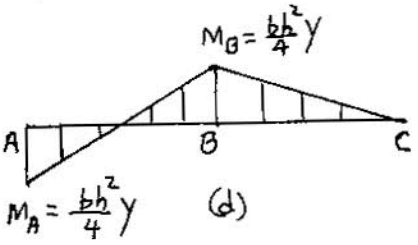
$\therefore \Sigma M_A = M_B + \left(\frac{2M_B}{L}\right)L - P_{PC}\frac{L}{2} = 0$

$\therefore P_{PC} = \frac{6M_B}{L} = \frac{3}{2}\frac{bh^2}{L}Y$



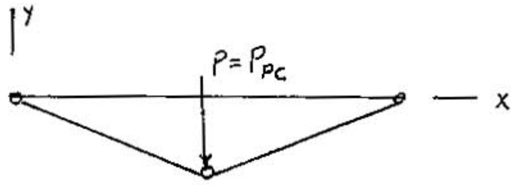
(c)

(d) The moment diagram for $P = P_{PC}$ is shown in Fig. d.



(d)

(e) For $P = P_{PC}$, the deformed shape of the beam is shown in Fig. e.



(e)

6.1 (a) The cross section of the member is shown in Fig. a. By Fig. a,

$$J = \frac{\pi}{32}(d_o^4 - d_i^4) = \frac{\pi}{32}(0.120^4 - 0.100^4)$$

or

$$J = 1.054 \times 10^{-5}\ m^4 \qquad (a)$$

Therefore, by Eq. (6.15) with Eq. (a),

$\tau_{max} = 50\ MPa$, and $r = r_o = \frac{d_o}{2} = 0.06\ m$,

$$T = \frac{\tau_{max} J}{r_o} = \frac{(50\times10^{6})(1.054\times10^{-5})}{0.060}$$

or

$$T = 8.783\ kN\cdot m \qquad (b)$$

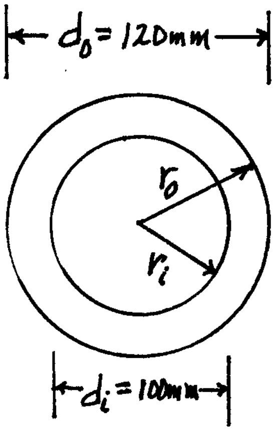


Figure a

(b) By Eq. (6.15), the shear stress at $r = r_i = \frac{d_i}{2} = 0.050\ m$ is

$$\tau = \frac{T r_i}{J} = \frac{(8.783\times10^{3})(0.050)}{1.054\times10^{-5}} = 41.67\ MPa$$

6.2 The cross sections of the two shafts are shown in Figs. a and b. By Fig. a, the area of the cross section of the solid shaft is

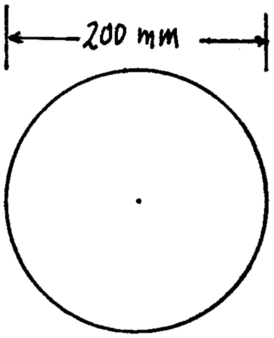


Figure a

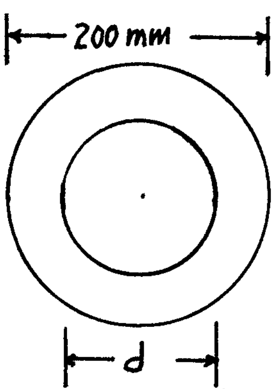


Figure b

$$A_s = \frac{\pi}{4}(0.20)^2 = 0.031416\ m^2 \qquad (a)$$

(cont.)

$\boxed{6.2}$ Since the lengths of the shafts are equal, the weights are proportional to the cross-sectional areas. Therefore, the area of the hollow shaft is, by Fig. b,

$$A_H = \frac{1}{2} A_S = \frac{1}{2}(0.031416) = \frac{\pi d^2}{4}$$

or

$$d = 0.1414 \text{ m} \qquad \text{(b)}$$

(a) For a design torque T = 80.0 kN·m, the maximum shear in the solid shaft is, by Eq. (6.15) and Fig. a,

$$\tau_S = \frac{Tr}{J_S} = \frac{(80\times10^3)(0.100)}{\frac{\pi}{32}(0.200)^4} = 50.930 \text{ MPa}$$

Similarly, by Eq. 6.15 and Fig. b, the maximum shear stress in the hollow shaft is, with Eq. (b),

$$\tau_H = \frac{Tr}{J_H} = \frac{(80\times10^3)(0.100)}{\frac{\pi}{32}(0.200^4 - 0.1414^4)} = 67.892 \text{ MPa}$$

Hence,

$$\frac{\tau_H}{\tau_S} = 1.333 \qquad \text{(c)}$$

(b) The solid steel shaft has a flexural stiffness $GJ_S = (77.5\times10^9)[\frac{\pi}{32}(0.200^4)] = 12.174 \text{ MN}\cdot\text{m}^2$. The hollow shaft has a flexural stiffness $GJ_H = (77.5\times10^9)[\frac{\pi}{32}(0.200^4 - 0.1414^4)] = 9.132 \text{ MN}\cdot\text{m}^2$. Therefore,

$$\frac{GJ_H}{GJ_S} = 0.750 \qquad \text{(d)}$$

(c) By Eqs. (c) and (d), we see that reducing the weight of one of the shafts increases the maximum shear stress in that shaft by 33% and decreases its flexural stiffness by 25%. Thus the advantage of less weight is balance by a higher stress and less stiffness.

6.3 By Eq. (6.15), including the safety factor, we have

$$(S.F.)T = \frac{\tau J}{r} \qquad (a)$$

where $r = 0.100$ m, $T = 80.0$ kN·m and $\tau = 150$ MPa.

For the solid shaft,

$$J_S = \frac{\pi}{32}(0.200)^4 = 1.5708 \times 10^{-4} \text{ m}^4 \qquad (b)$$

For the hollow shaft, with $r_i = 0.07071$ m,

$$J_H = \frac{\pi}{2}(0.100^4 - 0.07071^4) = 1.1781 \times 10^{-4} \text{ m}^4 \quad (c)$$

Hence, by Eqs. (a) and (b), the safety factor for the solid shaft is

$$(S.F)_S = \frac{(150 \times 10^6)(1.5708 \times 10^{-4})}{(80.0 \times 10^3)(0.100)} = 2.945$$

Similarly, by Eqs. (a) and (c), the safety factor for the hollow shaft is

$$(S.F)_H = \frac{(150 \times 10^6)(1.1781 \times 10^{-4})}{(80.0 \times 10^3)(0.100)} = 2.209$$

6.4 The original and modified torsion members are shown in Figs. a and b, respectively.

(a) By Fig. a, the polar moments of the middle and end sections are, respectively

$$J_{OM} = \frac{\pi}{32}(0.050)^4 = 6.136 \times 10^{-7} \text{m}^4$$

$$J_{OE} = \frac{\pi}{32}(0.080)^4 = 4.021 \times 10^{-6} \text{m}^4$$

(cont.)

6.4 Cont. Then, by Eq. (6.15), the torques in the middle and end sections are, respectively,

$$T_{OM} = \frac{\tau J_{OM}}{r} = \frac{(60\times10^{6})(6.136\times10^{-7})}{0.025} = 1.473 \text{ kN}\cdot\text{m}$$

$$T_{OE} = \frac{\tau J_{OE}}{r} = \frac{(60\times10^{6})(4.021\times10^{-6})}{0.040} = 6.031 \text{ kN}\cdot\text{m}$$

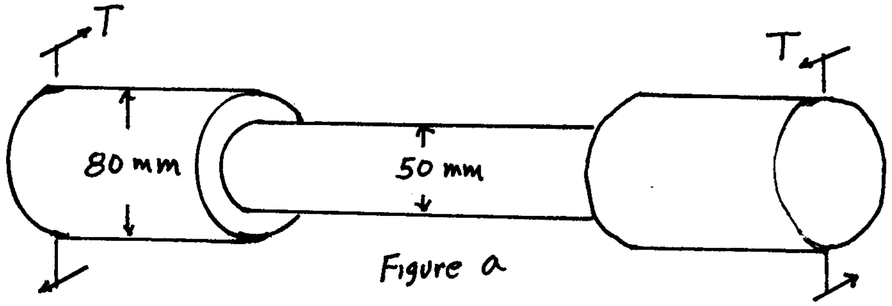


Figure a

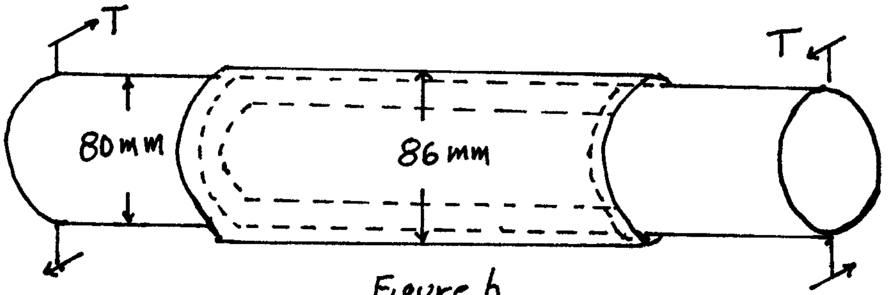


Figure b

Therefore, the design torque for the original member is $T_{OM} = 1.473$ kN·m.

By Fig. b, the polar moments of inertia for the modified member are

$$J_{MM} = J_{OM} + \frac{\pi}{32}(0.086^4 - 0.080^4) = 1.963\times10^{-6} \text{ m}^4$$

$$J_{ME} = J_{OE} = 4.021\times10^{-6} \text{ m}^4$$

The corresponding torques are

$$T_{MM} = \frac{\tau J_{MM}}{r} = \frac{(60\times10^{6})(1.963\times10^{-6})}{0.043} = 2.738 \text{ kN}\cdot\text{m}$$

$$T_{ME} = T_{OE} = 6.031 \text{ kN}\cdot\text{m}$$

Hence, the design torque is $T_{MM} = 2.738$ kN·m

(cont.)

6.4 Cont (b) The increase of torque due to the sleeve is

$$\Delta T = T_{MM} - T_{OM} = 2.738 - 1.473 = 1.265 \text{ kN}\cdot\text{m}$$

or an increase of 85.9%

6.5 The hollow aluminum shaft is shown in Fig. a. It has three lengths OA, AB, and BC in which different torques act, as follows.

$$\text{Length OA: } T_{OA} = 10 \text{ kN}\cdot\text{m} \ \twoheadleftarrow$$
$$\text{Length AB: } T_{AB} = 15 \text{ kN}\cdot\text{m} \ \twoheadrightarrow \qquad (a)$$
$$\text{Length BC: } T_{BC} = 14 \text{ kN}\cdot\text{m} \ \twoheadleftarrow$$

(a) The maximum torque acts in length AB, and the corresponding maximum shear stress in length AB is

$$\tau_{max} = \frac{Tr}{J} = \frac{(15\times10^3)(0.050)}{\frac{\pi}{32}(0.100^4 - 0.080^4)} = 129.4 \text{ MPa}$$

(b) By Eq. (6.12), the angle of twist of a torsion member of length L is

$$\psi = \theta L = \frac{TL}{GJ} \qquad (b)$$

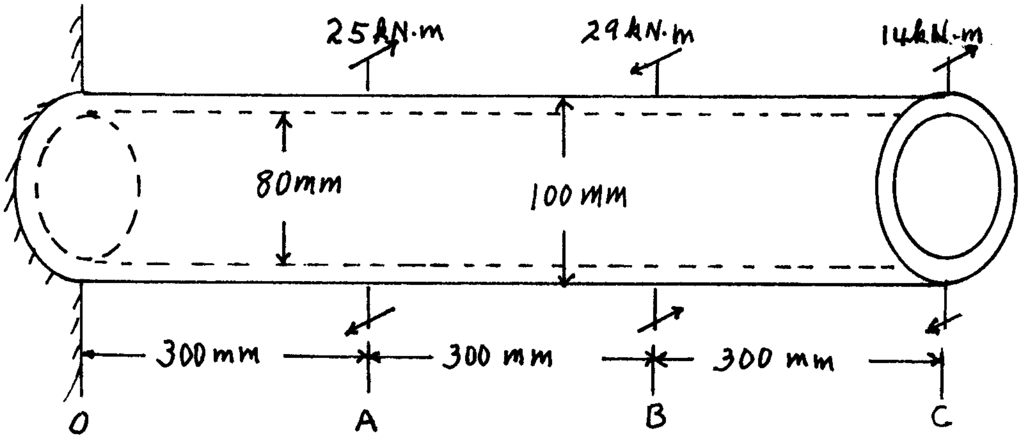


Figure a

(Cont.)

$\boxed{6.5 \text{ cont'd}}$ The angle of twist of the member at C (Fig. a) is

$$\varphi_C = \varphi_{CB} + \varphi_{BA} + \varphi_{AO} \qquad (c)$$

Where φ_{CB} is the angle of twist of section C relative to section B, φ_{BA} is the angle of twist of section B relative to section A, and φ_{AO} is the angle of twist of section A relative to O.

By Eqs. (a), (b), and (C), with Fig. a, we find

$$\varphi_{CB} = \frac{T_{BC} L_{BC}}{GJ} = \frac{(14\times10^3)(0.300)}{(27\times10^9)\left[\frac{\pi}{32}(0.100^4 - 0.080^4)\right]}$$

$$= \frac{(14\times10^3)(0.300)}{(27\times10^9)(5.796\times10^{-6})} = 0.0268 \text{ rad } \twoheadleftarrow$$

Similarly,

$$\varphi_{BA} = \frac{T_{AB} L_{AB}}{GJ} = \frac{(15\times10^3)(0.300)}{(27\times10^9)(5.796\times10^{-6})} = 0.0288 \text{ rad} \twoheadrightarrow$$

$$\varphi_{AO} = \frac{T_{OA} L_{OA}}{GJ} = \frac{(10\times10^3)(0.300)}{(27\times10^9)(5.796\times10^{-6})} = 0.0192 \text{ rad } \twoheadleftarrow$$

Hence,

$$\varphi_C = \varphi_{CB} + \varphi_{BA} + \varphi_{AO}$$

$$= 0.0268 \twoheadleftarrow + 0.0288 \twoheadrightarrow + 0.0192 \twoheadleftarrow$$

or

$$\varphi_C = 0.0172 \text{ rad } \twoheadleftarrow$$

6.6 The torsion member is shown in Fig. a. It has two lengths OA and AB in which the torques are different. They are

$$T_{OA} = 1.70 + 0.30 = 2.0 \text{ kN}\cdot\text{m} \twoheadrightarrow \qquad \text{(a)}$$

$$T_{AB} = 0.30 \text{ kN}\cdot\text{m} \twoheadrightarrow$$

By Fig. a, the polar moments of inertia are

$$J_{OA} = \frac{\pi}{32}(0.050^4) = 6.136 \times 10^{-7} \text{ m}^4 \qquad \text{(b)}$$

$$J_{AB} = \frac{\pi}{32}(0.030^4) = 7.952 \times 10^{-8} \text{ m}^4$$

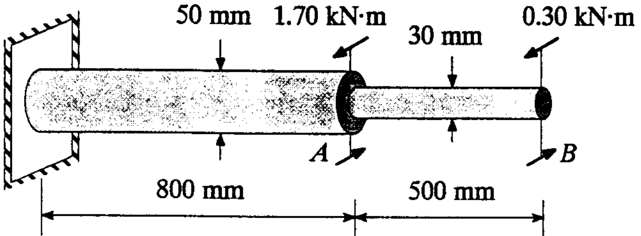


Figure a

(a) The corresponding shear stress in lengths OA and AB are, with Eqs. (a) and (b),

$$\tau_{OA} = \frac{T_{OA} r_{OA}}{J_{OA}} = \frac{(2.0\times10^3)(0.025)}{6.136\times10^{-7}} = 81.49 \text{ MPa}$$

$$\tau_{AB} = \frac{T_{AB} r_{AB}}{J_{AB}} = \frac{(0.30\times10^3)(0.015)}{7.952\times10^{-8}} = 56.59 \text{ MPa}$$

Therefore, the maximum shear stress is $\tau_{max} = 81.49$ MPa. It occurs in the length OA.

(b) The safety factor is, given that $\tau_y = 160$ MPa,

$$S.F. = \frac{\tau_y}{\tau_{max}} = \frac{160}{81.49} = 1.96$$

6.7 The angle of twist of the free end of the torsion member of Problem 6 is (see Fig. a)

$$\psi_B = \psi_{BA} + \psi_{AO} \qquad (a)$$

where ψ_{BA} is the angle of twist of section B relative to section A and ψ_{AO} is the angle of twist of section A relative to O. In general, by Eq. (6.12),

$$\psi = \theta L = \frac{TL}{GJ} \qquad (b)$$

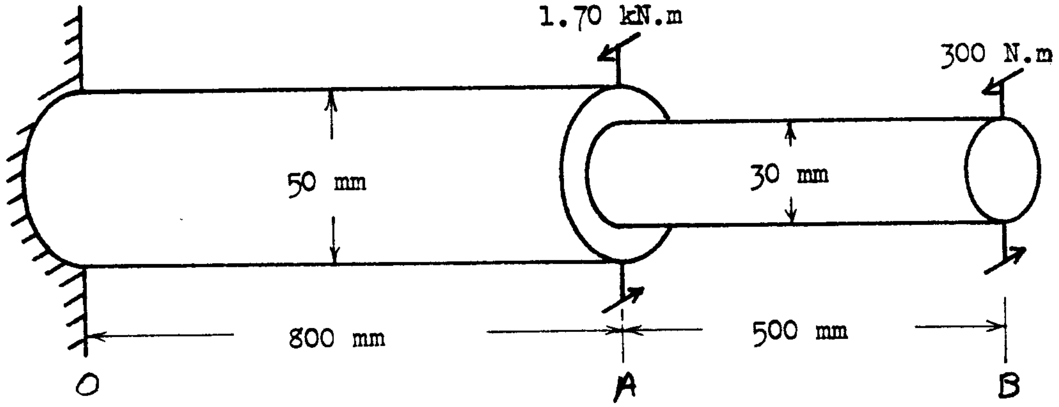


Figure a

Hence, by Eq. (b) and Fig. a,

$$\psi_{BA} = \frac{T_{AB} L_{AB}}{G J_{AB}} = \frac{(300)(0.500)}{(77.5\times10^9)(7.952\times10^{-8})} = 0.0243 \text{ rad} \rightarrow$$

$$\psi_{AO} = \frac{T_{OA} L_{OA}}{G J_{OA}} = \frac{(2.0\times10^3)(0.800)}{(77.5\times10^9)(6.136\times10^{-7})} = 0.0336 \text{ rad} \rightarrow \qquad (c)$$

Hence, by Eqs. (a) and (c),

$$\psi_B = \psi_{BA} + \psi_{AO} = 0.0243 + 0.0336 = 0.058 \text{ rad} \rightarrow$$

6.8 Since the motor rotates at a speed of 1780 rpm = 186.4 rad/s $= \omega$, the speed of the gear B, at the pitch radius $r_B = 0.030$ m, is (Fig. a)

$$v_B = \omega r_B = (186.4)(0.030) = 5.592 \text{ m/s} \qquad (a)$$

Hence, the speed of rotation of shaft CD is given by the relation

$$v_B = \omega r_B = \omega_{CD} r_C \qquad (b)$$

or by Eqs. (a) and (b) and Fig. a ($r_C = 0.120$ m)

$$\omega_{CD} = \frac{5.592}{0.120} = 46.60 \text{ rad/s} \qquad (c)$$

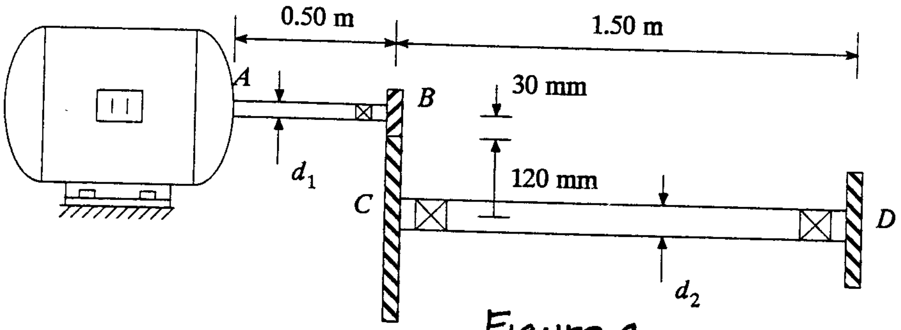


Figure a

Since power is related to torque and angular speed ω by the relation

$$P = T\omega \qquad (d)$$

and since the motor delivers $P = 15$ kW of power, by Eq. (d), the torque in shaft AB is

$$T_{AB} = \frac{15 \times 10^3}{186.4} = 80.47 \text{ N}\cdot\text{m} \qquad (e)$$

Similarly, with Eq. (c), the torque in shaft CD is

$$T_{CD} = \frac{15 \times 10^3}{46.60} = 321.89 \text{ N}\cdot\text{m} \qquad (f)$$

(cont.)

6.8 cont. Hence, the maximum shear stress in shaft AB is, with $T_{AB} = 80.47$ N·m and $d_1 = 0.020$ m,

$$\tau_{max} = \frac{Tr}{J} = \frac{(80.47)(0.010)}{\frac{\pi}{32}(0.020^4)} = 51.23 \text{ MPa}$$

The maximum shear stress in shaft CD, with $T_{CD} = 321.89$ N·m and $d_2 = 0.040$ m, is

$$\tau_{max} = \frac{Tr}{J} = \frac{(321.89)(0.020)}{\frac{\pi}{32}(0.040^4)} = 25.62 \text{ MPa}$$

6.9 Since gear D and hence shaft CD rotate at a frequency $f_{CD} = 6$ Hz, by Fig. a, the frequency of gear B and shaft AB is

$$f_{AB} = \frac{r_C}{r_B} f_{CD} = \frac{120}{30}(6) = 24 \text{ Hz}$$

Hence, the angular frequencies of the shafts CD and AB are, respectively,

$$\omega_{CD} = 2\pi f_{CD} = 37.699 \text{ rad/s} \qquad (a)$$

$$\omega_{AB} = 2\pi f_{AB} = 150.796 \text{ rad/s} \qquad (b)$$

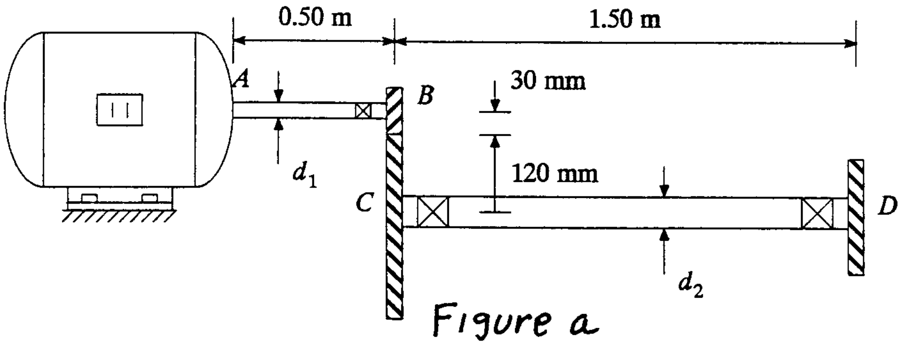


Figure a

(cont.)

6.9 cont. In terms of power P and angular frequency ω, the torque in a shaft is

$$T = \frac{P}{\omega} \qquad (c)$$

So, since the motor delivers $P = 15\,kW$, by Eqs. (a) and (c), the torque in shaft CD is

$$T_{CD} = \frac{15\times10^3}{37.699} = 397.89\ N\cdot m \qquad (d)$$

Similarly, by Eqs. (b) and (c), the torque in shaft AB is

$$T_{AB} = \frac{15\times10^3}{150.796} = 99.47\ N\cdot m \qquad (e)$$

Then, for shaft CD (Fig. a), by Eqs. (6.15) and (d), we have

$$\tau_{CD} = 55\times10^6 \frac{N}{m^2} = \frac{T_{CD}\,(d_2/2)}{\frac{\pi}{32}\,d_2^4} = \frac{397.89(16)}{d_2^3}$$

or

$$d_2 = 0.0333\ m = 33.3\ mm$$

Similarly for shaft AB,

$$\tau_{AB} = 55\times10^6 \frac{N}{m^2} = \frac{T_{AB}\,(d_1/2)}{\frac{\pi}{32}(d_1^4)} = \frac{(99.47)(16)}{\pi\, d_1^3}$$

or

$$d_1 = 0.0210\ m = 21.0\ mm$$

6.10 Since the 18 kW motor rotates at 3600 rpm, the shaft rotates at an angular speed of (see Fig. a)

$$\omega = 3600\left(\frac{100}{200}\right) = 1800 \text{ rpm} = 188.5 \text{ rad/s} \qquad (a)$$

The torque in each length of the shaft is

$$T = \frac{P}{\omega} \qquad (b)$$

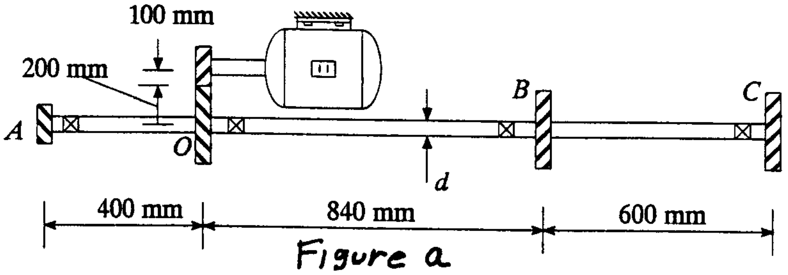


Figure a

Hence, by Fig. a, since the power delivered by the shaft to the gear A is $P = 5$ kW, the torque in length OA is, with Eqs. (a) and (b),

$$T_{OA} = \frac{5\times10^3}{188.5} = 26.525 \text{ N.m} \qquad (c)$$

Since the power delivered by the length OB is 13 kW, the torque in length OB is

$$T_{OB} = \frac{13\times10^3}{188.5} = 68.966 \text{ N.m} \qquad (d)$$

Similarly, for length BC,

$$T_{BC} = \frac{8\times10^3}{188.5} = 42.440 \text{ N.m} \qquad (e)$$

By Eqs. (c), (d), and (e), the maximum torque in the shaft is $T_{max} = 68.97$ N.m. Hence, by Eq. (6.15), with $\tau_y = 180$ MPa and S.F. = 1.80, the minimum required diameter d of the shaft is given by the relation

(cont.)

6.10 cont.

$$(S.F.)T_{max} = 1.80(68.966) = \frac{\tau_Y J}{r} = \frac{(180\times10^6)(\pi d^4/32)}{d/2}$$

or

$$d = 0.0152m = 15.2\ mm$$

6.11 The deflection at C (Fig. a) can be determined by beam theory or by energy methods. We may add equal and opposite forces of magnitude P at B so that it is clear that member AB is subjected to a couple of magnitude PL_2 and a downward force P.

By beam theory, the deflection of B of member AB due to P is

$$\delta_B = \frac{1}{3}\frac{PL_1^3}{EI_x} = \frac{64PL_1^3}{3\pi Ed^4} \quad (a)$$

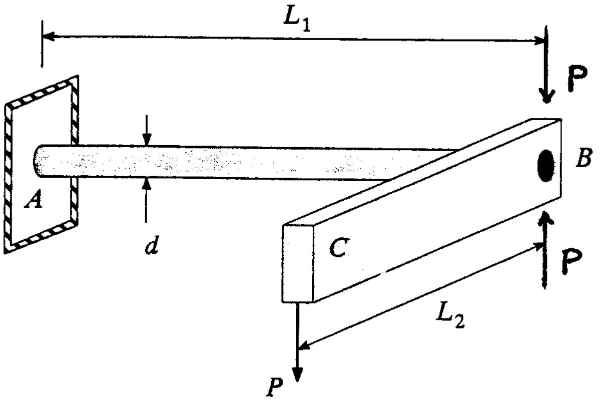


Figure a

Member AB also undergoes a rotation of [see Eq.(6.12)]

$$\psi = L_1\theta = \frac{TL_1}{GJ} = \frac{32PL_2}{\pi Gd^4}L_1 \qquad (b)$$

The deflection δ_ψ at C due to rotation ψ is, by Eq. (b), since BC is rigid,

$$\delta_\psi = L_2\psi = \frac{32PL_1L_2^2}{\pi Gd^4} \qquad (c)$$

Hence, the total deflection Δ at point C is

$$\Delta = \delta_B + \delta_\psi = \frac{32PL_1}{3\pi d^4}\left(\frac{2GL_1^2 + 3EL_2^2}{EG}\right)$$

(Cont.)

6.11 cont. or

$$P = \frac{3\pi d^4 EG}{32L_1(2GL_1^2+3EL_2^2)}\Delta = K\Delta$$

Hence,

$$K = \frac{3\pi d^4 EG}{32L_1(2GL_1^2+3EL_2^2)} \qquad (d)$$

By the energy method, the bending strain energy of member AB is [Eq.(5.13)]

$$U_B = \int \frac{M_x^2}{2EI_x}dz = \int_0^{L_1} \frac{Pz}{2E(\frac{\pi}{64}d^4)}dz = \frac{32P^2L_1^3}{3\pi Ed^4} \qquad (e)$$

The torsion strain energy of member AB is [Eq.(5.16)]

$$U_T = \int \frac{T^2}{2GJ}dz = \int_0^{L_1} \frac{(PL_2)^2}{2G(\frac{\pi}{32}d^4)}dz = \frac{16P^2L_1L_2^2}{\pi Gd^4} \qquad (f)$$

By Eqs.(e) and (f), the total strain energy of the system is, since member BC is rigid,

$$U = U_B + U_T = \frac{32P^2L_1^3}{3\pi Ed^4} + \frac{16P^2L_1L_2^2}{\pi Gd^4} \qquad (g)$$

Then, by Eq.(g) and Castigliano's theorem [Eq.(5.17)]

$$\Delta = \frac{\partial U}{\partial P} = P\left[\frac{32L_1(2GL_1^2+3EL_2^2)}{3\pi d^4 EG}\right]$$

or

$$P = \frac{3\pi d^4 EG\Delta}{32L_1(2GL_1^2+3EL_2^2)} = K\Delta$$

or

$$k = \frac{3\pi d^4 EG}{32L_1(2GL_1^2+3EL_2^2)} \qquad (h)$$

Equations (d) and (h) are identical.

6.12 The maximum shear stress in the shaft AB occurs at A (Fig. a) on the top (and the bottom) of the shaft, and is due to the bending stress determined by the flexure formula

$$\sigma = \pm \frac{M(d/2)}{I_x} \qquad (a)$$

and to the torsional stress

$$\tau = \frac{T(d/2)}{J} \qquad (b)$$

where (Fig. a)

$$M = PL_1,\; T = PL_2 \quad (c)$$

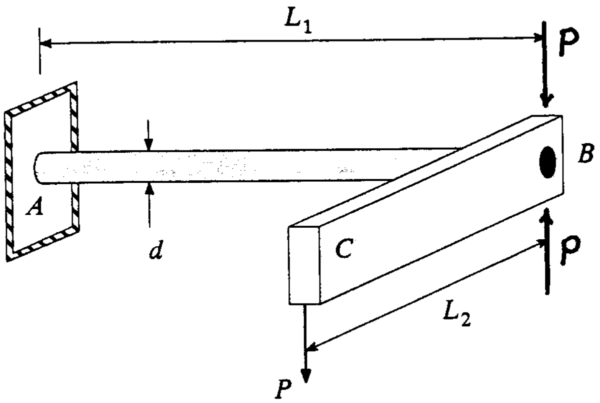


Figure a

With $L_1 = 1.50\ m$, $L_2 = 0.80\ m$, and $d = 0.050\ m$, Eqs. (a), (b), and (c) yield

$$\sigma = P\frac{(1.50)(0.050/2)}{\frac{\pi}{64}(0.050^4)} = 0.122231\,P$$

$$(d)$$

$$\tau = P\frac{(0.80)(0.050/2)}{\frac{\pi}{32}(0.050^4)} = 0.032595\,P$$

The stresses σ and τ acting on a surface element on the top of the shaft at A are shown in Fig. b.

By Fig. b, and Eqs. (d) and (4.45), with $\tau_{max} = 45\ MPa$, we have

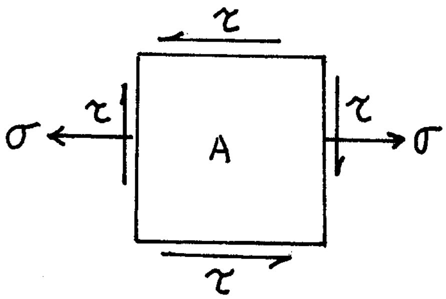


Figure b

$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = 45\ MPa \qquad (e)$$

By Eqs. (d) and (e), we obtain

$$P = 650\ N \qquad (f)$$

(cont.)

6.12 cont. By beam theory, the deflection of B of member AB (Fig. a) due to P is

$$\delta_B = \frac{1}{3}\frac{PL_1^3}{EI_x} = \frac{1}{3}\frac{(650)(1.50)^3}{(200\times10^9)(\frac{\pi}{64})(0.05^4)} = 0.01192\text{ m} \quad (g)$$

Member AB undergoes a rotation at B due to torque T of

$$\psi = L_1\theta = \frac{TL_1}{GJ} = \frac{(650\times0.800)(1.50)}{(77.5\times10^9)(\frac{\pi}{32})(0.05^4)} = 0.016403\text{ rad} \quad (h)$$

The deflection δ_ψ at C (Fig. a) due to the rotation of the rigid member BC is, with Eq. (h),

$$\delta_\psi = L_2\,\psi = (0.800)(0.016403) = 0.013122\text{ m} \quad (i)$$

So, by Eqs. (g) and (i), the downward deflection at C is

$$\Delta = \delta_B + \delta_\psi = 0.0250\text{ m} \quad (j)$$

Note also, by the solution of Problem 6.11, $P = K\Delta$ where, with the given data,

$$K = \frac{3\pi d^4 EG}{32L_1(2GL_1^2 + 3EL_2^2)} = 25959\ \frac{\text{N}}{\text{m}}$$

with P = 650 N, we find

$$\Delta = \frac{P}{K} = \frac{650}{25959} = 0.0250\text{ m}$$

as in Eq. (j).

6.13 The shear forces in the bolts of the coupling must resist the torque T (Fig. a, where 4 bolts are shown). For n bolts, equilibrium of torques yields

$$\sum \text{Torques} = T - (n\tau A)R = 0 \qquad (a)$$

where

$$A = \frac{\pi}{4}D^2 \qquad (b)$$

is the cross-sectional area of each bolt and τ is average shear stress in each bolt. By Eqs. (a) and (b), we obtain for n bolts

$$T = \frac{\pi}{4} n R D^2 \tau$$

or

$$\tau = \frac{4T}{\pi n R D^2}$$

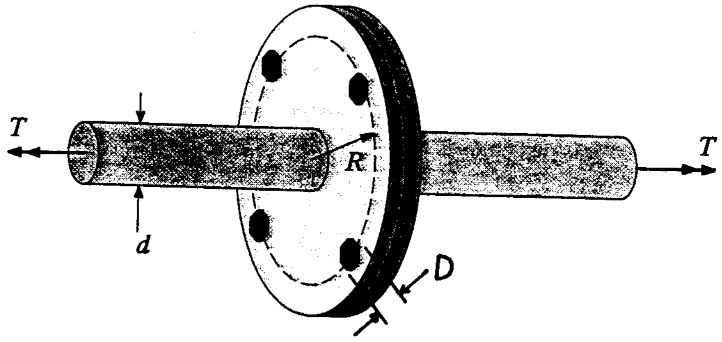


Figure a

6.14 In Fig. a, Prob. 6.13, let $R = 100$ mm and $D = 15$ mm. The shaft transmits power at $f = 5$ Hz, and the allowable shear stress in each bolt and the shaft is $\tau = 45$ MPa. To determine the minimum required diameter d of the shafts and the power that can be transmitted, we may proceed as follows:

(cont.)

6.14 cont. By equilibrium of torques resisted by the four bolts and exerted by the shafts, we have

$$\Sigma \text{Torques} = T - 4(\tau_{max} A)R = 0$$

where $A = \pi D^2/4$. Therefore, the maximum allowable torque in the shaft is, for $\tau_{max} = 45$ MPa,

$$T = 4(45 \times 10^6)\left[\frac{\pi}{4}(0.0150^4)\right](0.100) = 3180.9 \text{ N·m} \quad (a)$$

By the torsion formula for a solid circular shaft,

$$\tau_{max} = \frac{T(d/2)}{J} = \frac{T(d/2)}{\frac{\pi}{32}d^4} \qquad (b)$$

By Eqs. (a) and (b),

$$d^3 = \frac{16T}{\pi \tau_{max}} = \frac{16(3180.9)}{\pi(45 \times 10^6)} = 0.000360 \text{ m}^3$$

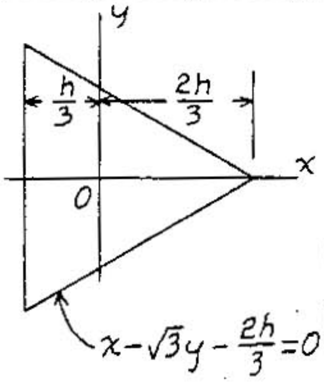
Therefore, the minimum required diameter of the shafts is

$$d = 0.07114 \text{ m} = 71.14 \text{ mm}$$

The power P that can be transmitted with this diameter at a frequency $f = 5$ Hz and a torque $T = 3180.9$ N·m is

$$P = T\omega = T(2\pi f) = 3180.9(2\pi)(5) = 99.9 \text{ kW}$$

6.15



$$\phi = B\left(x-\sqrt{3}y-\frac{2h}{3}\right)\left(x+\sqrt{3}y-\frac{2h}{3}\right)\left(x+\frac{h}{3}\right)$$

$$= B\left(x^3-hx^2-3xy^2-hy^2+\frac{4h^3}{27}\right)$$

$$\frac{\partial\phi}{\partial x} = B(3x^2-2hx-3y^2);\quad \frac{\partial^2\phi}{\partial x^2} = B(6x-2h)$$

$$\frac{\partial\phi}{\partial y} = B(-6xy-2hy);\qquad \frac{\partial^2\phi}{\partial y^2} = B(-6x-2h)$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = -2G\theta = B(-4h);\quad B = \frac{G\theta}{2h}$$

$$\sigma_{zy} = -\frac{\partial\phi}{\partial x} = \underline{-\frac{G\theta}{2h}(3x^2-2hx-3y^2)}$$

Along x-axis $y=0$; therefore

$$\sigma_{zy(y=0)} = \underline{\frac{G\theta}{2h}(2hx-3x^2)}$$

6.16

$$T = 2\iint\phi\,dx\,dy = \frac{2G\theta}{h}\int_{-\frac{h}{3}}^{\frac{2h}{3}}\int_{0}^{\left(\frac{2h}{3\sqrt{3}}-\frac{x}{\sqrt{3}}\right)}\left(x^3-hx^2-3xy^2-hy^2+\frac{4h^3}{27}\right)dx\,dy$$

$$= \frac{2G\theta}{9\sqrt{3}h}\int_{-\frac{h}{3}}^{\frac{2h}{3}}\left(-6x^4+10hx^3-4h^2x^2-\frac{8}{9}h^3x+\frac{16}{27}h^4\right)dx$$

$$= \frac{G\theta h^4}{15\sqrt{3}}$$

$$\tau_{max} = \sigma_{zy(y=0)(x=-\frac{h}{3})} = \frac{G\theta h}{2}$$

$$\theta = \frac{15\sqrt{3}\,T}{Gh^4} = \underline{\frac{2\tau_{max}}{Gh}}$$

$$\tau_{max} = \underline{\frac{15\sqrt{3}\,T}{2h^3}}$$

6.17 The maximum torque that can be applied to the member is obtained from the relation between torque and shear stress, namely

$$(S.F.)T_{max} = \tfrac{1}{2}\pi b h^2 \tau_{max} = \tfrac{1}{2}\pi b h^2 \frac{Y}{2} \qquad (a)$$

With the given data, Eq. (a) yields

$$T_{max} = \frac{\pi}{4}\frac{bh^2 Y}{(S.F.)} = \frac{\pi}{4}\frac{(25)(15)^2(400)}{1.85} = 955.2\ N\cdot m$$

6.18 The stepped shaft is shown in Fig. a. It is made of steel ($G = 77 GPa$) and has a yield stress $Y = 450$ MPa. Torque T is applied at section B. Ends A and C are fixed. Let $L_1 = 1.0 m$, $L_2 = 1.27 m$, $d_1 = 25.4 mm$, and $d_2 = 19.05 mm$. (a) Determine T to cause yielding. (b) Determine the associate rotation ψ_B at B.

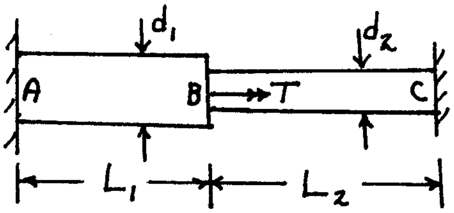


Figure a

(a) With the given data the polar moments of inertia for sections AB and BC are, respectively,

$$J_1 = \frac{\pi d_1^4}{32} = \frac{\pi (25.4)^4}{32} = 408634\ mm^4,\ J_2 = \frac{\pi d_2^4}{32} = \frac{\pi (19.05)^4}{32} = 12929.4\ mm^4$$

By equilibrium, the total torque $T = T_1 + T_2$, where T_1, T_2 are the torques in sections AB and BC, respectively. The angle of rotation at section B is, by Eq. (6.12),

$$\psi_B = \frac{T_1 L_1}{G J_1} = \frac{T_2 L_2}{G J_2} \text{ or } \frac{T_1 (1.0)}{408634} = \frac{T_2 (1.27)}{12929.4}$$

(cont.)

6.18 cont.

Therefore, $T_2 = 0.24914\,T_1$ and $T_1 = 0.8006T$, $T_2 = 0.1994T$.

The maximum shear stress in section AB is

$$\tau_{max(1)} = \frac{T_1 c_1}{J_1} = \frac{16T_1}{\pi d_1^3} = \frac{16(0.8006T)}{\pi(25.4)^3} = 0.00024882T$$

T in N.mm

The maximum shear stress in section BC is

$$\tau_{max(2)} = \frac{T_2 c_2}{J_2} = \frac{16T_2}{\pi d_2^3} = \frac{16(0.1994T)}{\pi(19.05)^3} = 0.00014690T$$

Therefore, yield will occur first in section AB.

At yield, $\tau_Y = 450\text{ MPa} = 450\frac{\text{N}}{\text{mm}^2} = 0.00024882T$,

or $T = 1.809$ kN·m

6.19 Let $\phi = A\left(\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1\right)$. The constant A is determined by the Condition (Boresi and Chong, 2000)

$$T = \iint_R 2\phi\, dA + 2K_1 A_1$$

where T is the twisting moment, K_1 is the value of ϕ at the inner elliptic surface, R is the area bound by the inner and outer ellipses, and A_1 is the area bound by the inner ellipse (Figure A)

By Fig. A, $A_1 = \pi h b k^2$. By the equation for ϕ, $K_1 = A(k^2 - 1)$. Therefore,

$$T = 8A\int_0^{kb}\int_{\frac{h}{b}\sqrt{k^2b^2-y^2}}^{\frac{h}{b}\sqrt{b^2-y^2}} \left(\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1\right) dx\,dy$$

$$+ 2A\pi h b k^2(k^2 - 1)$$

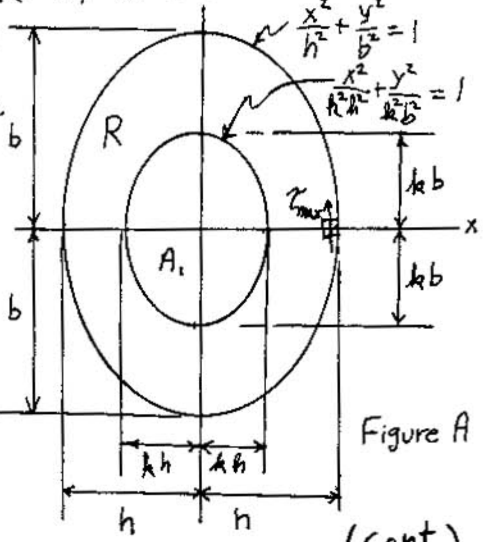

$\frac{x^2}{h^2}+\frac{y^2}{b^2}=1$
$\frac{x^2}{k^2h^2}+\frac{y^2}{k^2b^2}=1$
R
A_1
τ_{max}
x
b
b
kb
kb
kh
kh
h
h

Figure A

(cont.)

6.19 cont.

Integration yields (see a good table of integrals)

$$A = -\frac{T}{\pi h b(1-k^4)} . \text{ Thus, } \phi = -\frac{T}{\pi h b(1-k^4)}\left(\frac{x^2}{h^2}+\frac{y^2}{b^2}-1\right)$$

By Eq. (6.38) of the text, $\theta = -\frac{1}{2G}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) = \frac{(h^2+b^2)T}{\pi h^3 b^3(1-k^4)G}$

By Eqs. (6.43) and (6.44),

$$\sigma_{zx} = \frac{\partial\phi}{\partial y} = -\frac{2Ty}{\pi h b^3(1-k^4)}$$

$$\sigma_{zy} = -\frac{\partial\phi}{\partial x} = \frac{2Tx}{\pi h^3 b(1-k^4)}$$

With $b > h$,

$$\tau_{max} = \sigma_{zy}\Big|_{x=h} = \frac{2T}{\pi h^2 b(1-k^4)}$$

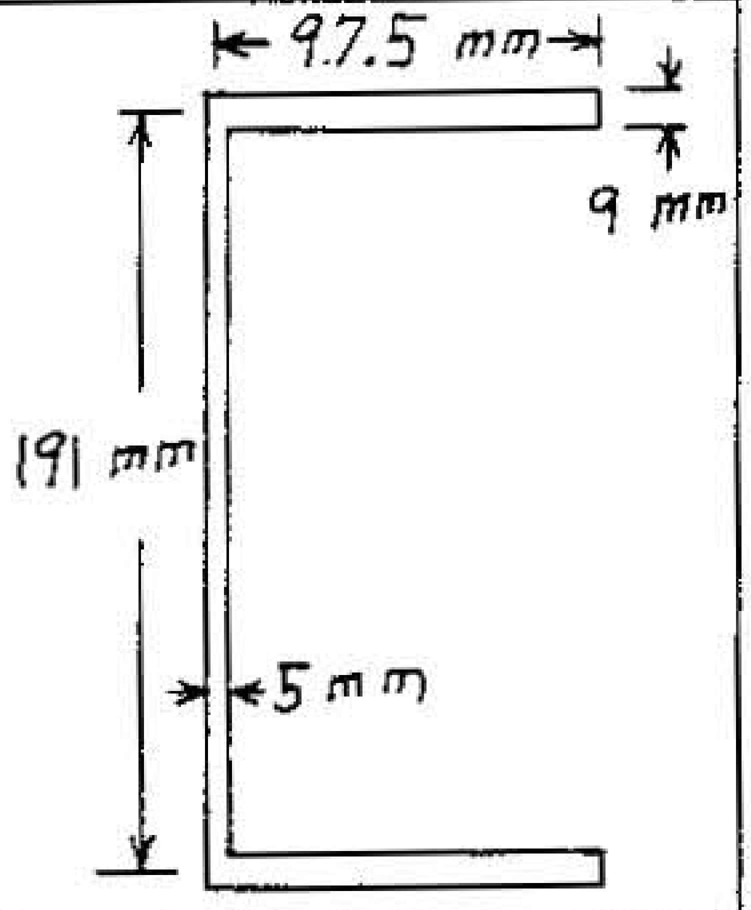
6.20

$$J = \frac{1}{3}\sum_1^3 (2b_i)(2h_i)^3 = \frac{1}{3}\left[2(97.5)(9)^3 + 191(5)^3\right] = 55{,}300 \text{ mm}^4$$

$$\tau_{max} = \frac{2T h_{max}}{J} = \frac{2(600{,}000)(4.5)}{55{,}300} = \underline{97.6 \text{ MPa}}$$

$$\theta = \frac{T}{GJ} = \frac{600{,}000}{77{,}500(55{,}300)} = 0.000140 \text{ rad/mm}$$

$$= \underline{0.140 \text{ rad/m}}$$

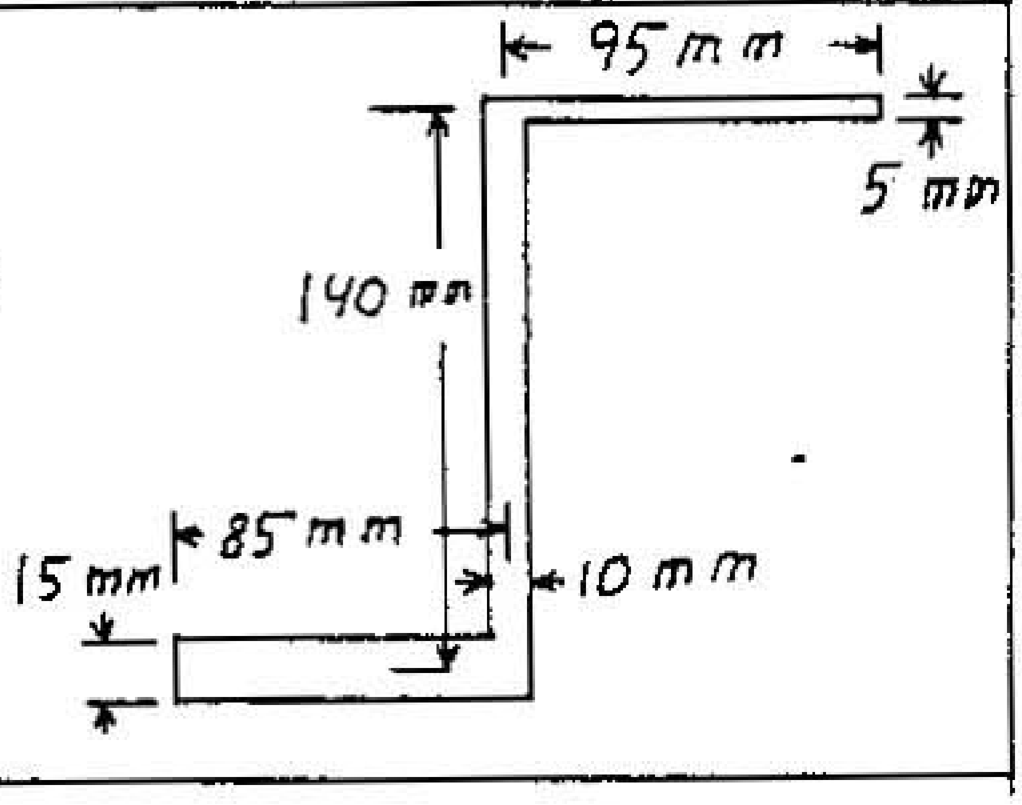


6.21

$$J = \frac{0.91}{3}\sum_1^3 (2b_i)(2h_i)^3$$

$$= \frac{0.91}{3}\left[95(5)^3 + 140(10)^3 + 85(15)^3\right] = 133{,}100 \text{ mm}^4$$

$$T = \frac{\tau_{max} J}{2h_{max}} = \frac{75(133{,}100)}{2(7.5)} = \underline{665.4 \text{ N.m}}$$



6.22 By Example 6.7, a torque $T = 5000$ N·m is applied to a W 760×220 I-Section (Fig. a). Determine the maximum shear stress, its location, and the angle of twist per unit length for the following cases (Ignore the fillets and stress concentrations):

(a) use Eq. (6.60) to calculate J.

(b) use Eq. (6.62) to calculate J.

(c) Use the value $J = 6.077 \times 10^6$ mm listed in AISC, 1997.

(d) Compare results obtained in Parts (a), (b), and (c) to the results obtained in Example 6.7. Comment on the differences.

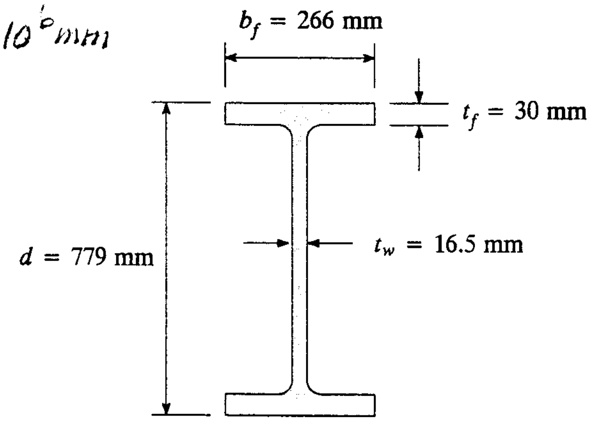


Figure a

(a) By Eq. (6.60),

$$J = \frac{1}{3}(2b)(2h)^3 \qquad (a)$$

For the flanges (Fig. a), $2b = 266$ mm and $2h = 30$ mm. Therefore, for two flanges

$$J_f = \frac{2}{3}(266)(30)^3 = 4,788,000 \text{ mm}^4 \qquad (b)$$

For the web, $2b = d - 2t_f = 719$ mm and $2h = 16.5$ mm. Then, for the web,

$$J_w = \frac{1}{3}(719)(16.5)^3 = 1,076,600 \text{ mm}^4 \qquad (c)$$

So, the torsional constant for the section is, by Eqs (b) and (c),

$$J = J_w + J_f = 5.865 \times 10^{-6} \text{ m}^4 \qquad (d)$$

(cont.)

6.22 cont. Hence, with this value of J, the maximum shear stress is (located in the flanges)

$$\tau_{max} = \frac{2T\,h_{max}}{J} = \frac{2(5000)(0.015)}{5.865\times10^{-6}} = 25.58\,\text{MPa} \quad (e)$$

and the angle of twist per unit length is

$$\theta = \frac{T}{GJ} = \frac{5000}{(200\times10^9)(5.865\times10^{-6})} = 0.00426\ \text{rad/m} \quad (f)$$

(b) By Eq.(6.62),

$$J = \frac{C}{3}\sum_{i=1}^{3}(2b_i)(2h_i)^3$$

Since for the flanges $b/h = 8.867 < 10$, we take $C = 0.91$. Hence, with Eq.(d), the torsional constant is

$$J = 0.91\,(5.865\times10^{-6}) = 5.337\times10^{-6}\ m^4$$

and as in Part a,

$$\tau_{max} = \frac{2(5000)(0.015)}{5.337\times10^{-6}} = 28.11\ \text{MPa}$$

$$(g)$$

$$\theta = \frac{5000}{(200\times10^9)(5.337\times10^{-6})} = 0.00468\ \text{rad/m}$$

(c) With $J = 6.077\times10^{6}\ mm = 6.077\times10^{-6} m$ we have

$$\tau_{max} = \frac{2(5000)(0.015)}{6.077\times10^{-6}} = 24.68\ \text{MPa}$$

$$(h)$$

$$\theta = \frac{5000}{(200\times10^9)(6.077\times10^{-6})} = 0.00411\ \text{rad/m}$$

(d) By comparing the results to those of Example 6.7 ($\tau_{max} = 27.27$ MPa and $\theta = 0.00454$ rad/m) we see that the AISC value of J yields the more conservative

(cont.)

6.22 cont values of τ_{max} and θ (smaller values). The results of Part b compare most nearly to those of Example 6.7, predicting slightly larger values of τ_{max} and θ. The values of Part a are somewhat less than those of Example 6.7. Note that for the largest (Part b) and the smallest (Part c) values the difference is

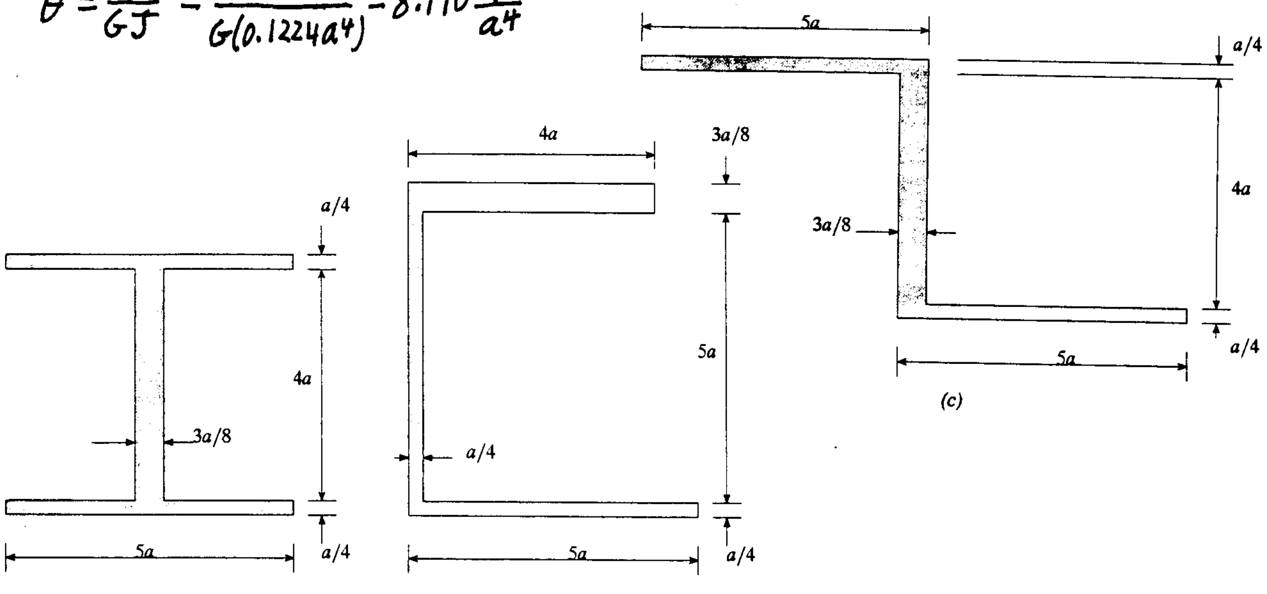
$$\left(\frac{28.11 - 24.68}{24.68}\right) \times 100 = 13.9\%$$

a rather significant difference. The results of Example 6.7 appear to be reasonable estimates, provided code requirements do not specify the AISC value of J.

6.23 For each section, $b/h > 10$. Therefore, for each section, J is equal to $\frac{1}{3}[2(5a)(\frac{1}{4}a)^3 + 4a(\frac{3a}{8})^2] = 0.1224\,a^4$. Hence,

$$\tau_{max} = \frac{2T h_{max}}{J} = \frac{2T(3a/16)}{0.1224a^4} = 3.064\frac{T}{a^3}$$

$$\theta = \frac{T}{GJ} = \frac{T}{G(0.1224a^4)} = 8.170\frac{T}{a^4}$$



(a)

(b)

(c)

6.24 By Fig. a, for the flange,

$$\frac{2b}{2h} = \frac{100}{12} = 8.33 < 10. \text{ Therefore, } C = 0.91,$$

and

$$J = \frac{0.91}{3}\left[176(10^3) + 2(100)(12^3)\right] = 158{,}220 \text{ mm}^4.$$

Then, by Eq. (6.63), with T = 1356 N·m

$$\tau_{max} = \frac{2Th_{max}}{J} = \frac{2(1356)(0.006)}{1.5822\times10^{-7}} = 102.8 \text{ MPa}$$

and

$$\theta = \frac{T}{GJ} = \frac{1356}{(26.7\times10^9)(1.5822\times10^{-7})} = 0.321 \text{ rad/m}$$

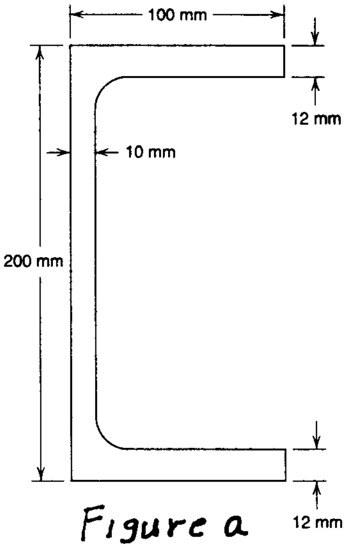


Figure a

6.25 By Fig. a, for the flange, $\frac{2b}{2h} = \frac{100}{12} = 8.33 < 10$. Therefore, $C = 0.91$, and

$$J = \frac{0.91}{3}\left[138(6^3) + 100(12^3)\right] = 61{,}457.8 \text{ mm}^4$$

(a) Since L = 1.2 m, T = 226 N·m and G = 79 MPa, Eq. (6.63) yields

$$\psi = \theta L = \frac{TL}{GJ} = \frac{226(1.2)}{(79\times10^9)(6.1458\times10^{-8})} = 0.0559 \text{ rad}$$

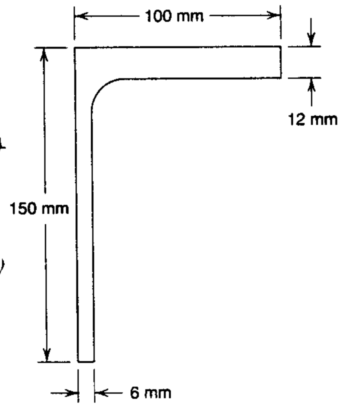


Figure a

(b) also, by Fig. a and Eq. (6.63),

$$\tau_{max} = \frac{2Th_{max}}{J} = \frac{2(226)(0.012)}{6.1458\times10^{-8}} = 88.255 \text{ MPa}$$

$\boxed{6.26}$ assume $c = 1$ for both sections (Figs. a and b). Ordinarily $t << b$. Therefore, by Eq. (6.60),

$$J_a = \frac{1}{3}\left[2bt^3 + 2(b-t)t^3\right] \approx \frac{4}{3}bt^3$$

and

$$J_b = \frac{1}{3}\left[2bt^3 + (b-t)(2t)^3\right] \approx \frac{10}{3}bt^3$$

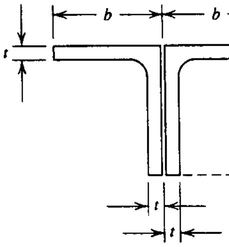


Figure a

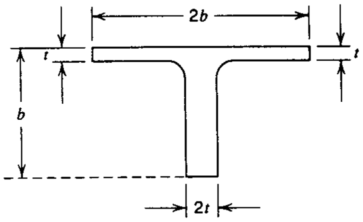


Figure b

(a) For $\theta_a = \theta_b$, $\frac{T_a}{GJ_a} = \frac{T_b}{GJ_b}$ or $T_a = \frac{J_a}{J_b} T_b = 0.40 T_b$

(b) For $(\tau_a)_{max} = (\tau_b)_{max}$, $\frac{2T_a(h_a)_{max}}{J_a} = \frac{2T_b(h_b)_{max}}{J_b}$

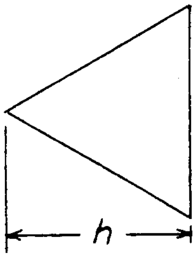
or $\quad T_a = \frac{J_a}{J_b}\frac{(h_b)_{max}}{(h_a)_{max}} T_b = 0.80 T_b$

6.27

The failure torque is $(SF)T = k_2(2h)^3\tau_{max} = k_2(2h)^3\frac{Y}{\sqrt{3}}$

$$T = \frac{0.208(25)^3(380)}{2.00\sqrt{3}} = \underline{356.5\ \mathrm{N.m}}$$

6.28



$A = 42^2 = 1764\ \mathrm{mm}^2 = \pi r^2(\text{circle}) = 0.5774\ h^2(\text{triangle})$

$r = 23.70$ mm for circle

$h = 55.27$ mm for equilateral triangle

$$\tau_{(square)} = \frac{T}{k_2(2h)^3} = \frac{1{,}000{,}000}{0.208(42)^3} = \underline{64.89\ \mathrm{MPa}}$$

$$\tau_{(circle)} = \frac{Tc}{J} = \frac{1{,}000{,}000(23.70)(2)}{\pi(23.70)^4} = \underline{47.82\ \mathrm{MPa}}$$

$$\tau_{(triangle)} = \frac{15\sqrt{3}\,T}{2h^3} = \frac{15\sqrt{3}(1{,}000{,}000)}{2(55.27)^3} = \underline{76.94\ \mathrm{MPa}}$$

6.29

$$\theta_{(square)} = \frac{T}{k_1(2h)^4G} = \frac{1{,}000{,}000}{0.141(42)^4(77{,}500)} = 0.0000294\ \mathrm{rad/mm} = \underline{0.0294\ \mathrm{rad/m}}$$

$$\theta_{(circle)} = \frac{T}{JG} = \frac{1{,}000{,}000(2)}{\pi(23.70)^4(77{,}500)} = 0.0000260\ \mathrm{rad/mm} = \underline{0.0260\ \mathrm{rad/m}}$$

$$\theta_{(triangle)} = \frac{15\sqrt{3}\,T}{h^4G} = \frac{15\sqrt{3}(1{,}000{,}000)}{(55.27)^4(77{,}500)} = 0.0000359\ \mathrm{rad/mm} = \underline{0.0359\ \mathrm{rad/m}}$$

6.30

Assume failure in 2 m length (rectangle).

$$T(40\times 60) = T_1 + T_2 = k_2(2b)(2h)^2\tau_{max} = 0.231(60)(40)^2(45) = 997.9\ \mathrm{N.m}$$

$$T(30\times 40) = 997.9 - 350.0 = \underline{647.9\ \mathrm{N.m}}$$

Assume failure in 1m length (30x40).

$$T(30\times 40) = k_2(2b)(2h)^2\tau_{max} = 0.223(40)(30)^2(45) = 361.26\ \mathrm{N.m}$$

$$T(40\times 60) = 350 + 361.26 = 711.26$$

Failure occurs in 1 m length.

$$\text{Angle of twist} = \frac{T(40\times 60)\ L(40\times 60)}{k_1(\text{rectangle})(2b)(2h)^3G} + \frac{T(30\times 40)\ L(30\times 40)}{k_1(30\times 40)(2b)(2h)^3G}$$

$$= \frac{711{,}260\ (2000)}{0.196(60)(40)^3(27{,}100)} + \frac{361{,}260(1000)}{0.178\ (30)^3(27{,}100)(40)}$$

$$= 0.06974 + 0.06934$$

$$= \underline{0.13908\ \mathrm{rad}}$$

6.31 (a) By Eq. (6.64),

$$\tau_{max} = \frac{T}{k_2(2b)(2h)^2} \qquad (a)$$

For $b/h = 38.1/12.7 = 3$, by Table 6.1, $k_2 = 0.267$. Therefore, with $T = 135.7$ N·m, $2b = 38.1$ mm, and $2h = 12.7$ mm, Eq. (a) yields

$$\tau_{max} = \frac{135.7}{0.267(0.0381)(0.0127)^2} = 82.7 \text{ MPa}$$

Since τ_{max} is equal to the yield stress, yield is just reached at the centers of the long sides (Fig. a).

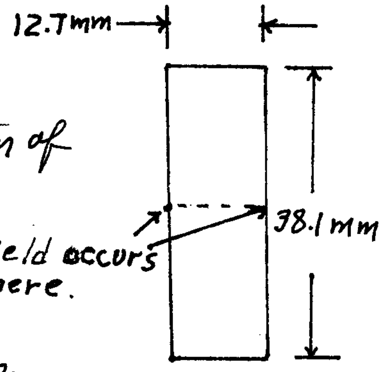


Figure a

(b) To obtain the shear stress at the center of the short side, interchange (2b) and (2h) in Eq. (6.64). Thus at the center of the short side, by Eq. (a),

$$\tau = \frac{135.7}{0.267(0.0127)(0.0381)^2} = 27.6 \text{ MPa}$$

6.32 By Eq. (6.64), with $2h = 19$ mm, $\tau_{max} = \frac{2}{3}\tau_Y = \frac{2}{3}(207) = 138$ MPa, and $T = 565$ N·M, we have

$$\tau_{max} = 138 \text{ MPa} = \frac{565}{k_2\left(\frac{2b}{1000}\right)\left(\frac{19}{1000}\right)^2}; \text{ or } \quad 2bk_2 = 11.3413 \quad (a)$$

Equation (a) may be solved by trial and error and interpolation as follows:

Table P6.32

2b	b/h	k_2 (By Table 6.1)	$2bk_2$
19	1.00	0.208	3.952
38	2.00	0.246	9.348
57	3.00	0.267	15.219
47.5	2.50	0.256	12.160
43.7	2.30	0.252	11.012
44.5	2.34	0.253	11.250
44.8	2.36	0.2532	11.343

By Table P6.32, Eq. (a) is satisfied approximately by $2b = 44.8$ mm, or by $b \approx 22.4$ mm.

6.33 For the rectangular cross section

$$\tau_{max} = \frac{T}{k_2(2b)(2h)^2} = \frac{T}{8k_2 k h^3} \; ; \; k = \frac{b}{h}.$$

For the circular cross section

$$\tau_{max} = \frac{Tc}{J} = \frac{T(d/2)}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

Therefore, $\dfrac{16}{\pi d^3} = \dfrac{1}{8k_2 k h^3}$ or $d^3 = 40.744\, h^3 k k_2$

or

$$d = 3.441\, h (k k_2)^{1/3} \qquad (a)$$

For a given value of k, Eq. (a) may be solved by trial and error (see Prob. 6.32)

6.34 Given $d = 51$ mm and $2b = 38.1$ mm ($b = 19.05$ mm), by Eq. (6.64) assuming that $b > h$, we have

$$\tau_{max} = \frac{T}{k_2(2b)(2h)^2} = \frac{T}{8k_2(19.05)h^2} = \frac{16T}{\pi d^3}.$$

Therefore,

$$h^2 k_2 = \frac{\pi(51)^3}{(16)(8)(19.05)} = 170.91 \qquad (A)$$

Equation (A) may be solved by trial and error. Try $h = 19.05$. Then $b/h = 1$ and by Table 6.1, $k_2 = 0.208$. Hence, $k_2 h^2 = 75.48 < 170.91$. Therefore, $h > b$. Consequently τ_{max} is obtained by Eq. (6.64) with b and h interchanged. Also b and h must be interchanged in Table 6.1. Thus,

$$\tau_{max} = \frac{T}{k_2(2h)(2b)^2} = \frac{16T}{\pi d^3} \; ; \; h k_2 = \frac{\pi(51)^3}{16(38.1)^2(2)} = 8.971 \qquad (B)$$

Solving Eq. (B) by trial and error, we have

(cont.)

6.34 cont.

h	h/b	k_2 (by Table 6.1)	hk_2
19.05	1.00	0.208	3.962
28.58	1.50	0.231	6.602
31.10	2.00	0.246	9.373
36.75	1.929	0.2439	8.962
36.77	1.930	0.2439	8.969
36.78	1.931	0.2440	8.973

Therefore $h \simeq 36.775$ mm.

6.35 For equal areas (with $b = h$ for the square bar and d = diameter of the circular bar),

$$(2b)(2h) = 4h^2 = \frac{\pi d^2}{4} \quad \text{or} \quad d = 2.2567h$$

For $b = h$, $b/h = 1$, $k_2 = 0.208$ by Table 6.1.

Hence,

$$\tau_{max}(\text{square bar}) = \frac{T}{k_2(2b)(2h)^2} = \frac{T}{(0.208)(8)(h)^3} = 0.60096\frac{T}{h^3}$$

$$\tau_{max}(\text{circular bar}) = \frac{16T}{\pi d^3} = \frac{16T}{\pi(2.2567)^3(h)^3} = 0.44312\frac{T}{h^3}$$

$$\tau_{max}(\text{square bar}) = 1.356\,\tau_{max}(\text{circular bar})$$

6.36 (a) By Figs a and 6.15, $2h = 25$ mm and $2b = 50$ mm. Then, by Eq. (m) of Section 6.6,

$$J = 2.6042\times10^5\left[1 - \frac{96}{\pi^5}\sum_{n=1,3,5,\cdots}\frac{1}{n^5}\tanh n\pi\right] \simeq 1.787\times10^5\ \text{mm}^4 \quad (a)$$

or by Table 6.1, with $b/h = 2$, $J = k_1(2b)(2h)^3 = 1.789\times10^5\ \text{mm}^4$

(b) By Eqs. (k) of Section 6.6, with $2h = 25$ mm, $2b = 50$ mm, and $G\theta = T/J$ [by Eq. (6.63)], we have

(Cont.)

$\boxed{6.36}$

$$\sigma_{zx} = -\frac{200}{\pi^2}\frac{T}{J}\left[\sum_{m=1,3,5,\cdots} \frac{(-1)^{(m-1)/2}\cos\frac{n\pi x}{25}\sinh\frac{n\pi y}{25}}{n^2\cosh n\pi}\right]$$

$$\sigma_{zy} = \frac{2Tx}{J} - \frac{200}{\pi^2}\frac{T}{J}\left[\sum_{n=1,3,5,\cdots} \frac{(-1)^{(n-1)/2}\sin\frac{n\pi x}{25}\cosh\frac{n\pi y}{25}}{n^2\cosh n\pi}\right] \qquad (b)$$

where by Eq(a), with $T = 100$ N.m,

$$\frac{T}{J} = 0.5596 \text{ N/mm}^3 \qquad (c)$$

For point A ($x = 12.5$ mm, $y = 0$, Fig. a), Eqs. (b) and (c) yield

$\sigma_{zx} = 0$

$$\sigma_{zy} = 0.5596\left\{25 - \frac{200}{\pi^2}\left[\frac{1}{\cosh\pi} + \frac{1}{9\cosh 3\pi} + \frac{1}{25\cosh 5\pi} + \right]\right\}$$

or

$\sigma_{zy} = 13.89$ MPa

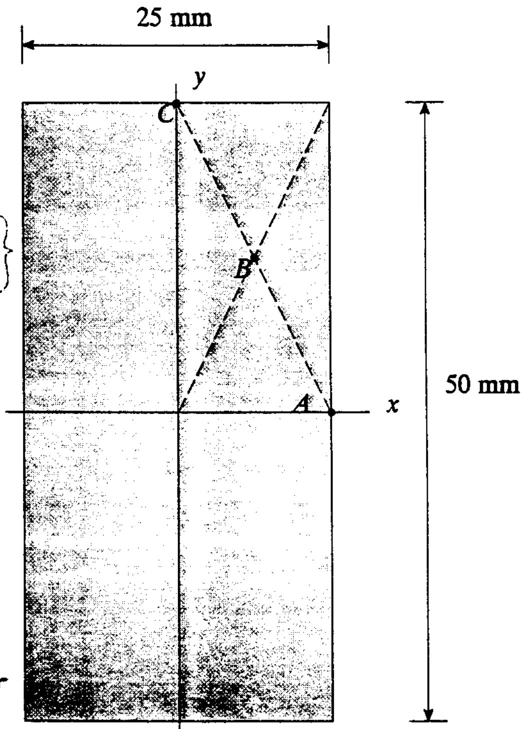


Figure a

For point B, ($x = 6.25$ mm, $y = 12.5$mm, Fig. a), Eqs. (b) and (c) yield

$$\sigma_{zx} = -\frac{200}{\pi^2}(0.5596)\left[\frac{\cos\frac{\pi}{4}\sinh\frac{\pi}{2}}{\cosh\pi} - \frac{\cos\frac{3\pi}{4}\sinh\frac{3\pi}{2}}{9\cosh 3\pi} + \frac{\cos\frac{5\pi}{4}\sinh\frac{5\pi}{2}}{25\cosh 5\pi} \text{----}\right]$$

or

$\sigma_{zx} \approx -1.744$ MPa

$$\sigma_{zy} = 12.5(0.5596) - \frac{200}{\pi^2}(0.5596)\left[\frac{\sin\frac{\pi}{4}\cosh\frac{\pi}{2}}{\cosh\pi} - \frac{\sin\frac{3\pi}{4}\cosh\frac{3\pi}{2}}{9\cosh 3\pi} + \frac{\sin\frac{5\pi}{4}\cosh\frac{5\pi}{2}}{25\cosh 5\pi} \text{---}\right]$$

or

$\sigma_{zy} \approx 5.267$ MPa

(cont.)

6.36 cont. For point C ($x=0$, $y=25$ mm; Fig. a), Eqs. (b) and (c) yield

$$\sigma_{zx} = -\frac{200}{\pi^2}(0.5596)\left[\tanh\pi - \frac{1}{9}\tanh 3\pi + \frac{1}{25}\tanh 5\pi - \frac{1}{49}\tanh 7\pi \cdots\right]$$

or

$$\sigma_{zx} \approx 10.40 \text{ MPa}$$

and

$$\sigma_{zy} = 0$$

6.37 By Fig. a, the torsional constant for the member is calculated as follows:

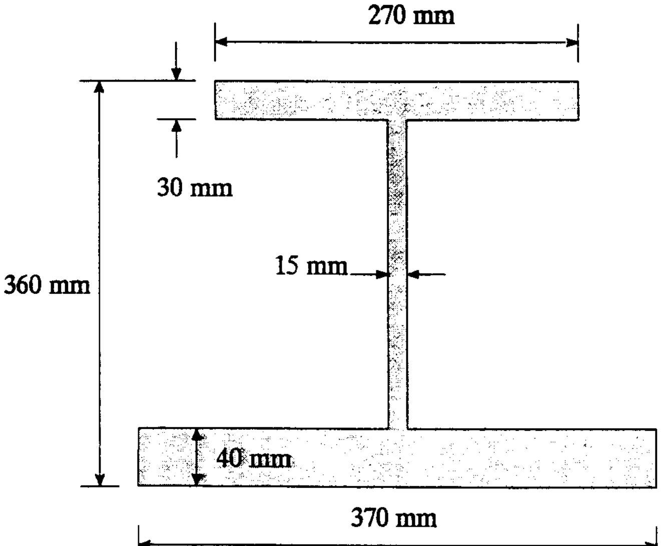


Figure a

For the top flange,

$$\frac{b}{h} = \frac{135}{15} = 9.0 < 10.$$

Therefore, by Table 6.1,

$$k_1 = 0.0309$$

and the torsional constant for the top flange is

$$J_{TF} = k_1(2b)(2h)^3$$
$$= 0.309(270)(30)^3$$

or

$$J_{TF} = 2,252,610 \text{ mm}^4.$$

For the bottom flange, $b/h = 185/20 = 9.25$; $k_1 = 0.310$.
So, the torsional constant for the bottom flange is

$$J_{BF} = k_1(2b)(2h)^3 = 0.310(370)(40)^3 = 7,340,800 \text{ mm}^4$$

For the web, $b/h = 145/7.5 = 19.33$. Therefore, $k_1 \approx 0.333$.
Then, the torsional constant for the web is

$$J_W = k_1(2b)(2h)^3 = 0.333(290)(15)^3 = 325,924 \text{ mm}^4$$

(cont.)

6.37 cont. Therefore, the torsional constant of the cross section is

$$J = J_{TF} + J_{BF} + J_W = 9{,}919{,}334 \text{ mm}^4 = 9.919\times10^{-6} \text{ m}^4 \quad (a)$$

(a) By Eqs. (a) and (6.63), with $T = 5000$ N·m and $h_{max} = 20$ mm $= 0.020$ m, the maximum shear stress is

$$\tau_{max} = \frac{2Th_{max}}{J} = \frac{2(5000)(0.020)}{9.919\times10^{-6}} = 20.16 \text{ MPa}$$

(b) By Eqs. (a) and (6.63), or (6.64), with $T = 5000$ N·m and $G = 200$ GPa, the angle of twist per unit length is

$$\theta = \frac{T}{GJ} = \frac{5000}{(200\times10^9)(9.919\times10^{-6})} = 0.00252 \text{ rad/m}$$

6.38 By Fig. a, $b/h = 100/50 = 2.0$. Hence, by Table 6.1, $k_1 = 0.229$ and $k_2 = 0.246$.

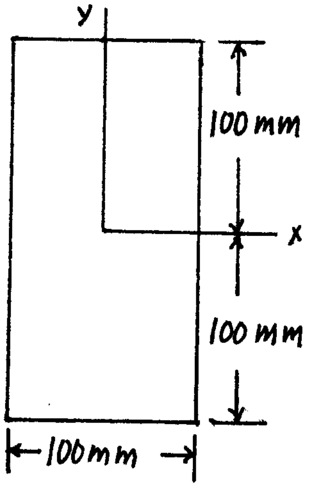


Figure a

(a) By Eq. (6.64), with $\tau_{max} = 20$ MPa,

$$\tau_{max} = \frac{T}{k_2(2b)(2h)^2} = \frac{T}{0.246(0.20)(0.10)^2} = 20\times10^6 \text{ N·m}$$

Hence,

$$T = 9840 \text{ N·m}$$

(b) By Eq. (6.64),

$$\theta = \frac{T}{k_1 G(2b)(2h)^3} = \frac{9840}{0.229(200\times10^9)(0.20)(0.10)^3}$$

or

$$\theta = 0.00107 \text{ rad/m}$$

6.39 The free-body diagram of an element of the member of length dz and arch length $r d\theta$ at the junction of thicknesses t_1 and t_2 is shown in Fig. a

By Fig. a, summation of forces in the z direction yields

$$\sum F_z = \tau_1 (t_1 dz) - \tau_2 (t_2 dz) = 0$$

or

$$\tau_1 t_1 = \tau_2 t_2$$

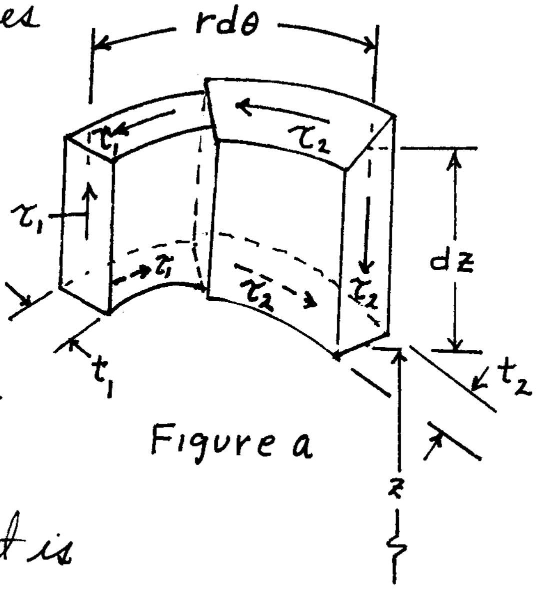


Figure a

Hence, the shear flow $q = \tau t$ is constant around the mean circumference. That is

$$q = \tau_1 t_1 = \tau_2 t_2 = \text{constant}$$

6.40 The cross section of the torsion member is shown in Fig. a.

(a) Given that $t/D = 0.10$, the length of the bar is $L = 1.50$ m and the shear stress at the mean diameter D is $\tau = 40$ MPa, we obtain the torque T, by Eq. (6.66), as

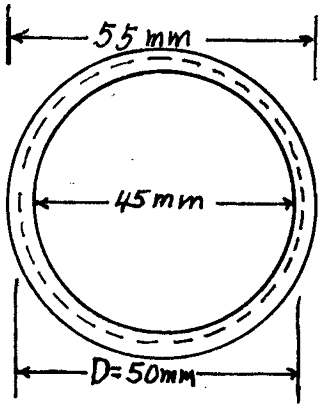


Figure a

$$T = 2A\tau t = 2\left(\frac{\pi}{4}D^2\right)(40\times10^6)(0.10D)$$

Then, since $D = 50$ mm, $T = 785.4$ N.m

also, by Eq. (6.67), with $G = 77.5$ GPa

$$\theta = \frac{1}{2GA}\int_0^{\pi D} \tau\, dl = \frac{(40\times10^6)(\pi)(0.050)}{2(77.5\times10^9)\left[\frac{\pi}{4}(0.050)^2\right]} = 0.0206 \text{ rad/m}$$

So, $\psi = \theta L = (0.0206)(1.5) = 0.0310$ rad

(cont.)

6.40 cont. By elasticity theory, Eq. (6.15) yields, with Fig. a,

$$T = \frac{\tau J}{r} = \frac{40\times 10^6}{0.025}\frac{\pi}{32}\left[(0.055)^4 - (0.045)^4\right] = 793.25 \text{ N}\cdot\text{m}$$

and by Eq. (6.12),

$$\theta = \frac{T}{GJ} = \frac{1}{GJ}\cdot\frac{\tau J}{r} = \frac{\tau}{Gr} = \frac{40\times 10^6}{(77.5\times 10^9)(0.025)} = 0.0206 \text{ rad/m}$$

Hence, $\psi = \theta L = 0.0206(1.5) = 0.0310$ rad

Therefore, for this case, the advanced mechanics of materials method agrees closely with the elasticity theory.

6.41 Two cross sections for a proposed design of an aircraft torsion member are shown in Figs. a and b. The allowable shear stress and the shear modulus are, respectively, $\tau_{max} = 60$ MPa and $G = 27.0$ GPa.

(a) The allowable torques and torsional stiffnesses of the two cross sections are, by Figs. a and b,

$$J_a = \frac{\pi}{32}\left[(0.041)^4 - (0.039)^4\right]$$
$$= 5.030\times 10^{-8} \text{ m}^4$$

$$GJ_a = (27.0\times 10^9)(5.030\times 10^{-8})$$
$$= 1358.0 \text{ N}\cdot\text{m}^2 \quad (a)$$

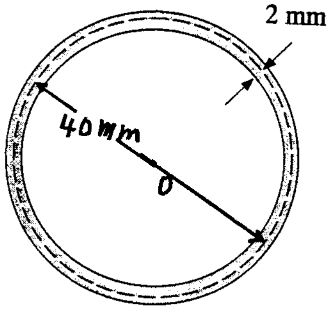


Figure a

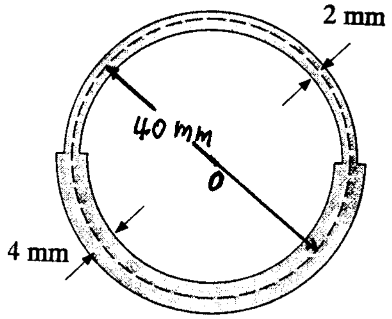


Figure b

Then, by Eq. (6.66), $T_a = 2A\tau t = 2\left(\frac{\pi}{4}D^2\right)\tau t = \frac{\pi}{2}(0.040)^2(60\times 10^6)(0.002)$
or

$$T_a = 301.59 \text{ N}\cdot\text{m} \quad (b)$$

Likewise,

$$J_b = \frac{1}{2}\frac{\pi}{32}\left[(0.041)^4 - (0.039)^4\right] + \frac{1}{2}\frac{\pi}{32}\left[(0.042)^4 - (0.038)^4\right]$$

or

$$J_b = 2.515\times 10^{-8} + 5.039\times 10^{-10} = 7.554\times 10^{-8} \text{ m}^4$$

(cont.)

6.41 cont. Hence, the torsional constant for Fig. b is

$$GJ_b = (27.0\times 10^9)(7.554\times 10^{-8}) = 2039.6 \text{ N}\cdot\text{m}^2 \qquad (c)$$

Since the shear flow is constant, by Fig. b, we have

$$q = \tau_2 t_2 = \tau_4 t_4 \qquad (d)$$

where subscripts 2 and 4 denote the 2 mm and 4 mm sections, respectively. Hence, by Eq. (6.66) and (d), the torque is

$$T_b = 2Aq = 2A\tau_2 t_2 \,(\text{or } 2A\,\tau_4 t_4) = 2\left(\tfrac{\pi}{4}D^2\right)(60\times 10^6)(0.002)$$

or with $D = 0.040$ m,

$$T_b = 301.59 \text{ N.m} \qquad (e)$$

Comparison of Eqs. (a) and (c) yields

$$\frac{J_b}{J_a} = 1.50 \qquad (f)$$

Similarly, by Eqs. (b) and (e),

$$\frac{T_b}{T_a} = 1 \qquad (g)$$

(b) By Eq. (d), $\tau_2 = 60$ MPa $> \tau_4$, when $T = 301.59$ N.m. Hence, with $t_2 = 2$ mm and $t_4 = 4$ mm, Eq. (d) yields

$$\tau_4 = \frac{t_2}{t_4}\tau_2 = \frac{2}{4}(60) = 30 \text{ MPa}$$

Hence,

$$\frac{\tau_4}{\tau_2} = 0.50 \qquad (h)$$

(cont.)

6.41 cont. (c) Equation (f) shows that the torsional stiffness of the cross section in Fig. b is 50% larger than that of the cross section in Fig. a. However, Eq. (g) shows that each section attains a maximum shear stress of 60 MPa when $T = 301.5$ N·m. So, there is no advantage, relative to allowable torque for the section of Fig. b. Hence, based upon the initial requirement of a 30% improvement in both torque and torsional stiffness, we would recommend the cross section of Fig. a. Note, however, that if the angle of twist is important, one would have to take that into account, since the cross section of Fig. b is stiffer and hence reduces the angle of twist.

6.42 By Fig a, the area enclosed by the mean circumference is

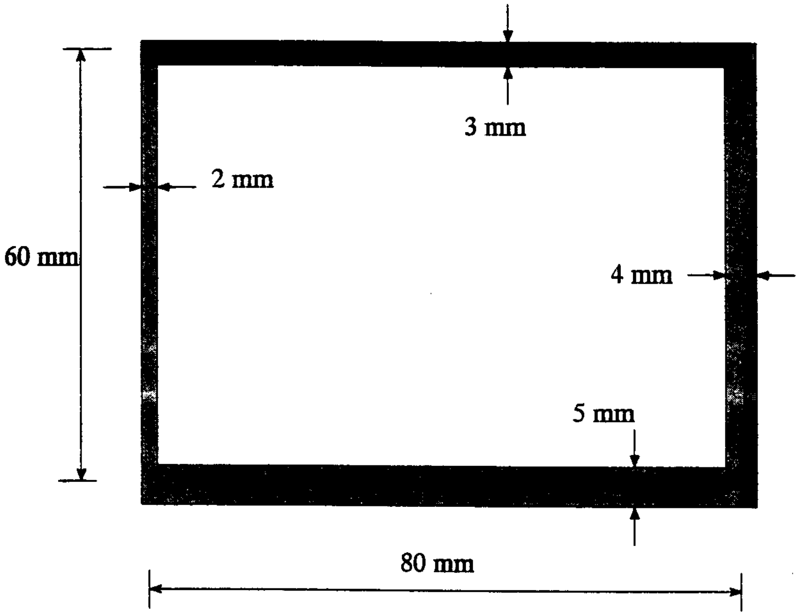


Figure a

$A = 0.080(0.060) = 0.0048$ m.

The shear flow $q = \tau t$ is constant in the wall of the member. Hence, the maximum shear stress occurs in the smallest wall thickness $t = 0.002$ m. Hence, with $t = 0.002$ m,

$$q = \tau_{max} t = 0.002\tau_{max}$$

Then, by Eq (6.66), with $T = 9500$ N·m,

$$T = 2Aq = 2(0.0048)(0.002\tau_{max}) = 9500 \text{ N·m}$$

or

$$\tau_{max} = 494.8 \text{ MPa}$$

6.43 (a) By Fig. a, the area enclosed by the mean perimeter is

$$A \simeq 0.100(0.150) + \frac{1}{2}\frac{\pi}{4}(0.100)^2 = 0.0189\ m^2 \qquad (a)$$

Also, since the shear flow $q = \tau t$ is constant, the maximum shear stress $\tau_{max} = 55\,MPa$ occurs in the 0.004 m thick wall. Hence, by Eq.(6.66),

$T = 2Aq = 2A\tau_{max}t$

or

$T = 2(0.0189)(55\times10^6)(0.004)$
$= 8316\ N\cdot m.$

(b) Since the shear flow is constant,

$q = (55)(0.004) = \tau_5(0.005)$

where τ_5 is the shear stress in the 0.005 m thick wall. So in the 0.005 m thick wall, the shear stress is $\tau_5 = 44\,MPa$, with the shear stress in the 0.004 m thick wall $\tau_4 = 55\,MPa$.

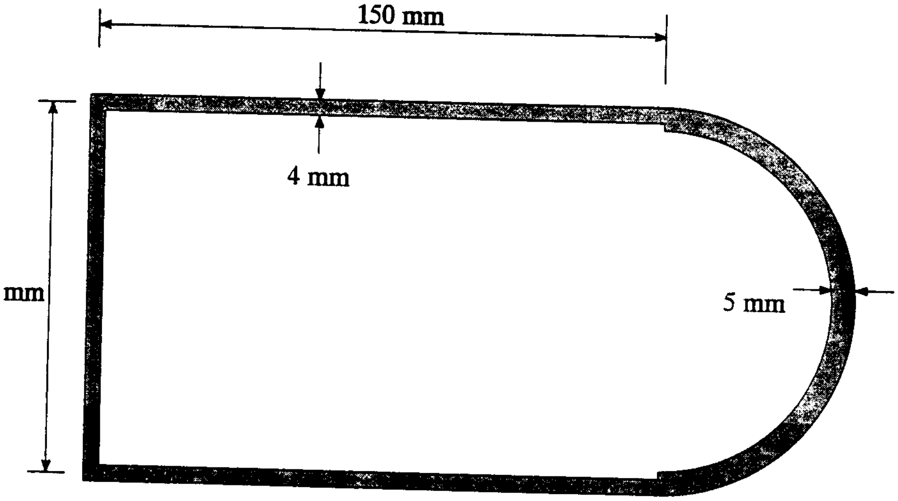


Figure a

6.44 The area enclosed by the mean perimeter is (see Fig. a)

$$A = 0.050(0.150) + 0.040(0.050) = 0.0095\ m^2 \qquad (a)$$

By Eq.(6.66) and (a), $T = 2Aq$, or

$$q = \frac{T}{2A} = \frac{T}{0.019} \qquad (b)$$

By Eq.(6.67), with $\psi = L\theta = 0.12$ rad and $\tau = q/t =$ constant,

$$L\theta = 0.12\ rad = L\left(\frac{q\ell}{2GAt}\right) \qquad (c)$$

where $L = 2\,m$ is the length of the member, and $t = 0.0025\,m$, $\ell = 0.500\,m$ (Fig. a).
(Cont.)

6.44 cont. Therefore, by Eqs. (a) and (c),

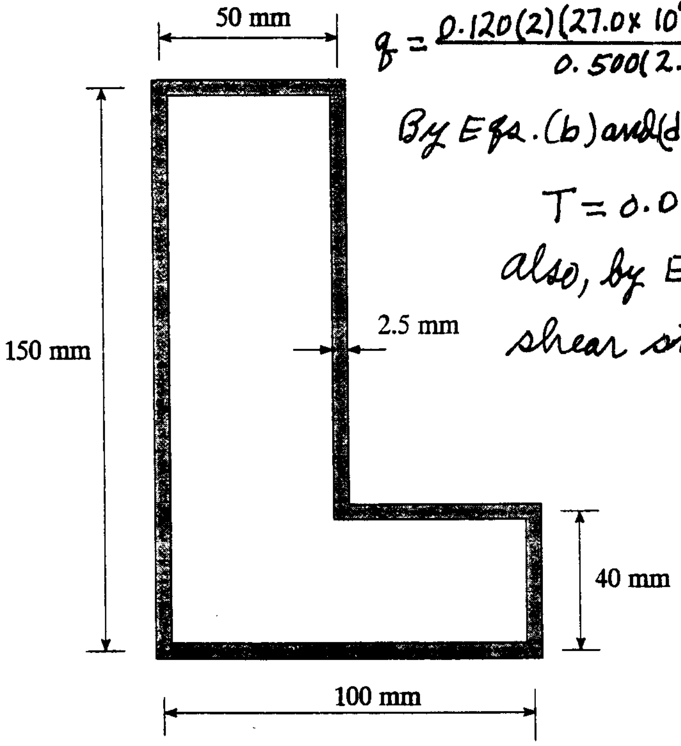


Figure a

$$q = \frac{0.120(2)(27.0\times 10^9)(0.0095)(0.0025)}{0.500(2.0)} = 153.9\ \text{kN/m} \quad (d)$$

By Eqs. (b) and (d), the torque T is

$$T = 0.019\,q = 2924.1\ \text{N}\cdot\text{m}$$

also, by Eq. (d), with $q = \tau t$, the average shear stress is

$$\tau = \frac{q}{t} = \frac{153.9}{0.0025} = 61.56\ \text{MPa}$$

6.45 By Fig. a, the area enclosed by the mean perimeter is

$$A = 0.080(0.140) - \frac{1}{2}\frac{\pi}{4}(0.060)^2$$
$$= 0.0076\ m^2 \quad (a)$$

Since the maximum shear stress occurs in the 0.004 m thick wall, by Eq. (6-66), with $T = 3000\ \text{N}\cdot\text{m}$,

$$T = 2A\tau_{max}t_1 = 3000\ \text{N}\cdot\text{m} \quad (b)$$

Then, by Eqs. (a) and (b),

$$\tau_{max} = \frac{3000}{2(0.0076)(0.004)} = 49.34\ \text{MPa}$$

The shear flow is

$$q = \tau_{max}t_1 = (49.34\times 10^6)(0.004) = 197.4\ \text{kN/m} \quad (c)$$

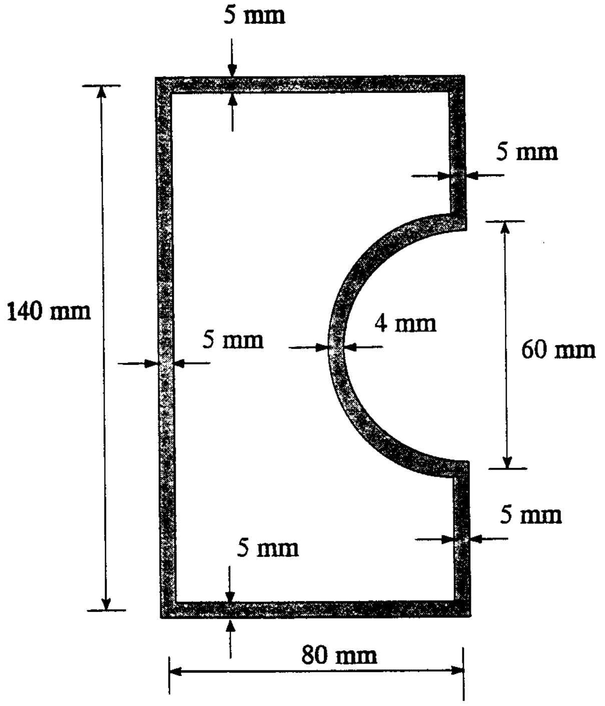


Figure a

(cont.)

6.45 cont. By Fig. a, the mean perimeter length of the wall of thickness $t_1 = 0.004$ m is

$$l_1 = \frac{1}{2}\pi D = 0.030\pi = 0.09425 \text{ m} \quad (d)$$

and the mean perimeter length of the wall thickness $t_2 = 0.005$m is

$$l_2 = 0.380 \text{ m} \quad (e)$$

By Eqs. (a), (c), (d), (e) and (6.69),

$$\theta = \frac{q}{2GA}\left(\frac{l_1}{t_1} + \frac{l_2}{t_2}\right) = \frac{197.4\times10^3}{2(27\times10^9)(0.0076)}\left(\frac{0.09425}{0.004} + \frac{0.380}{0.005}\right)$$

or

$$\theta = 0.0479 \text{ rad/m}$$

6.46 By Fig. a, the area enclosed by the mean perimeter is

$$A = \frac{1}{2}(0.090)(0.045) + \frac{1}{2}\frac{\pi}{4}(0.090)^2$$

or

$$A = 0.008387 \text{ m}^2 \quad (a)$$

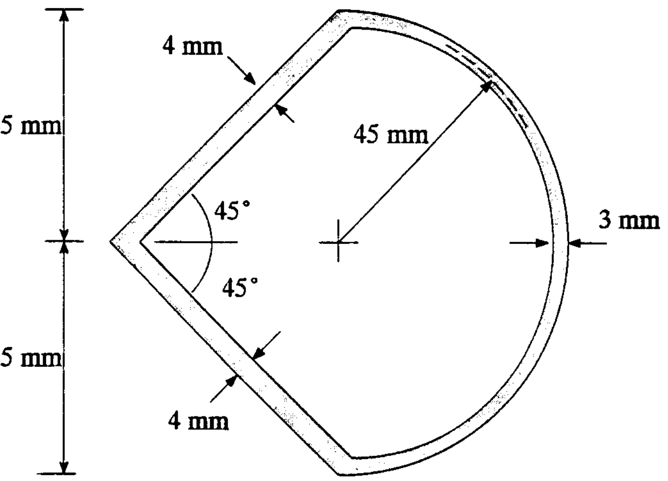


Figure a

Also since the shear flow $q = \tau t$ is constant, the maximum shear stress occurs in the thinnest section ($t = 0.003$ m).

Therefore, with $\tau_{max} = 60$ MPa, the shear flow is

$$q = \tau_{max} t = (60\times10^6)(0.003) = 180 \text{ kN/m} \quad (b)$$

By Eqs. (a), (b), and (6.66), the design torque is

$$T = 2Aq = 2(0.008387)(180\times10^3) = 3.019 \text{ kN}\cdot\text{m}$$

6.47 By Fig. a, $A_1 = b^2$, $A_2 = b^2$.

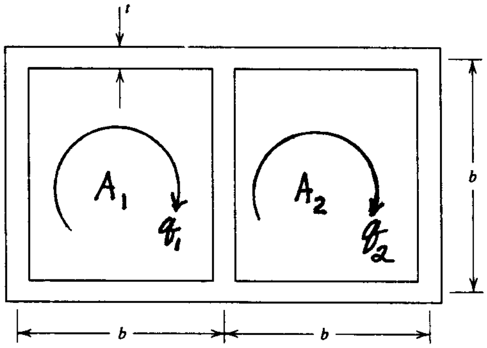


Figure a

Then, by Eq. (6.68),

$$T = 2\sum_{i=1}^{2} A_i q_i = 2(b^2 q_1 + b^2 q_2) = 4b^2 q$$

where $q = q_1 = q_2$, since cells 1 and 2 are identical. Hence,

$$q = \frac{T}{4b^2}$$

(a) The shear stresses in the outer walls and the interior web are, respectively,

$$\tau_{wall} = \frac{q}{t} = \frac{T}{4tb^2}\ ;\quad \tau_{web} = \frac{q_2 - q_1}{t} = \frac{q - q}{t} = 0$$

(b) The unit angle of twist is, by Eq. (6.69),

$$\theta = \frac{1}{2A_1 G}\left[\frac{q_1 l_1}{t} + \left(\frac{q_1 - q_2}{t}\right) l_2\right]\ ;\ q = q_1 = q_2$$

or

$$\theta = \frac{q}{2A_1 G}\left(\frac{l_1}{t}\right) = \frac{1}{2b^2 G}\left(\frac{T}{4b^2}\right)\left(\frac{3b}{t}\right) = \frac{3}{8}\frac{T}{Gb^3 t}$$

6.48 By Fig. a, of Problem 6.47, without the web, $A = 2b^2$. Then, by Eq. (6.66), $T = 2A\tau t = 4b^2 \tau t$, or $\tau = \frac{T}{4b^2 t}$.

By Eq. (6.67), with t = constant,

$$\theta = \frac{\tau l}{2GA} = \frac{1}{2GA}\left(\frac{T}{4b^2 t}\right)(6b) = \frac{3}{4}\frac{T}{Gb^3 t}$$

6.49 By Fig. a, of Problem 6.47, without the web and with a longitudinal slit, Eq. (6.62) yields the torsional constant $J = \frac{1}{3}(6b)t^3 = 2bt^3$, assuming that $6b/t > 10$. Then, by Eqs. (6.63),

$$\tau_{max} = \frac{2T h_{max}}{J} = \frac{2T(t/2)}{2bt^3} = \frac{T}{2bt^2}$$

and

$$\theta = \frac{T}{GJ} = \frac{T}{2Gbt^3}$$

6.50 The perimeter area of an equilateral triangle with mean length of a side equal to 25.4 mm is

$$A = \frac{1}{2}(25.4)(25.4 \times \sin 60^\circ) = 279.36\,\text{mm}^2 \qquad (a)$$

By Eq. (6.66),

$$\tau = \frac{T}{2At} \qquad (b)$$

With $T = 20$ N·m and $t = 2.54$ mm, by Eqs. (a) and (b), we have

$$\tau = \frac{20}{2(279.36\times10^{-6})(2.54\times10^{-3})} = 14.09\ \text{MPa}$$

also, by Eq. (6.67),

$$\theta = \frac{\tau l}{2GA} = \frac{(14.09\times10^{6})(3\times0.0254)}{2(27.6\times10^{9})(279.36\times10^{-6})} = 0.0696\ \text{rad/m}$$

6.51 For equal perimeters of the circular tube (Fig. a), the square tube (Fig. b), and the triangular tube (Fig. c),

$$\pi D = 4L_S = 3L_T \qquad (a)$$

With Eqs. (a), the areas enclosed by the perimeters are in terms of D,

$$(b)\quad A_C = \frac{\pi D^2}{4},\ A_S = L_S^2 = \frac{\pi^2}{16}D^2,\ A_T = \frac{1}{2}L_T\left(\frac{\sqrt{3}}{2}L_T\right) = \frac{\sqrt{3}\pi^2 D^2}{36}$$

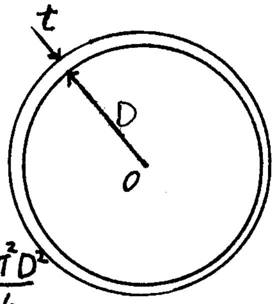
where subscripts C, S, and T, denote the circular, square, and triangular tubes, respectively.

The shear stresses for the circle, square, and triangular tube, respectively, are by Eqs. (b) and (6.66),

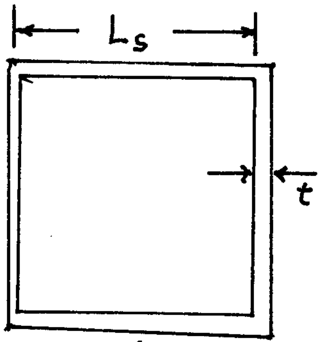
$$\tau_C = \frac{T}{2A_C t} = \frac{2T}{\pi D^2 t};\ \tau_S = \frac{T}{2A_S t} = \frac{8T}{\pi^2 D^2 t};\ \tau_T = \frac{T}{2A_T t} = \frac{18T}{\sqrt{3}\pi^2 D^2 t}$$

Hence,

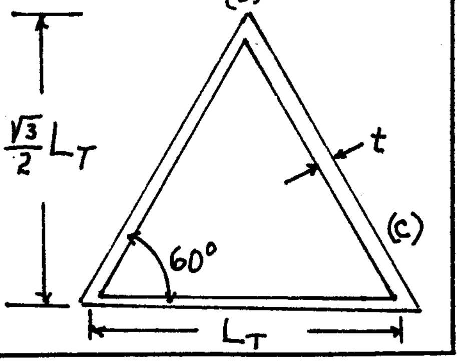
$$\frac{\tau_S}{\tau_C} = \frac{4}{\pi} = 1.27;\ \frac{\tau_T}{\tau_C} = \frac{9}{\sqrt{3}\pi} = 1.65$$



(a)



(b)



(c)

(cont.)

6.51 cont. Likewise, by Eq. (6.67), with $l = \pi D$, the unit angles of twist are

$$\theta_c = \frac{\tau_c \pi D}{2GA_c} = \frac{\pi D}{2G}\left(\frac{2T}{\pi D^2 t}\right)\left(\frac{4}{\pi D^2}\right) = \frac{T}{\pi G t}\,\frac{4}{D^3}$$

$$\theta_s = \frac{\tau_s \pi D}{2GA_s} = \frac{\pi D}{2G}\left(\frac{8T}{\pi^2 D^2 t}\right)\left(\frac{16}{\pi^2 D^2}\right) = \frac{T}{\pi G t}\,\frac{64}{\pi^2 D^3}$$

$$\theta_T = \frac{\tau_T \pi D}{2GA_T} = \frac{\pi D}{2G}\left(\frac{18T}{\sqrt{3}\pi^2 D^2 t}\right)\left(\frac{36}{\sqrt{3}\pi^2 D^2}\right) = \frac{T}{\pi G t}\,\frac{324}{3\pi^2 D^3}$$

Therefore,

$$\frac{\theta_s}{\theta_c} = \frac{16}{\pi^2} = 1.62; \quad \frac{\theta_T}{\theta_c} = \frac{27}{\pi^2} = 2.74$$

6.52 By Fig. a, the areas enclosed by the perimeters of the circle and the square are, respectively $A_c = \frac{\pi b^2}{4}$, $A_s = b^2$. Then, by Eq. (6.66), for equal shear stresses in the members, $\tau_c = \frac{T_c}{2A_c t} = \tau_s = \frac{T_s}{2A_s t}$. Therefore, $T_s / T_c = A_s / A_c = \frac{4}{\pi} = 1.273$

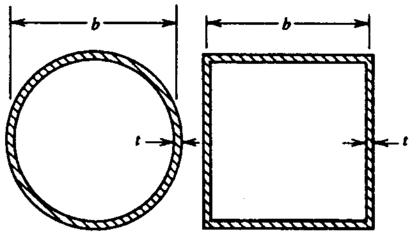


Figure a

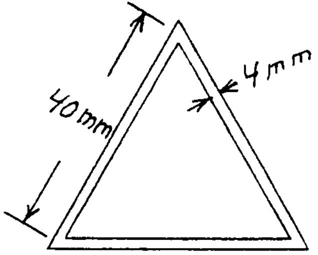
Likewise by Eq. (6.67), with $\tau_s = \tau_c = \tau$, $l_c = \pi b$, and $l_s = 4b$,

$$\theta_c = \frac{\tau l_c}{2GA_c} = \frac{2\tau}{Gb} \text{ and } \theta_s = \frac{\tau l_s}{2GA_s} = \frac{2\tau}{Gb}. \text{ Therefore, } \theta_s / \theta_c = 1.$$

6.53 To compare the shear stresses in the two torsion members in Problem 6.52, when they are subjected to equal twisting moments, we have by Eq. (6.66), $T = 2A_c \tau_c t = 2A_s \tau_s t$. Therefore $\frac{\tau_s}{\tau_c} = \frac{A_c}{A_s}$. But by Problem 6.52, $A_c = \frac{\pi b^2}{4}$ and $A_s = b^2$. Hence,

$$\frac{\tau_s}{\tau_c} = \frac{\pi}{4} = 0.785$$

6.54



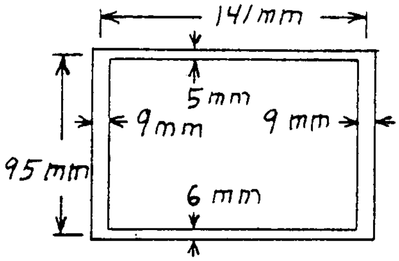
$$A = \frac{1}{2}(40)^2 \sin\frac{\pi}{3} = 692.8 \text{ mm}^2$$

$$T = 2A\tau t = 2(692.8)(20)(4) = \underline{110.8 \text{ N.m}}$$

$$\theta = \frac{\tau L}{2GA} = \frac{20(3)(40)}{2(31{,}000)(692.8)} = 0.0000559 \text{ rad/mm}$$

$$= \underline{0.0559 \text{ rad/m}}$$

6.55



$$A = 95(141) = 13{,}395 \text{ mm}^2$$

$$(SF)T = 2A\tau_Y t_{min} = 2A\frac{Y}{\sqrt{3}} t_{min}$$

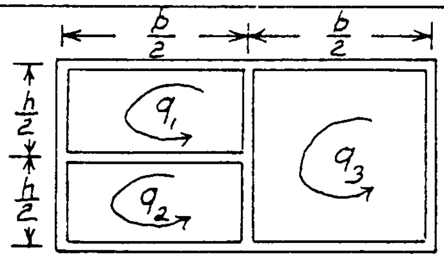
$$T = \frac{2(13{,}395)(360)(5)}{2.00\sqrt{3}} = \underline{13.92 \text{ kN.m}}$$

$$(SF)q = \tau_Y t_{min}; \quad q = \frac{360(5)}{2.00\sqrt{3}} = 519.6 \text{ N/mm}$$

$$\theta = \frac{q}{2AG}\sum_{i=1}^{n}\frac{\ell_i}{t_i} = \frac{519.6}{2(13{,}395)(77{,}500)}\left[\frac{(141)}{6} + \frac{(141)}{5} + \frac{2(95)}{9}\right] = \underline{0.0000182 \text{ rad/mm}}$$

$$= \underline{0.0182 \text{ rad/m}}$$

6.56 $2A_1 = 2A_2 = A_3$; thickness $t = \text{const.}$



$$\theta = \frac{1}{2GA_1 t}\left[q_1\left(\frac{b}{2} + \frac{h}{2}\right) + (q_1 - q_2)\frac{b}{2} + (q_1 - q_3)\frac{h}{2}\right]$$

$$= \frac{1}{2GA_2 t}\left[q_2\left(\frac{b}{2} + \frac{h}{2}\right) + (q_2 - q_1)\frac{b}{2} + (q_2 - q_3)\frac{h}{2}\right]$$

$$= \frac{1}{2GA_3 t}\left[q_3(b + h) + (q_3 - q_1)\frac{h}{2} + (q_3 - q_2)\frac{h}{2}\right]$$

$$q_1\left(\frac{b}{2} + \frac{h}{2}\right) + (q_1 - q_2)\frac{b}{2} + (q_1 - q_3)\frac{h}{2} = q_2\left(\frac{b}{2} + \frac{h}{2}\right) + (q_2 - q_1)\frac{b}{2} + (q_2 - q_3)\frac{h}{2}$$

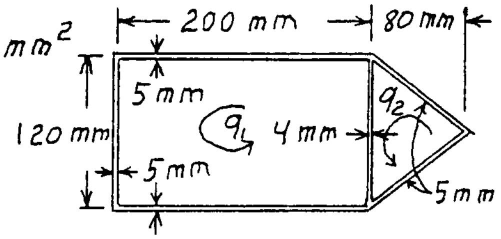
$$q_1 = q_2$$

$$q_1\left(\frac{b}{2} + \frac{h}{2}\right) + (q_1 - q_2)\frac{b}{2} + (q_1 - q_3)\frac{h}{2} = \frac{1}{2}\left[q_3(b + h) + (q_3 - q_1)\frac{h}{2} + (q_3 - q_2)\frac{h}{2}\right]$$

$$q_1 = q_3$$

Since $q_1 = q_2 = q_3$, the shearing stress in the inner walls must be zero.

6.57



$A_1 = 200(120) = 24{,}000\text{ mm}^2;\ A_2 = \frac{120(80)}{2} = 4800\text{ mm}^2$

$T = 2A_1q_1 + 2A_2q_2 = 48{,}000q_1 + 9600q_2$

$$\theta = \frac{1}{2G(24{,}000)}\left[\frac{520q_1}{5} + \frac{120(q_1-q_2)}{4}\right]$$

$$= \frac{1}{2G(4800)}\left[\frac{200q_2}{5} + \frac{120(q_2-q_1)}{4}\right]$$

$q_1 = 1.3380\,q_2$

$T = 11{,}000{,}000 = 48{,}000(1.3380\,q_2) + 9600\,q_2$

$q_2 = 149.0\text{ N/mm};\quad q_1 = 1.3380(149.0) = 199.4\text{ N/mm}$

$\tau_{max} = \frac{q_1}{5} = \frac{199.4}{5} = \underline{39.9\text{ MPa}}$

$$\beta = \theta L = \frac{3000}{2(27{,}100)(24{,}000)}\left[\frac{520(199.4)}{5} + \frac{120(199.4-149.0)}{4}\right] = \underline{0.0513\text{ rad}}$$

"PROBLEM 6.58 Follows PROBLEM 6.60"

6.59

$I_{min} = \frac{40(270)^3}{12} + \frac{260(15)^3}{12} = 65{,}680{,}000\text{ mm}^4$

$J = \frac{1}{3}\left[270(20)^3 + 270(20)^3 + 260(15)^3\right] = 1{,}733{,}000\text{ mm}^4$

$a = \frac{h}{2}\sqrt{\frac{EI_{min}}{JG}} = \frac{280}{2}\sqrt{\frac{200{,}000(65{,}680{,}000)}{1{,}733{,}000(77{,}500)}} = 1385\text{ mm}$

$\frac{L}{a} = \frac{8000}{1385} = 5.78 > 2.5;\quad M_{max} = \frac{Ta}{h} = \frac{7{,}000{,}000(1385)}{280} = 34.63\text{ kN.m}$

$\tau_{max} = \frac{Tt_{max}}{J} = \frac{7{,}000{,}000(20)}{1{,}733{,}000} = \underline{80.78\text{ MPa}};\quad \sigma_{max} = \frac{M_{max}\frac{b}{2}}{\frac{1}{2}I_{min}} = \frac{34{,}630{,}000(135)}{32{,}840{,}000} = \underline{142.4\text{ MPa}}$

6.60

$I_x = \frac{60(80)^3}{12} - \frac{54(68)^3}{12} = 1{,}145{,}000\text{ mm}^4;\quad I_y = \frac{12(60)^3}{12} + \frac{68(6)^3}{12} = 217{,}200\text{ mm}^4$

$J = \frac{1}{3}\left[60(6)^3 + 60(6)^3 + 68(6)^3\right] = 13{,}536\text{ mm}^4$

$a = \frac{h}{2}\sqrt{\frac{EI_y}{JG}} = \frac{74}{2}\sqrt{\frac{72{,}000(217{,}200)}{13{,}536(27{,}100)}} = 241.6\text{ mm}$

$\frac{L}{2a} = \frac{800}{241.6} = 3.31 > 2.5;\quad M_y = \frac{Ta}{h} = \frac{100P(241.6)}{74} = 326.5P\ (\text{N.mm})$

$\sigma_{max} = \frac{M_y\frac{b}{2}}{\frac{1}{2}I_y} + \frac{M_xc}{I_x} = \frac{326.5P(30)}{108{,}600} + \frac{1600P(40)}{1{,}145{,}000} = 0.1461P = 160\text{ MPa}$

$P = \underline{1.095\text{ kN}}$

$\boxed{6.58}$ (a) Given that $bt_1 = ht_2$, for equal cross-sectional areas of the two cross sections (see Figs. a and b), we have

$$A = 2(240)(8) + 2(120)(8) = 2(240)t_1 + 2(120)t_2 \qquad \text{(a)}$$

$$240t_1 = 120t_2$$

The solution of Eqs (a) is

$$t_1 = 6\ \text{mm}, \quad t_2 = 12\ \text{mm} \qquad \text{(b)}$$

(b) Since the working shear stress for each member is

$$\tau = 70.0\ \text{MPa} \qquad \text{(c)}$$

for the member in Fig. a, we have by Eq. (6.66), with $A = (240)(120) = 28{,}800\ \text{mm}^2$, the area enclosed by the perimeter,

$$T_a = 2A\tau t_a = 2(0.0288)(70\times10^6)(0.008) = 32.26\ \text{kN}\cdot\text{m}$$

Similarly, by Fig. b, with $q = \tau_{t_1}t_1 = \tau_{t_2}t_2 = \text{constant}$, by Eq. (6.66), we have

$$T_b = 2A\tau_{t_1}t_1 = 2(0.0288)(70\times10^6)(0.006) = 24.19\ \text{kN}\cdot\text{m}$$

where $\tau_{max} = \tau_{t_1} = 70\ \text{MPa}$, since $t_1 = 6\ \text{mm}$ is less than $t_2 = 12\ \text{mm}$.

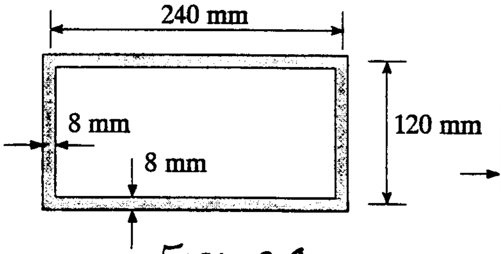


Figure a

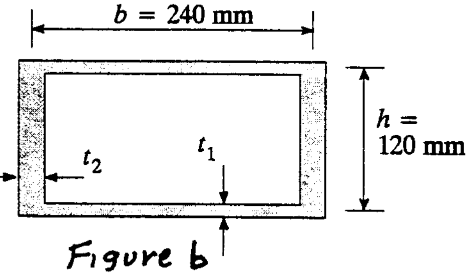


Figure b

6.61 $$\tau_{max} = \frac{2T h_{max}}{J} = \frac{T t_{max}}{J} = \frac{150{,}000\,(6)}{13{,}536} = 66.49 \text{ MPa}$$

By Table 6.2 and the data in Prob. 6.60,

$$\beta = \frac{T}{GJ}(L - 2\alpha) = \frac{150{,}000}{27{,}100\,(13{,}536)}\left[1600 - 2(241.6)\right] = 0.4567 \text{ rad.}$$

6.62 (a)

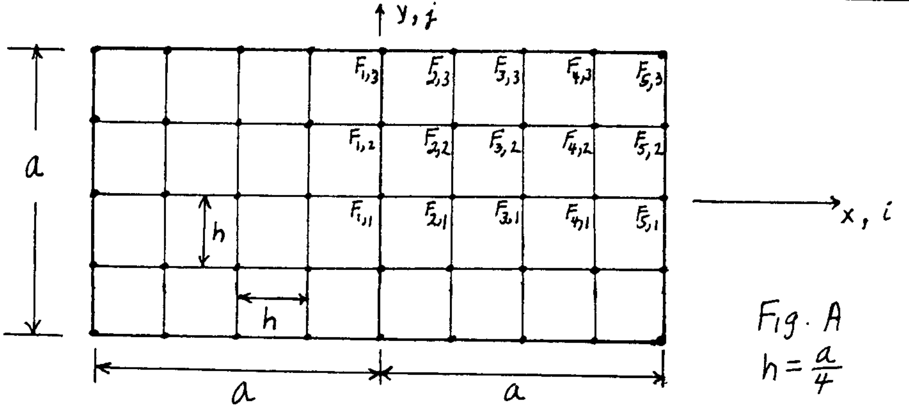


Fig. A
$h = \frac{a}{4}$

In terms of the stress function nodal values $F_{i,j}$, the torsional finite-difference equation is [Eqs. (6.92) and (6.93)]

$$(\nabla^2\phi)_{i,j} = \frac{1}{h^2}\left[F_{i-1,j} + F_{i+1,j} - 4F_{i,j} + F_{i,j-1} + F_{i,j+1}\right] = -2G\theta \quad \text{(a)}$$

By symmetry, only a quarter of the cross section need be considered (Fig. A). Equation (a) must be evaluated at the node points, (1,1), (1,2), --- . Thus, noting that $F_{1,3}, \text{--}, F_{5}$ --- on the boundary are zero [see Eq. (6.92)], we have

$$i=1, j=1: \quad F_{2,1} + F_{2,1} - 4F_{1,1} + F_{1,2} + F_{1,2} = -2G\theta h^2 \quad \text{or}$$

$$-4F_{1,1} + 2F_{1,2} + 2F_{2,1} = -2G\theta h^2 \quad \text{(b)}$$

Similarly, we obtain

$$i=1, j=2: \quad F_{11} - 4F_{1,2} + 2F_{2,2} = -2G\theta h^2 \quad \text{(c)}$$

$$i=2, j=1: \quad F_{1,1} - 4F_{2,1} + 2F_{2,2} + F_{3,1} = -2G\theta h^2 \quad \text{(d)}$$

$$i=2, j=2: \quad F_{1,2} + F_{2,1} - 4F_{2,2} + F_{3,2} = -2G\theta h^2 \quad \text{(e)}$$

$$i=3, j=1: \quad F_{2,1} - 4F_{3,1} + 2F_{3,2} + F_{4,1} = -2G\theta h^2 \quad \text{(f)}$$

$$i=3, j=2: \quad F_{2,2} + F_{3,1} - 4F_{3,2} + F_{4,2} = -2G\theta h^2 \quad \text{(g)}$$

$$i=4, j=1: \quad F_{3,1} - 4F_{4,1} + 2F_{4,2} = -2G\theta h^2 \quad \text{(h)}$$

$$i=4, j=2: \quad F_{3,2} - 4F_{4,2} + F_{4,1} = -2G\theta h^2 \quad \text{(i)}$$

(Cont.)

6.62a cont.

With the notation $F_{1,1} = a$, $F_{1,2} = b$, $F_{2,1} = c$, $F_{2,2} = d$, $F_{3,1} = e$, $F_{3,2} = f$, $F_{4,1} = g$, and $F_{4,2} = h$, we may write Eqs.(b)-(i) as

$$\begin{aligned}
-4a + 2b + 2c &= -2G\theta h^2 \\
a - 4b + 2d &= -2G\theta h^2 \\
a - 4c + 2d + e &= -2G\theta h^2 \\
b + c - 4d + f &= -2G\theta h^2 \\
c - 4e + 2f + g &= -2G\theta h^2 \\
d + e - 4f + h &= -2G\theta h^2 \\
e - 4g + 2h &= -2G\theta h^2 \\
f + g - 4h &= -2G\theta h^2
\end{aligned} \qquad (j)$$

By TK solver or any other equation solver, the solution of Eqs.(f) is

$$\begin{aligned}
a &= F_{1,1} = 3.5873956 & e &= F_{3,1} = 3.0368974 \\
b &= F_{1,2} = 2.7077835 & f &= F_{3,2} = 2.3126852 \\
c &= F_{2,1} = 3.4670078 & g &= F_{4,1} = 2.0552114 \\
d &= F_{2,2} = 2.6218691 & h &= F_{4,2} = 1.5919741
\end{aligned}$$

The maximum shear stress occurs at node $i=1, j=3$. Therefore, with a fourth-order backward-difference formula (Boresi and Chong, 1991),

$$\sigma_{zx} = \frac{\partial \phi}{\partial y} = \frac{1}{24h}\left[6F_{i,j-4} - 32F_{i,j-3} + 72F_{i,j-2} - 96F_{i,j-1} + 50F_{i,j}\right]$$

For $i=1, j=3$, $\sigma_{zx} = \sigma_{zx(max)}$. Therefore, with $h = a/4$,

$$\begin{aligned}
\sigma_{zx}\Big|_{\substack{i=1\\j=3}} = \sigma_{zx(max)} &= \frac{1}{96}\left[6F_{1,-1} - 32F_{1,2} + 72F_{1,1} - 96F_{1,2} + 50F_{1,3}\right]G\theta a \\
&= \frac{G\theta a}{96}\left[6\times 0 - 32\times 2.7077835 + 72\times 3.5873965 - 96\times 2.7077835 + 50\times 0\right]
\end{aligned}$$

or $\sigma_{zx(max)} = -0.9198313\ G\theta a$ or $\tau_{max} = 0.9198313G\theta a$

(cont.)

6.62 cont. (b) For $h = a/8$, the mesh is subdivided further, with the result that the number of node points in both the x and y directions is double. Hence, we obtain 32 equations (8×4, compared to Part a) that must be solved for nodal values of ϕ. Analogous to Part a, Eq. (a) yields

$i=1, j=1$: $-4a + 2b + 2e = -2G\theta h^2$
$i=1, j=2$: $a - 4b + c + 2f = -2G\theta h^2$
$i=1, j=3$: $b - 4c + d + 2g = -2G\theta h^2$
$i=1, j=4$: $c - 4d + 2h = -2G\theta h^2$
$i=2, j=1$: $a - 4e + 2f + i = -2G\theta h^2$
$i=2, j=2$: $b + e - 4f + g + j = -2G\theta h^2$
$i=2, j=3$: $c + f - 4g + h + k = -2G\theta h^2$
$i=2, j=4$: $d + g - 4h + l = -2G\theta h^2$
$i=3, j=1$: $e - 4i + 2j + m = -2G\theta h^2$
$i=3, j=2$: $f + i - 4j + k + n = -2G\theta h^2$
$i=3, j=3$: $g + j - 4k + l + o = -2G\theta h^2$
$i=3, j=4$: $h + k - 4l + p = -2G\theta h^2$
$i=4, j=1$: $i - 4m + 2n + q = -2G\theta h^2$
$i=4, j=2$: $j + m - 4n + o + r = -2G\theta h^2$
$i=4, j=3$: $k + n - 4o + p + s = -2G\theta h^2$
$i=4, j=4$: $l + o - 4p + t = -2G\theta h^2$
$i=5, j=1$: $m - 4q + 2r + u = -2G\theta h^2$
$i=5, j=2$: $n + q - 4r + s + v = -2G\theta h^2$
$i=5, j=3$: $o + r - 4s + t + w = -2G\theta h^2$
$i=5, j=4$: $p + s - 4t + x = -2G\theta h^2$
$i=6, j=1$: $q - 4u + 2v + y = -2G\theta h^2$
$i=6, j=2$: $r + u - 4v + w + z = -2G\theta h^2$
$i=6, j=3$: $s + v - 4w + x + (aa) = -2G\theta h^2$
$i=6, j=4$: $t + w - 4x + (bb) = -2G\theta h^2$
$i=7, j=1$: $u - 4y + 2z + (cc) = -2G\theta h^2$
$i=7, j=2$: $v + y - 4z + (aa) + (dd) = -2G\theta h^2$
$i=7, j=3$: $w + z - 4(aa) + (bb) + (ee) = -2G\theta h^2$
$i=7, j=4$: $x + (aa) - 4(bb) + (ff) = -2G\theta h^2$

$F_{1,1} = a$
$F_{1,2} = b$
$F_{1,3} = c$
$F_{1,4} = d$
$F_{2,1} = e$
$F_{2,2} = f$
$F_{2,3} = g$
$F_{2,4} = h$
$F_{3,1} = i$
$F_{3,2} = j$
$F_{3,3} = k$
$F_{3,4} = l$
$F_{4,1} = m$
$F_{4,2} = n$
$F_{4,3} = o$
$F_{4,4} = p$
$F_{5,1} = q$
$F_{5,2} = r$
$F_{5,3} = s$
$F_{5,4} = t$
$F_{6,1} = u$
$F_{6,2} = v$
$F_{6,3} = w$
$F_{6,4} = x$
$F_{7,1} = y$
$F_{7,2} = z$
$F_{7,3} = (aa)$
$F_{7,4} = (bb)$
$F_{8,1} = (cc)$
$F_{8,2} = (dd)$
$F_{8,3} = (ee)$
$F_{8,4} = (ff)$

(cont.)

6.62 b cont.

$i=8,\ j=1:\ y-4(cc)+2(dd)=-2G\theta h^2$

$i=8,\ j=2:\ z+(cc)-4(dd)+(ee)=-2G\theta h^2$

$i=8,\ j=3:\ (aa)+(dd)-4(ee)+(ff)=-2G\theta h^2$

$i=8,\ j=4:\ (bb)+(ee)-4(ff)=-2G\theta h^2$

By TK Solver or any other equation solver, the solution of these 32 equations is*

a	14.518555
b	13.631189
e	14.40592
c	10.952108
f	13.527047
d	6.4327478
g	10.872248
h	6.3894416
i	14.051032
j	13.198832
k	10.620394
l	6.2527709
m	13.400543
n	12.596854
o	10.157724
p	6.0012483
q	12.357434
r	11.630318
s	9.4123999
t	5.5944985
u	10.768556
v	10.154583
w	8.2670597
x	4.9643459
y	8.4076236
z	7.9523977
aa	6.5369101
bb	3.9958255
cc	4.9571432
dd	4.7104747
ee	3.9323576
ff	2.4820458

*These numerical values are for $G\theta h^2=1$

The maximum shear stress is

$$\sigma_{zx}\Big|_{i=1,j=5}=\sigma_{zx(max)}=\frac{1}{24h}\left[6F_{1,1}-32F_{1,2}+72F_{1,3}-96F_{1,4}+50F_{1,5}\right]$$

or

$$\sigma_{zx(max)}=\frac{1}{24h}\left[6a-32b+72c-96d+50\times\overset{\overset{F_{1,5}=\text{zero}}{\downarrow}}{0}\right]G\theta h^2$$

For $h=a/8$ and with the numerical values for a, b, c, and d, we have

$$\sigma_{zx(max)}=-0.92749\,G\theta a$$

or

$$\tau_{max}=0.92749\,G\theta a$$

An improvement on this result may be obtained by Eq.(6.100), with

$m_2=8,\ S_2=\tau_{max}=0.92749\,G\theta a,$

$m_1=4,\ S_1=\tau_{max}=0.91983\,G\theta a$

Thus

$$\tau_{max}=\frac{m_2^2S_2-m_1^2S_1}{m_2^2-m_1^2}=\frac{[(64)(0.92749)-16(0.91983)]G\theta a}{64-16}$$

$$=0.9300\,G\theta a$$

This result agrees with the exact solution $(0.9300\,G\theta a)$ given by Timoshenko and Goodier, Theory of Elasticity McGraw Hill 1970 (2nd Ed.)

6.63 Given that a bar, with hollow circular cross section with outer radius b and inner radius a, is subjected to a twisting moment T, which is increased until the bar is fully yielded. Repeat Example 6.12 for this bar

(a) The shear stress distribution at initial yield at $r = a$ is shown in Fig. a.

By Eq. (6.15), with

$$J = \frac{\pi}{2}(b^4 - a^4),$$

$$T_Y = \frac{\pi}{2}\tau_Y\left(\frac{b^4 - a^4}{b}\right) \quad (a)$$

or $$\tau_Y = \frac{2bT_Y}{\pi(b^4 - a^4)} \quad (b)$$

Hence, by Eq. (6.14),

$$\theta_Y = \frac{\tau_Y}{Gb} = \frac{2T_Y}{\pi G(b^4 - a^4)} \quad (c)$$

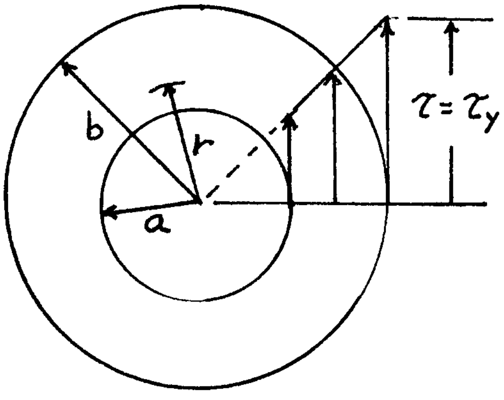


Figure a

(b) As the torque increases beyond T_Y, the section yields to radius b_Y (Fig. b). Since the cross section is elastic to radius b_Y, by Eq. (6.14), $b_Y = \frac{\tau_Y}{G\theta}$. Then, dividing by b, we have with Eq. (c),

$$\frac{b_Y}{b} = \frac{\tau_Y}{G\theta b} = \frac{\theta_Y}{\theta}$$

Hence, the angle of twist, when yield is to radius b_Y, is

$$\theta = \frac{b\,\theta_Y}{b_Y} \quad (d)$$

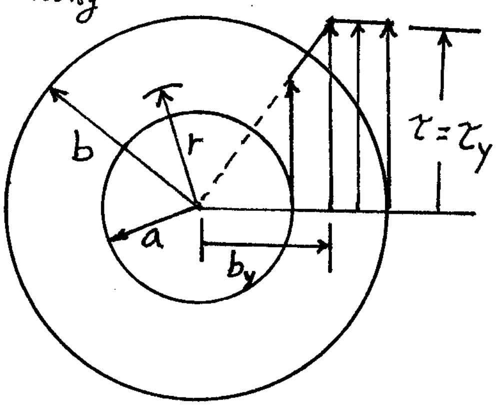


Figure b

(cont.)

6.63 cont.

With Eqs. (c) and (d), we find in terms of T_Y

$$\theta = \frac{2bT_Y}{\pi b_Y G(b^4-a^4)} \quad (e)$$

(c) The elastic twisting moment T_E is due to the shear stress from $r=a$ to $r=b_Y$ (Fig. b). By Eq. (6.15), with $J=\frac{\pi}{2}(b_Y^4-a^4)$, we have

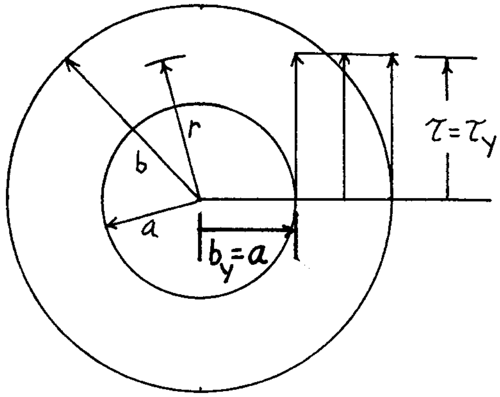


Figure c

$$T_E = \frac{\pi}{2}\frac{(b_Y^4-a^4)}{b_Y}\tau_Y \quad (f)$$

The plastic torque T_P due to the shear stress in the yielded (plastic) region of the bar $(r \geq b_Y)$ is

$$T_P = \int_{b_Y}^{b} r\tau_Y(2\pi r)dr = \frac{2\pi}{3}\tau_Y(b^3-b_Y^3) = \frac{2\pi}{3}\tau_Y\left(\frac{b^4-bb_Y^3}{b}\right) \quad (g)$$

Therefore,

$$T_{EP} = T_E + T_P = \frac{3\pi}{3}\tau_Y\left[\frac{b^4-bb_Y^3}{b} + \frac{3}{4}\left(\frac{b_Y^4-a^4}{b_Y}\right)\right] \quad (h)$$

(d) The fully plastic moment T_L is given by letting $b_Y \to a$, in Eq. (h), Fig. c. Thus, since $T_{EP} \to T_L$ as $b_Y \to a$, Eq. (h) yields

$$T_L = \frac{2\pi\tau_Y}{3}\left(\frac{b^4-ba^3}{b}\right) = \frac{2\pi\tau_Y}{3}(b^3-a^3) \quad (i)$$

Dividing Eq. (i) by Eq. (a) and reassembling, we obtain

$$T_L = \frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}T_Y \quad (j)$$

Note that as $a \to 0$, $T_L \to \frac{4}{3}T_Y$ [See Eq. (i), Example 6.12].

(cont.)

6.63 cont.

(e) When the limiting value of the moment T_L is released, the tube springs back (unwinds) elastically, and the shear stress-strain curve will follow a path similar to CA′ in Fig. 4.4a.

Since $T_L = \frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}T_Y$, the elastic stress recovery at $r=b$ is $\tau_b = \frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}\tau_Y$. Since the recovery is elastic, the recovery at $r=a$ is $\tau_a = \frac{a}{b}\tau_b = \frac{4}{3}\frac{a(b^3-a^3)}{b^4-a^4}\tau_Y$.

Hence, the residual stress at $r=b$ is

$$\tau_{b(residual)} = -\frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}\tau_Y + \tau_Y$$

or

$$\tau_{b(residual)} = \left[\frac{-4b(b^3-a^3)+3(b^4-a^4)}{3(b^4-a^4)}\right]\tau_Y \qquad (k)$$

and the residual stress at $r=a$ is

$$\tau_{a(residual)} = \tau_Y - \frac{4}{3}\frac{a(b^3-a^3)}{b^4-a^4}\tau_Y$$

$$= \left[\frac{3(b^4-a^4)-4a(b^3-a^3)}{3(b^4-a^4)}\right]\tau_Y \qquad (\ell)$$

Note that as $a \rightarrow 0$, Eqs. (k) and (ℓ) reduce to

$$\tau_{b(residual)} = -\frac{1}{3}\tau_Y \quad \text{(see Fig. E6.12c)}$$

where the − sign denotes a shear stress in the opposite sense of the original shear stress τ_Y,

and

$$\tau_{a(residual)} = \tau_Y \quad \text{(See Fig. E6.12c)}.$$

(cont.)

6.63 cont. (f) For $a = 0$ and $b = \frac{1}{2}b$, we have the following results

$$\text{Equation (b)} \rightarrow \tau_y = \frac{2\,T_y}{\pi b^3}$$

$$\text{Equation (h)} \rightarrow T_{EP} = \frac{2\pi}{3}\tau_y\left(\frac{31b^3}{32}\right) = 2.029\,b^3\tau_y$$

or

$$T_{EP} = (2.029\,b^3)\left(\frac{2T_y}{\pi b^3}\right) = 1.292\,T_y$$

This value of T_{EP} agrees with Eq.(c) of Section 6.10. Consequently, the resulting residual stress distribution is given by Fig. 6.30b.

6.64 (a) By Eq.(1.6), the maximum shear strain is, with $\psi = 10^\circ = \pi/18$ rad, $b = D/2 = 0.050$ m, and $L = 1.5$ m,

$$\gamma_{max} = \frac{b}{L}\psi = \frac{0.050}{1.5}\left(\frac{\pi}{18}\right) = 0.005818 \qquad (a)$$

The maximum elastic shear strain is, with $\tau_y = 150$ MPa and $G = 77.5$ GPa,

$$\gamma_y = \frac{\tau_y}{G} = \frac{150\times10^6}{77.5\times10^9} = 0.001935 \qquad (b)$$

Then, by Eq.(a) of Section 6.10, the radius of the elastic-plastic boundary is, with Eqs.(a) and (b),

$$r_y = b\frac{\gamma_y}{\gamma_{max}} = 0.01663 \text{ m} \qquad (c)$$

or

$$D_y = 2r_y = 0.03327 \text{ m} \qquad (d)$$

By Eqs.(6.103) and (d),

$$T = T_{EP} = \left(\frac{\pi}{12}D^3 - \frac{\pi}{48}D_y^3\right)\tau_y = 38.91 \text{ kN}\cdot\text{m} \qquad (e)$$

(cont.)

6.64 cont. (b) At $r=b$, the maximum elastic shear stress due to $T = T_{EP} = 38.91$ kN·m is

$$\tau_E = \frac{T_{EP}\, b}{J} = \frac{(38.9\times10^3)(0.050)}{\frac{\pi}{32}(0.100)^4} = 198.16 \text{ MPa}$$

The elastic-plastic stress distribution and the elastic (linear) stress distribution due to a torque of 38.91 kN·m are shown in Fig. a. The residual stresses at $r = r_y$ and $r=b$ after unloading are shown in Fig. b.

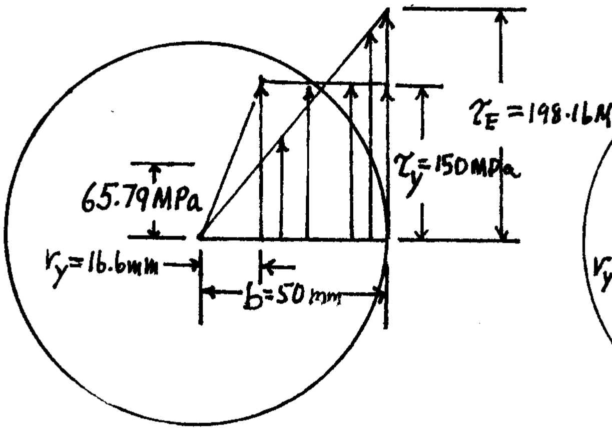


Figure a

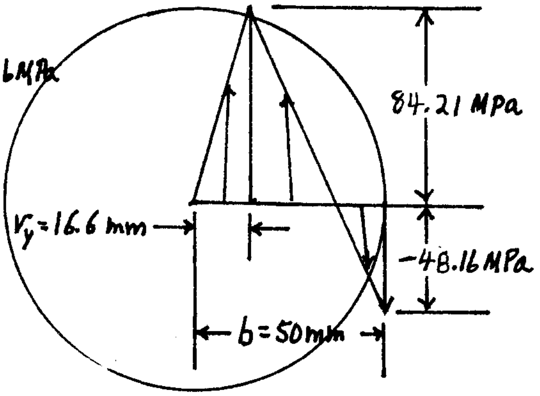


Figure b

(c) The permanent set is, with $\psi = \pi/18$ rad, $L = 1.5$ m, $T_{EP} = 38.91$ kN·m

$$\psi_{set} = \psi - \frac{T_{EP} L}{GJ} = \frac{\pi}{18} - \frac{(38910)(1.5)}{(77.5\times10^9)(\frac{\pi}{32})(0.100)^4} = 0.09782 \text{ rad}$$

The permanent set is in the same sense as ψ.

6.65 The solid shaft has a diameter $D = 200$ mm (Fig. a). The hollow shaft has an inside diameter $D_i = 150$ mm (Fig. b). Both shafts have the same elastic shear strength.

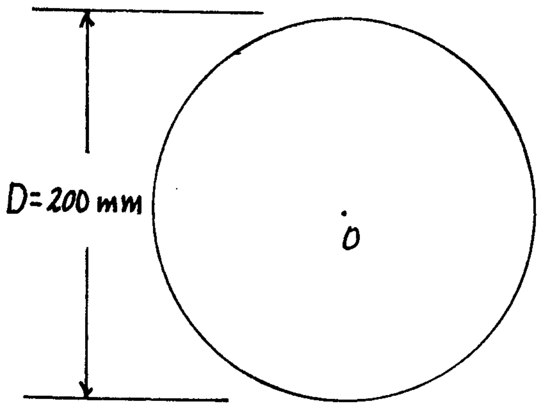


Figure a

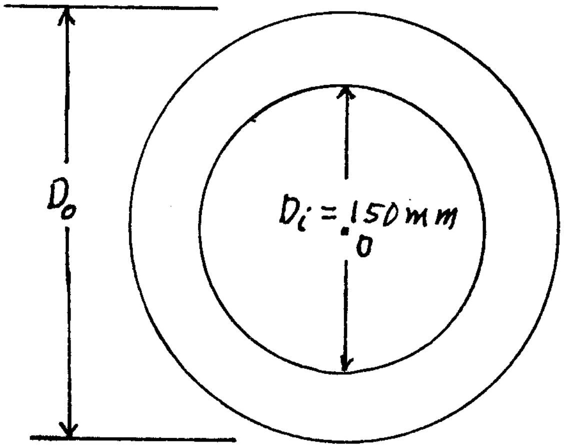


Figure b

(a) Both shafts are torqued to their elastic shear strength τ_Y by torques of equal magnitude T. To determine the external diameter D_0 of the hollow shaft, we proceed as follows: For Fig. a,

$$\tau_Y = \frac{T(D/2)}{J_a} = \frac{TD}{2(\frac{\pi}{32})D^4} = \frac{16T}{\pi D^3} \qquad (a)$$

By Fig. (b) and Eq. (a),

$$\tau_Y = \frac{T(D_0/2)}{J_b} = \frac{16TD_0}{\pi(D_0^4 - D_i^4)} = \frac{16T}{\pi D^3}$$

Hence,

$$D_0 D^3 = D_0^4 - D_i^4 \qquad (b)$$

Or, with $D = 0.200$ m,

$$125 D_0^4 - D_0 - 0.0632812 = 0 \qquad (c)$$

The solution of Eq. (c) is

$$D_0 = 0.2178 \text{ m} = 217.75 \text{ mm} \qquad (d)$$

(cont.)

6.65 cont. (b) To determine the ratio of the fully-plastic torques of the shafts, we note that for Fig. a, the fully-plastic torque is

$$(T_{FP})_a = \frac{\pi}{12}\tau_Y D^3 \qquad (e)$$

and for Fig. b,

$$(T_{FP})_b = \frac{\pi}{12}\tau_Y (D_o^3 - D_i^3) \qquad (f)$$

Hence, by Eqs. (d), (e), and (f), we obtain the ratio

$$\frac{(T_{FP})_a}{(T_{FP})_b} = \frac{D^3}{D_o^3 - D_i^3} = \frac{0.200^3}{(0.2178^3 - 0.150^3)} = 1.15$$

6.66 For Fig. a, the fully-plastic torque is

$$(T_{FP})_a = \frac{\pi}{12}\tau_Y D^3 = \frac{\pi}{12}\tau_Y (0.050)^3 = 3.2725\times 10^{-5}\tau_Y \qquad (a)$$

For the end sections of the modified member (Fig. b),

$$(T_{FP})_{ends} = \frac{\pi}{12}\tau_Y D^3 = \frac{\pi}{12}\tau_Y (0.080)^3 = 5.0265\times 10^{-5}\tau_Y \qquad (b)$$

For the modified section of Fig. b, for the sleeve,

$$(T_{FP})_{sleeve} = \frac{\pi}{12}\tau_Y (D_o^3 - D_i^3)$$

$$= \frac{\pi}{12}\tau_Y (0.086^3 - 0.080^3)$$

or

$$(T_{FP})_{sleeve} = 3.2478\times 10^{-5}\tau_Y \qquad (c)$$

For the solid section encased by the sleeve [as in Eq. (a)]

$$(T_{PP})_{solid} = 3.2725\times 10^{-5}\tau_Y \qquad (d)$$

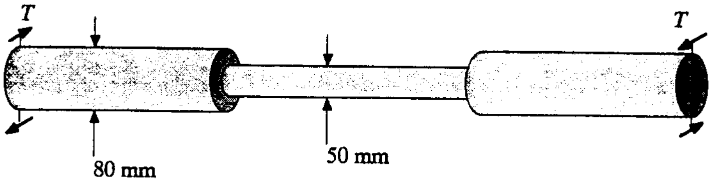


Figure a

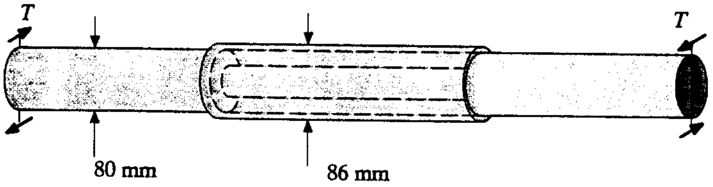


Figure b

(cont.)

6.66 cont. Since $(T_{FP})_{solid} > (T_{FP})_{sleeve}$, by Eqs. (c) and (d) the torque required to cause fully-plastic deformation in the modified section is

$$(T_{FP})_{modified} = (T_{FP})_{solid} = 3.2725 \times 10^{-5}\, \tau_Y \quad (e)$$

Hence, by Eqs. (a) and (e), the fully-plastic torque for Figs. (a) and (b) are the same; the ratio is one.

6.67 For the inner core $(0 \le r_i \le 22\text{ mm})$, the maximum elastic torque is, with $\tau_Y = 250$ MPa and $D_i = 44$ mm,

$$T_i = \frac{\pi}{16}\tau_Y D_i^3 = \frac{\pi}{16}(250 \times 10^6)(0.044)^3 = 4181.5\text{ N·m} \quad (a)$$

For the induction-hardened region $(22 \le r \le 25\text{ mm})$, the maximum elastic torque is, with $\tau_Y = 500$ MPa, $D_0 = 50$ mm and $D_i = 44$ mm,

$$T_{hardened} = \frac{\pi(D_0^4 - D_i^4)}{16 D_0}\tau_Y = \frac{\pi}{16}\frac{(0.050^4 - 0.044^4)}{(0.050)}(500 \times 10^6)$$

or

$$T_{hardened} = 4912.5\text{ N·m} \quad (b)$$

Since $T_{hardened} > T_i$, by Eqs. (a) and (b), the maximum elastic torque is $T_E = T_i = 4181.5$ N·m. A torque larger than T_i will cause the inner core to become plastic.

The torque that initiates yield at the outer surface of the member is

$$T = 4912.5\text{ N·m}$$

6.68

$$T_P = 2\left[\tfrac{1}{3}(4a^2)(a\tau_Y) + \tfrac{1}{2}(2b-2a)(2a)(a\tau_Y)\right] = \underline{\tfrac{8}{3}\tau_Y a^3 + 4(b-a)\tau_Y a^2}$$

6.69

$$T_P = 2\left[\tfrac{1}{3}(t^2)(\tfrac{t}{2}\tau_Y) + \tfrac{1}{2}\left\{2(b-t)+(a-t)\right\}(t)(\tfrac{t}{2}\tau_Y)\right] = \underline{\tau_Y t^2\left(\tfrac{a}{2} + b - \tfrac{7}{6}t\right)}$$

6.70

$$T_P = \frac{8(100)(50)^3}{3} + 4(75-50)(100)(50)^2 = 58.33 \text{ kN.m}$$

$$T_Y = 0.231(150)(100)^2(100) = 34.65 \text{ kN.m}$$

$$\frac{T_P}{T_Y} = \frac{58.33}{34.65} = \underline{1.683}$$

6.71 Use results of Problem 6.68.

$$T_P = \left[\tfrac{8}{3}(120)(100)^3 + 4(200-100)(120)(100)^2\right] - \left[\tfrac{8}{3}(120)(70)^3 + 4(170-70)(120)(70)^2\right]$$

$$= \underline{455.0 \text{ kN.m}}$$

6.72

(a) The torque at yield is (Fig. A)

$$T_Y = \frac{\tau_Y J}{b}, \quad \text{where } J = \frac{\pi}{2}(b^4 - a^4).$$

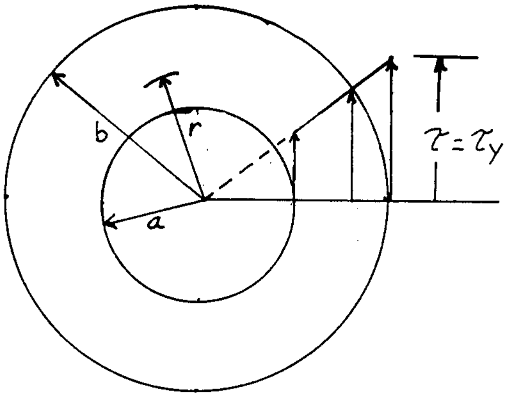
Therefore, $T_Y = \frac{\pi}{2}\tau_Y\left(\frac{b^4-a^4}{b}\right)$ (a)

or

$$\tau_Y = \frac{2bT_Y}{\pi(b^4-a^4)} \qquad \text{(b)}$$

Hence,

$$\theta_Y = \frac{\tau_Y}{Gb} = \frac{2T_Y}{\pi G(b^4-a^4)} \qquad \text{(c)}$$



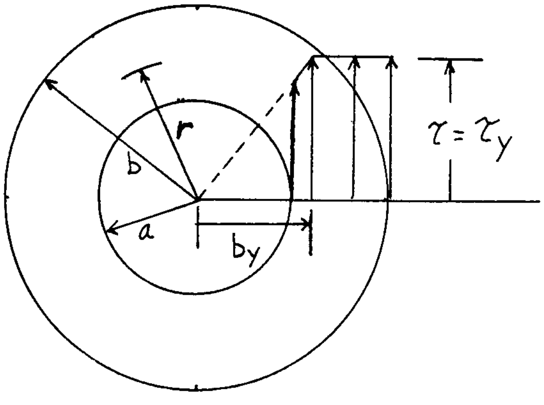
(A)

(b) Since the cross section is elastic to the radius b_Y (Fig. B), by Eq. (6.14),

$$b_Y = \frac{\tau_Y}{G\theta} \quad \text{or}$$

$$\frac{b_Y}{b} = \frac{\tau_Y}{G\theta b} = \frac{\theta_Y}{\theta}$$

$$\therefore \quad \theta = \frac{b\theta_Y}{b_Y} \qquad \text{(d)}$$



(B)

(cont.)

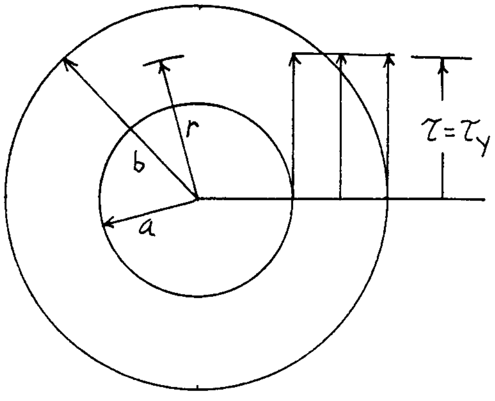
6.72 cont.

With Eqs. (c) and (d), we find in terms of T_Y

$$\theta = \frac{2bT_Y}{\pi b_Y G(b^4-a^4)} \quad (e)$$

(c) The elastic twisting moment T_E is due to the shear stress from $r=a$ to $r=b_Y$ (Fig. B). By Eq. (6.15), with $J=\frac{\pi}{2}(b_Y^4-a^4)$, we have

$$T_E = \frac{\pi}{2}\frac{(b_Y^4-a^4)}{b_Y}\tau_Y \quad (f)$$



(C)

The plastic torque T_P due to the shear stress in the yielded (plastic) region of the bar ($r \geq b_Y$) is

$$T_P = \int_{b_Y}^{b} r\,\tau_Y(2\pi r)dr = \frac{2\pi}{3}\tau_Y(b^3-b_Y^3) = \frac{2\pi}{3}\tau_Y\left(\frac{b^4-bb_Y^3}{b}\right) \quad (g)$$

Therefore,

$$T_{EP} = T_E + T_P = \frac{3\pi}{3}\tau_Y\left[\frac{b^4-bb_Y^3}{b} + \frac{3}{4}\left(\frac{b_Y^4-a^4}{b_Y}\right)\right] \quad (h)$$

(d) The fully plastic moment T_L is given by letting $b_Y \to a$, in Eq.(h). Thus, since $T_{EP} \to T_L$ as $b_Y \to a$, Eq.(h) yields

$$T_L = \frac{2\pi\tau_Y}{3}\left(\frac{b^4-ba^3}{b}\right) = \frac{2\pi\tau_Y}{3}(b^3-a^3) \quad (i)$$

Dividing Eq.(i) by Eq.(a) and reassembling, we obtain

$$T_L = \frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}T_Y \quad (j)$$

Note that as $a \to 0$, $T_L \to \frac{4}{3}T_Y$ [See Eq.(i), Example 6.12]

(cont.)

(e) When the limiting value of the moment T_L is released, the tube springs back (unwinds) elastically, and the shear stress-strain curve will follow a path similar to CA′ in Fig. 4.4.

Since $T_L = \frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}T_Y$, the elastic stress recovery at $r=b$ is $\tau_b = \frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}\tau_Y$. Since the recovery is elastic, the recovery at $r=a$ is $\tau_a = \frac{a}{b}\tau_b = \frac{4}{3}\frac{a(b^3-a^3)}{b^4-a^4}\tau_Y$.

Hence, the residual stress at $r=b$ is

$$\tau_{b(\text{residual})} = -\frac{4}{3}\frac{b(b^3-a^3)}{b^4-a^4}\tau_Y + \tau_Y$$

or

$$\tau_{b(\text{residual})} = \left[\frac{-4b(b^3-a^3)+3(b^4-a^4)}{3(b^4-a^4)}\right]\tau_Y \qquad (k)$$

and the residual stress at $r=a$ is

$$\tau_{a(\text{residual})} = \tau_Y - \frac{4}{3}\frac{a(b^3-a^3)}{b^4-a^4}\tau_Y$$

$$= \left[\frac{3(b^4-a^4)-4a(b^3-a^3)}{3(b^4-a^4)}\right]\tau_Y \qquad (\ell)$$

Note that as $a \to 0$, Eqs. (k) and (ℓ) reduce to

$$\tau_{b(\text{residual})} = -\frac{1}{3}\tau_Y \quad (\text{see Fig. E6.12c})$$

where the − sign denotes a shear stress in the opposite sense of the original shear stress τ_Y,

and

$$\tau_{a(\text{residual})} = \tau_Y \quad (\text{See Fig. E6.12c}).$$

7.1 By Fig. a, the area moment of inertia is

$$I_x = \frac{1}{12}(60)(80)^3 - \frac{1}{12}(30)(40)^3 = 2.4\times10^{-6}\,m^4 \qquad (a)$$

Since $(\sigma_{zz})_{max} = 75$ MPa, by Fig. a and the flexure formula, we obtain, with Eq. (a)

$$(M_x)_{max} = \frac{I_x(\sigma_{zz})_{max}}{c} = \frac{(2.4\times10^{-6})(75\times10^6)}{0.040}$$

or

$$(M_x)_{Max} = 4.50 \text{ kN}\cdot\text{m}$$

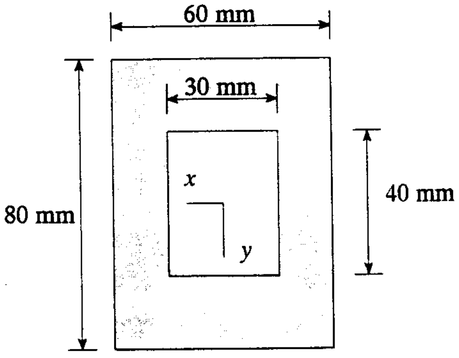


Figure a

7.2 The area A of the cross section (Fig. a) is

$$A = 2(20)(100) = 4000\,mm^2 = 4.0\times10^{-3}\,m^2$$

The centroid of the area is located at $\bar{y}$ above the bottom of the beam, where

$$A\bar{y} = (100)(20)(110) + (20)(100)(50) = 3.2\times10^5\,mm^3$$

or

$$\bar{y} = \frac{3.2\times10^5}{4.0\times10^3} = 80\text{ mm}$$

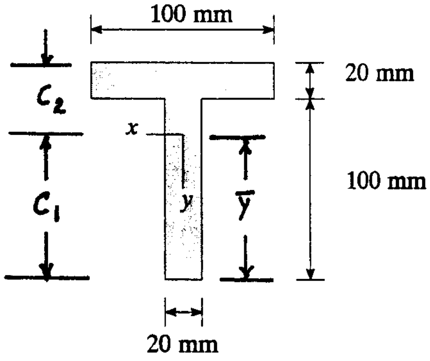


Figure a

Hence, by Fig. a,

$$c_1 = 80\,mm,\ c_2 = 40\,mm \qquad (a)$$

By Fig. a, Eqs (a), and the parallel axis theorem, we have

$$I_x = \frac{1}{12}(20)(80)^3 + (20)(80)(40)^2 + \frac{1}{12}(20)(20)^3 + (20)(20)(10)^2 + \frac{1}{12}(100)(20)^3 + (100)(20)(30)^2$$

or

$$I_x = 5.333\times10^6\,mm^4 = 5.333\times10^{-6}\,m^4 \qquad (b)$$

With Eqs. (a) and (b), and with $M_x = 5.0$ kN·m, the maximum tensile and compressive flexural stresses are (Cont.)

7.2 cont.

$$\sigma_{zz} = \frac{M_x c_1}{I_x} = \frac{5000(0.080)}{5.333\times10^{-6}} = 75\text{ MPa, Tension}$$

$$\sigma_{zz} = \frac{M_x c_2}{I_x} = \frac{5000(0.040)}{5.333\times10^{-6}} = 37.5\text{ MPa, Compression}$$

The stress distribution over the cross section is shown in Fig. b. By Figs. a and b, summation of tensile forces yields

$$\Sigma F_z = \frac{1}{2}(0.020)(0.080)(75\times10^6) = 60\text{ kN}$$

That is the total tensile force acting on the cross section is

$$F = 60\text{ kN}$$

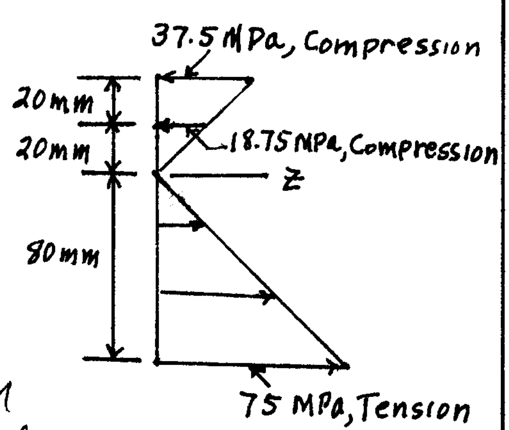


Figure b

Since the resultant force system on the section is a moment, the total compressive force is also 60 kN, as can be shown by Figs. a and b:

$$\Sigma F_z(\text{compression}) = \frac{1}{2}(0.020)(0.020)(18.75\times10^6) + \frac{1}{2}(0.100)(0.020)(37.5+18.75)\times10^6$$
$$= 60\text{ kN}$$

7.3 A positive bending moment M_x (Fig. a) causes a maximum compressive flexural stress of 50 MPa in the beam cross section. The magnitude of M_x is given by the flexure formula

$$M_x = \frac{|\sigma_{zz}|\, I_x}{c_2} \qquad (a)$$

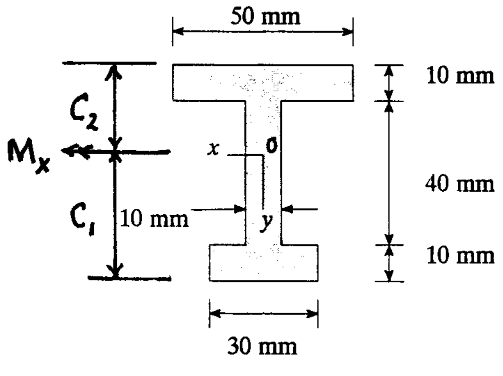


Figure a

By Fig. a, the area A and centroid $\bar{y} = c_1$ are determined as follows:

(Cont.)

7.3 cont.

$$A = 50(10) + 40(10) + 30(10) = 1200 \text{ mm}^2$$

Measuring from the bottom of the beam (Fig. a), we have

$$A\bar{y} = 1200C_1 = 30(10)(5) + (C_1 - 10)(10)\left(\frac{C_1+10}{2}\right) + (50 - C_1)(10)\left(\frac{50+C_1}{2}\right)$$
$$+ 50(10)(55) = 41{,}000$$

or

$$C_1 = 34.167 \text{ mm}, C_2 = 60 - C_1 = 25.833 \text{ mm} \quad (b)$$

Then, by Fig. a and Eqs. (b),

$$I_x = 5.3917 \times 10^5 \text{ mm}^4 = 5.3917 \times 10^{-7} \text{ m}^4 \quad (c)$$

By Eqs. (a), (b), and (c),

$$M_x = \frac{50 \times 10^6 (5.3917 \times 10^{-7})}{0.025833} = 1.044 \text{ kN}\cdot\text{m} \quad (d)$$

The maximum tensile flexural stress is

$$\sigma_{zz} = \frac{M_x C_1}{I_x} = \frac{(1044)(0.034167)}{5.3917 \times 10^{-7}} = 66.16 \text{ MPa}$$

7.4

The area of the cross section (Fig. a) is

$$A = 150(200) - 100(100) = 20{,}000 \text{ mm}^2$$

and

$$A\bar{y} = 20{,}000C_1 = 150(200)(100) - 100(100)(125)$$

or

$$C_1 = 87.5 \text{ mm} = 0.0875 \text{ m} \quad (a)$$
$$C_2 = 200 - C_1 = 112.5 \text{ mm} = 0.1125 \text{ m}$$

Then,

$$I_x = 82.292 \times 10^6 \text{ mm}^4 = 82.292 \times 10^{-6} \text{ m}^4 \quad (b)$$

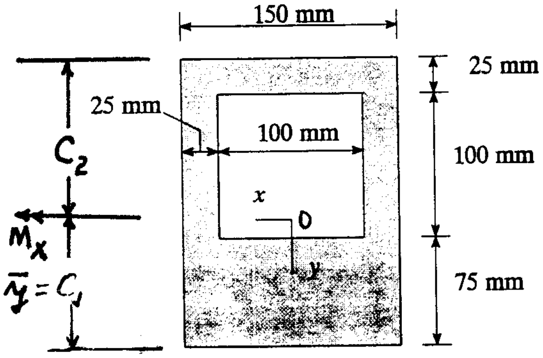


Figure a

By Eqs. (a) and (b), with the flexure formula, the magnitude of the maximum flexural stress (compression) is with $M_x = 80$ kN·m,

$$\sigma_{max} = \frac{M_x C_2}{I_x} = \frac{(80 \times 10^3)(0.1125)}{82.292 \times 10^{-6}} = 109.4 \text{ MPa}$$

$\boxed{7.5}$ By measurements, the axial strains in the beam (Fig. a) are $\epsilon_1 = 0.00080$ at 75 mm from the top and $\epsilon_2 = -0.000120$ at 55 mm from the top. Since the strains vary linearly with y, by Fig. a, we have relative to the centroidal axis x

$$\frac{\epsilon_2}{C_2-55} = \frac{\epsilon_1}{75-C_2} \text{ or } \frac{0.00012}{C_2-55} = \frac{0.0008}{75-C_2}$$

Hence,

$$C_2 = 57.61 \text{ mm} \qquad (a)$$

and

$$\bar{y} = C_1 = 100-57.61 = 42.39 \text{ mm} \quad (b)$$

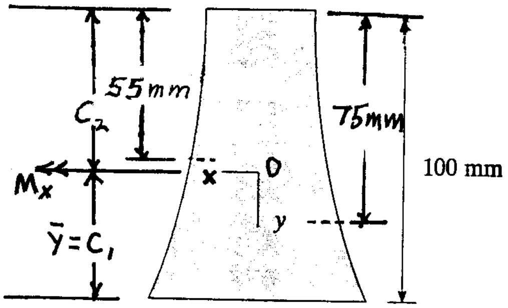


Figure a

that is, the neutral axis is located 42.39 mm from the bottom of the section, or 57.61 mm from the top. Then, since the strain varies linearly with y, the strain at the top is obtained from the relation

$$\frac{\epsilon_{TOP}}{-57.61} = \frac{\epsilon_2}{55-57.61} = \frac{-0.000120}{-2.61}$$

or

$$\epsilon_{TOP} = -0.00265$$

So, the stress at the top of the beam is compressive and equal to

$$\sigma_{TOP} = E\,\epsilon_{TOP} = (72\times10^9)(-0.00265) = -190.7 \text{ MPa} \quad (c)$$

Likewise, the strain at the bottom of the beam is given by

$$\frac{\epsilon_{BOTTOM}}{42.39} = \frac{\epsilon_1}{75-C_2} = \frac{0.0008}{17.39}\,;\ \epsilon_{BOTTOM} = 0.00195$$

and

$$\sigma_{BOTTOM} = E\,\epsilon_{BOTTOM} = (72\times10^9)(0.00195) = 140.4 \text{MPa} \quad (d)$$

By Eqs. (a), (b), (c), and (d), and the flexure formula, we obtain I_x as with $M_x = 6.0$ kN·m, $I_x = \dfrac{M_x\, y_{TOP}}{\sigma_{TOP}} = \dfrac{M_x\, y_{BOTTOM}}{\sigma_{BOTTOM}} = 1.812\times10^{-6} \text{ m}^4$

7.6 By the free-body diagram of the beam (Fig. a), the support reactions are given by

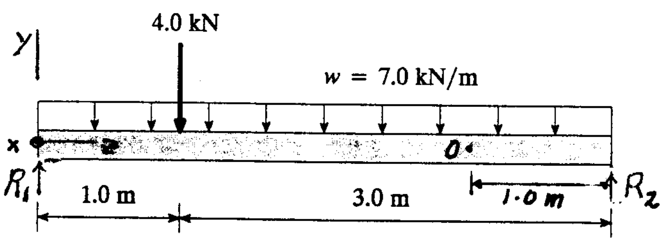


Figure a

$$\Sigma F_y = R_1 - 4 - 7(4) + R_2 = 0$$

$$\circlearrowleft \Sigma M_{R_1} = 4(1) + 7(4)(2) - 4R_2 = 0 \qquad (a)$$

The solution of Eqs. (a) is

$$R_1 = 17 \text{ kN}, \quad R_2 = 15 \text{ kN} \qquad (b)$$

Hence, with $R_2 = 15$ kN, the moment at the section 1.0 m from the right end of the beam is given by (Fig. a)

$$\circlearrowleft + \Sigma M_D = M_x + 7(1)\left(\tfrac{1}{2}\right) - R_2(1) = 0$$

or

$$M_x = 11.5 \text{ kN}\cdot\text{m} \qquad (c)$$

By Eq. (1.7), the flexural stress is

$$\sigma_{zz} = -\frac{M_x y}{I_x} \qquad (d)$$

where $I_x = \frac{1}{12}(0.150)(0.300)^3 = 3.375 \times 10^{-4} \text{ m}^4$

Hence, with Eqs. (c) and (d), the maximum tensile flexural stress at the section 1.0 m from the right end occurs at $y = -0.150$ m and is

$$\sigma_{zz} = -\frac{(11.5)(-0.150)}{3.375 \times 10^{-4}} = 5.11 \text{ MPa (Tension)}$$

The maximum compressive flexural stress occurs at $y = 0.150$ m and is

$$\sigma_{zz} = -\frac{(11.5)(0.150)}{3.375 \times 10^{-4}} = -5.11 \text{ MPa (Compression)}$$

$\boxed{7.7}$ The maximum moment in the beam due to P occurs at the wall ($z=0$; Fig. a). Summation of moments about 0 yields

$$\circlearrowleft + \Sigma M_0 = M_x + 8.0(2.5) = 0$$

or

$$M_x = -20.0 \text{ kN}\cdot\text{m} \qquad (a)$$

By Eq.(1.7), the flexural stress is

$$\sigma_{zz} = -\frac{M_x y}{I_x} \qquad (b)$$

where by Fig. a, with a board thickness of 50 mm,

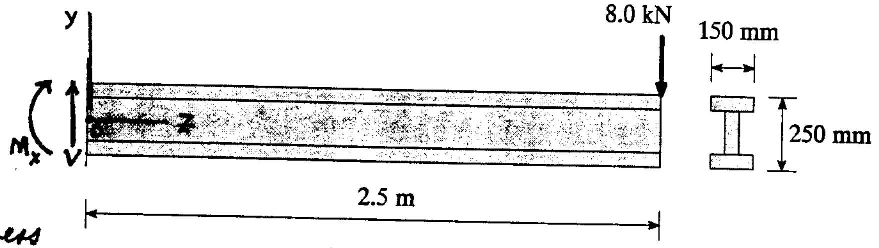


Figure a

$$I_x = \tfrac{1}{12}(0.050)(0.150)^3 + 2\left(\tfrac{1}{12}\right)(0.150)(0.050)^3 + 2(0.150)(0.050)(0.10)^2$$

or

$$I_x = 1.672 \times 10^{-4} \text{ m}^4 \qquad (c)$$

The maximum tensile stress occurs at the top of the beam at the wall ($z=0$) and is, with Eqs. (a), (b), and (c),

$$\sigma_{zz} = -\frac{(-20)(0.125)}{1.672\times 10^{-4}} = 14.95 \text{ MPa}$$

Likewise, the maximum tensile flexural stress in the center board occurs at $y = 0.075$ m. Hence,

$$\sigma_{zz} = -\frac{(-20)(0.075)}{1.672\times 10^{-4}} = 8.97 \text{ MPa}$$

7.8 For the beam of Problem 7.7, we want to determine the maximum tensile load in the top board. By Eq.(1.7), the stress distribution in the beam is

$$\sigma_{zz} = -\frac{M_x y}{I_x} \qquad (a)$$

Where $M_x = -20.0$ kN·m and $I_x = 1.672 \times 10^{-4}$ m^4. Hence, by Eq.(a),

$$\sigma_{zz} = (1.196 \times 10^5) y \qquad (b)$$

The tensile load in the top board is given by

$$P = \int_{0.075}^{0.125} \sigma_{zz} (0.150 dy) \qquad (c)$$

or by Eqs.(b) an (c),

$$P = 89.71 \text{ kN}$$

7.9 By Fig. a, the support reactions are determined by

$$\Sigma F_y = R_1 + R_2 - 3.0w = 0$$
$$\circlearrowleft + \Sigma M_{R_1} = 3.0w(1.5) - 3.0 R_2 = 0 \qquad (a)$$

The solution of Eq.(a) is, with $w = 60$ kN/m,

$$R_1 = 90 \text{ kN}, \quad R_2 = 90 \text{ kN}$$

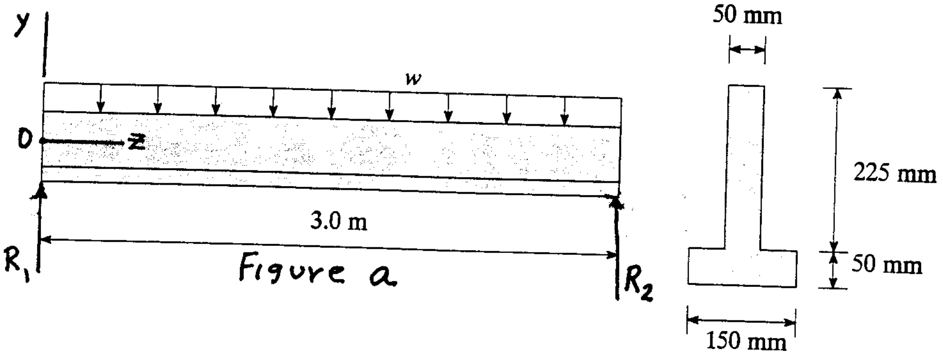


Figure a

cont.

7.9 Cont. The maximum bending moment occurs at the midlength ($z = 1.5\text{m}$) of the beam. By Fig. b,

$$\circlearrowleft + \Sigma M_0 = (1.5w)(0.075) - M = 0$$

or

$$M = 67.5\ \text{kN}\cdot\text{m} \qquad (a)$$

By Fig a, the area A of the cross section is

$$A = 0.050(0.225) + 0.050(0.150) = 0.01875\ \text{m}^2 \qquad (b)$$

The centroidal axis is located at $\bar{y}$ from the bottom and is given by

$$A\bar{y} = 0.01875\bar{y} = 0.150(0.050)(0.025) + 0.050(0.225)(0.1625)$$

or

$$\bar{y} = 0.1075\ \text{m} = 107.5\ \text{mm} \qquad (c)$$

Then, by Fig.a, the area moment of inertia of the cross section is

$$I_x = \tfrac{1}{12}(0.150)(0.050)^3 + 0.050(0.150)(0.1075 - 0.025)^2 + \tfrac{1}{12}(0.050)(0.225)^3$$
$$+ 0.050(0.225)(0.1625 - 0.1075)^2$$

or

$$I_x = 1.341 \times 10^{-4}\ \text{m}^4 \qquad (d)$$

Hence, by Eq.(1.7), the magnitude of the flexural stress is

$$|\sigma_{zz}| = \left| -\frac{My}{I_x} \right| = \left| \frac{67.5 \times 10^3}{1.341 \times 10^{-4}} y \right| = |5.034 \times 10^8 y| \qquad (e)$$

at the top of the beam, $y = 0.275 - 0.1075 = 0.1675\ \text{m}$. Then, by Eq. (e), the magnitude of the stress is $|\sigma_{zz}| = 84.32\ \text{MPa}$ (compression)

at the junction between the web and the flange, $y = -0.1075 + 0.050 = -0.0575\ \text{m}$. Then, by Eq.(e), the magnitude of the stress is $|\sigma_{zz}| = 28.95\ \text{MPa}$ (Tension)

(Cont.)

7.9 cont. at the bottom of the beam, $y = -0.1075$ m, and the magnitude of the stress is, by Eq. (e), $|\sigma_{zz}| = 54.12$ MPa (Tension).

7.10 By Fig. a, the support reactions are determined by the equilibrium conditions

$$\Sigma F_y = R_1 + R_2 - 3.0(8.0) - 30 = 0$$

$$\circlearrowleft + \Sigma M_0 = 3.0(8.0)(1.50) + 30.0(3.0) - 5.0 R_2 = 0$$

The solution of these equations is

$$R_1 = 28.8 \text{ kN}, \quad R_2 = 25.2 \text{ kN} \qquad (a)$$

The moment diagram is shown in Fig. b, where the maximum moment is $M_{max} = 50.4$ kN at $z = 3.0$ m. Also, by Fig. a, the area moment of inertia is $I_x = \pi d^4/64$ and the maximum distance from the neutral axis is $c = d/2$. Hence, the magnitude of the maximum flexure stress is, with $d = 0.150$ m,

$$\sigma_{Max} = \frac{M_{max} c}{I_x} = \frac{32 M_{max}}{\pi d^3} = 152.1 \text{ MPa}.$$

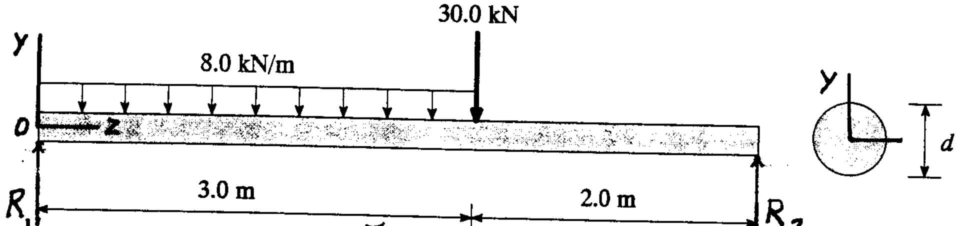


Figure a

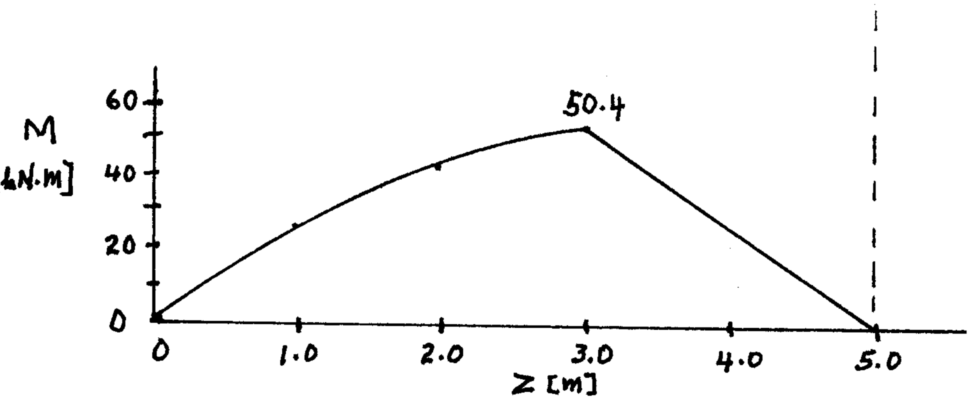


Figure b

7.11 The free-body diagram of the beam is shown in Fig. a, By Fig. a and equilibrium conditions, we have

$$\Sigma F_y = R_1 - 4.0(4.0) + R_2 - 16.0 = 0$$

$$\circlearrowleft + \Sigma M_O = 4.0(4.0)(2.0) - 16.0 - 6.0R_2 + 16.0(8.0) = 0$$

The solution of these equations is

$$R_1 = 8\,kN,\ R_2 = 24\,kN \qquad (a)$$

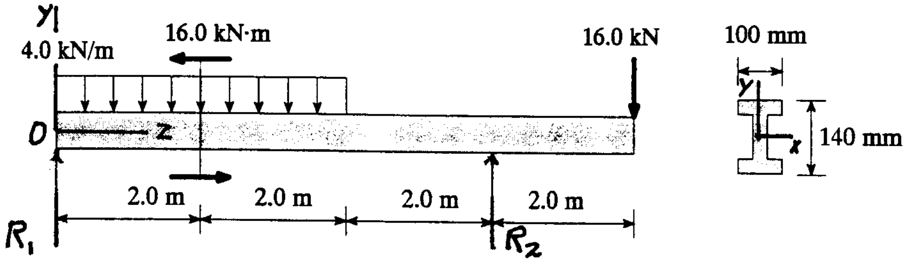


Figure a

Hence, by Eq. (a) and Fig. a, the moment to the left of the 16.0 kN·m couple is

$$M = 8\,kN\cdot m \qquad (b)$$

and to the right of the 16.0 kN·m couple is

$$M = -8\,kN\cdot m \qquad (c)$$

The area moment of inertia of the cross section is, by Fig. a, with the thickness of all flanges and the web of 20 mm,

$$I_x = \frac{1}{12}\left[0.100(0.140)^3 - 6.080(0.100)^3\right] = 1.620\times10^{-5}\,m^4 \quad (d)$$

Therefore, to the left of the couple, the maximum tensile stress is, at $y = -0.070$ m, the bottom of the beam,

$$\sigma_{zz} = -\frac{My}{I_x} = -\frac{8(-0.070)}{1.620\times10^{-5}} = 34.57\,MPa$$

Likewise, to the right of the couple, the maximum tensile stress occurs at the top of the beam ($y = +0.070$ m) and is

$$\sigma_{zz} = -\frac{My}{I_x} = -\frac{(-8)(0.070)}{1.620\times10^{-5}} = 34.57\,MPa$$

7.12 The free-body diagram of the beam is shown in Fig. a. By Fig. a and equilibrium conditions, we have

$$\Sigma F_y = R_1 - 4.0(2.0) - 16.0 + R_2 - 4.0 = 0$$

$$\circlearrowright + \Sigma M = -4.0(2.0)(1.0) + 16.0(3.0) - 4.0R_2 + 4.0(5.0) = 0$$

The solutions of these equations is

$$R_1 = 13 \text{ kN}, \quad R_2 = 15 \text{ kN} \qquad (a)$$

By Eqs. (a) and Fig. a, we may construct the moment diagram (Fig. b). By Fig. b, the magnitude of the largest moment is $M_{max} = 8$ kN·m at $z = 2$ m. The next largest moment is 7 kN·m at $z = 5$ m. To determine the maximum values for tensile and compressive flexural stresses, we must determine the neutral axis of the cross section.

By Fig a, the area A of the cross section is

$$A = 0.100(0.120) - 0.080(0.100) = 0.004 \text{ m}^2$$

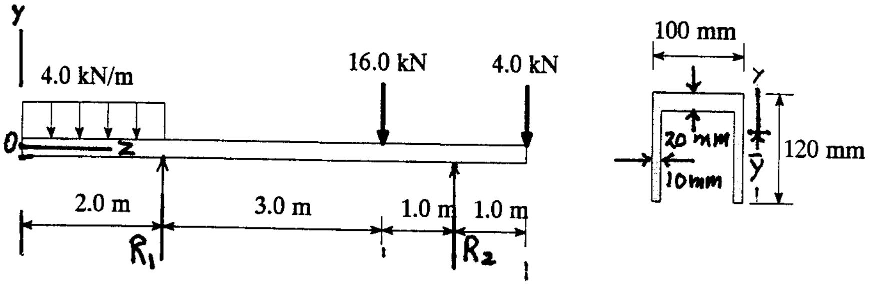


Figure a

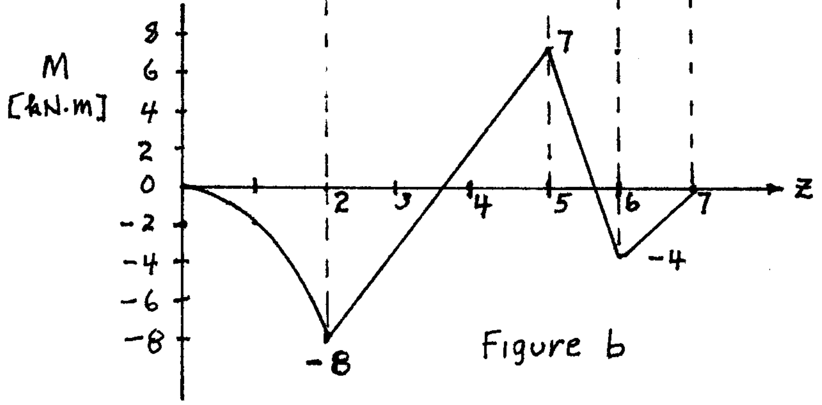


Figure b

(cont.)

7.12 Cont. The distance $\bar{y}$ to the neutral axis from the bottom of the beam (Fig. a) is given by

$$A\bar{y} = 0.004\bar{y} = 0.100(0.020)(0.110) + 2(0.100)(0.010)(0.050)$$

or

$$\bar{y} = 0.08\ \text{m} \qquad (b)$$

With Eq.(b) and Fig. a, the area moment of inertia is

$$I_x = \frac{1}{12}(0.100)(0.020)^3 + 0.100(0.02)(0.03)^2$$
$$+ 2\left[\frac{1}{12}(0.010)(0.100)^3 + 0.010(0.100)(0.03)\right]^2$$

or

$$I_x = 5.333 \times 10^{-6}\ \text{m}^4 \qquad (c)$$

At $z = 2$ m (Fig. b), the flexural stress is tensile at the top of the beam and compressive at the bottom. At the top, $y = 0.04$ m. So, with $M = -8$ kN·m,

$$\sigma_{zz} = -\frac{My}{I_x} = -\frac{(-8)(0.04)}{5.333 \times 10^{-6}} = 60\ \text{MPa, Tension}$$

At the bottom, $y = -0.08$ m, and

$$\sigma_{zz} = -\frac{My}{I_x} = -\frac{(-8)(-0.08)}{5.333 \times 10^{-6}} = -120\ \text{MPa, compression}$$

At $z = 5$ m (Fig. b), $M = 7$ kN·m and the flexural stress is compressive at the top of the beam and tensile at the bottom. At the top $y = 0.04$ m and hence,

$$\sigma_{zz} = -\frac{My}{I_x} = -\frac{(7)(0.04)}{5.333 \times 10^{-6}} = -52.50\ \text{MPa, compression}$$

At the bottom, $y = -0.08$ m and

$$\sigma_{zz} = -\frac{My}{I_x} = -\frac{(7)(-0.08)}{5.333 \times 10^{-6}} = 105\ \text{MPa, Tension}$$

(cont.)

7.12 cont. Therefore the maximum tensile stress (105 MPa) occurs at the bottom of the beam at $z = 5$ m, and the maximum compressive stress (−120 MPa) occurs at the bottom of the beam at $z = 2$ m.

7.13 By Table A.5 of Appendix A, $I_{x'}$ for the angle section (Fig. a) is

$$I_{x'} = 4.20 \times 10^{5} \text{ mm}^4 = 4.20 \times 10^{-7} \text{ m}^4 \qquad (a)$$

and its area is

$$A = 1120 \text{ mm}^2 = 0.00112 \text{ m}^2 \qquad (b)$$

Hence, the area moment of inertia of the cross section is, with Eqs (a) and (b) and Fig. a,

$$I_x = 4 I_{x'} + 4(0.00112)(0.0805)^2 + \tfrac{1}{12}(0.010)(0.200)^3$$

or

$$I_x = 3.738 \times 10^{-5} \text{ m}^4 \qquad (c)$$

Hence, by the flexure formula, with $\sigma_{zz} = 60$ MPa,

$$M_{max} = \frac{\sigma_{zz} I_x}{y} = \frac{(60 \times 10^6)(3.738 \times 10^{-5})}{0.100}$$

or

$$M_{max} = 22.43 \text{ kN}\cdot\text{m}$$

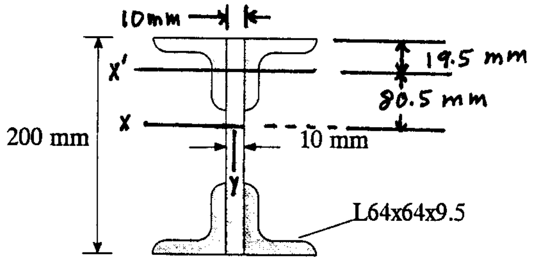


Figure a

7.14 The properties of the C150× 19 Channel are $t_{web} = 11.1$ mm, $A_c = 2470$ mm^2, $\bar{y}_c = 13.1$ mm, $I_{x'} = 0.437 \times 10^6$ mm^4. For the W310× 33 I-section, d = 313 mm, $\bar{y}_I = 156.5$ mm, $A_I = 4180$ mm^2, $I_X = 65.0 \times 10^6$ mm^4 (see Fig. a). Therefore, the total area of the cross section is $A = A_c + A_I = 6650$ mm^2. Then, the centroidal axis of the cross section located $\bar{y}$ from the bottom of the beam is obtained as (Fig. a)

$$A\bar{y} = 6650\bar{y} = 4180(156.5) + 2470(313 + 11.1 - 13.1)$$

or

$$\bar{y} = 213.9 \text{ mm}$$

Then, by Fig. a, the area moment of inertia of the cross section is

$$I_x = I_X + 4180(213.9 - 156.5)^2 + I_{x'} + 2470(97.1)^2$$

or

$$I_x = 1.025 \times 10^8 \text{ mm}^4 = 1.025 \times 10^{-4} \text{ m}^4 \qquad (a)$$

By Eq. (a) and Fig. a, with $M_x = 20.0$ kN·m, at the bottom of the beam, $y = \bar{y} = 0.2139$ mm, the maximum tensile stress is

$$\sigma_{zz} = \frac{M_x \bar{y}}{I_x} = 41.74 \text{ MPa, Tension}$$

at the top of the beam, $y = -110.2$ mm, the maximum compressive stress is

$$\sigma_{zz} = \frac{M_x y}{I_x} = -21.50 \text{ MPa, compression}$$

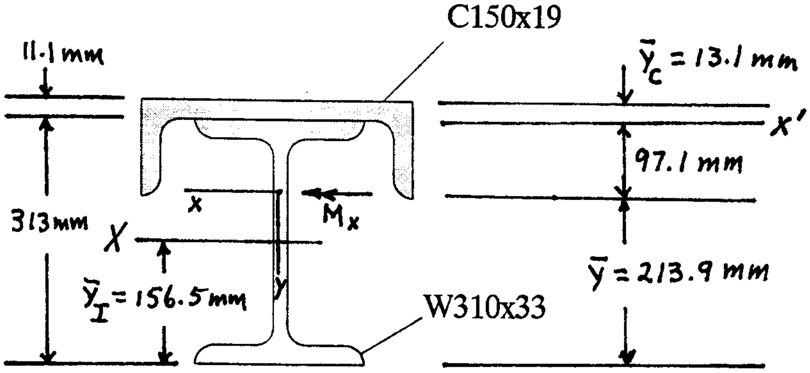


Figure a

7.15 The support reactions are $R_1 = R_2 = 1.5P$ (Fig. a). The moment diagram of the beam is shown in Fig. b. By the results of Problem 7.13, the area moment of the beam cross section (Fig. c) is $I_x = 3.738\times10^{-5}\,m^4$. Hence, by Figs. b and c and the flexure formula, the maximum flexural stress at the left support is (P in Newtons)

$$(\sigma_{zz})_{max} = -\frac{M_{max} y}{I_x} = -\frac{(-P)(0.100)}{3.738\times10^{-5}} = 2.675\times10^{3} P \;[N/m^2]$$

Since $M_x = 0$ at the center of the beam, the flexure stress is zero at the center of the beam.

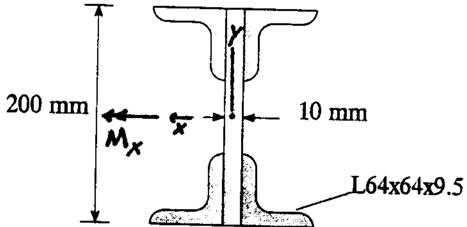


Figure c

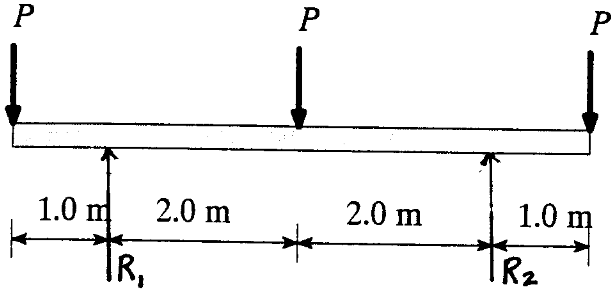


Figure a

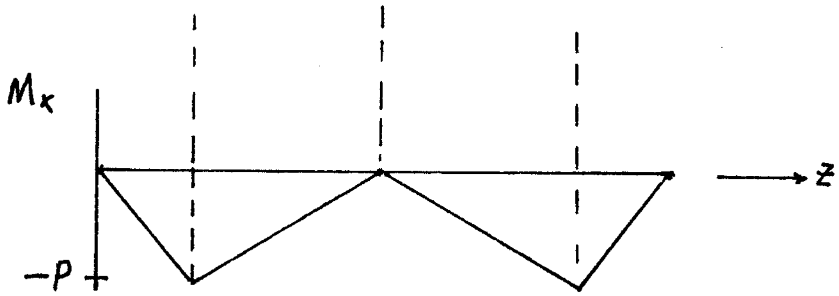


Figure b

7.16 By equilibrium of the beam (Fig. a), $2(w) = 20$ kN, or $w = 10$ kN/m. The moment diagram of the beam is shown in Fig. b.

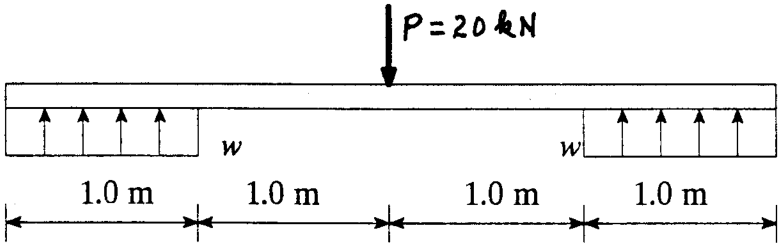


Figure a

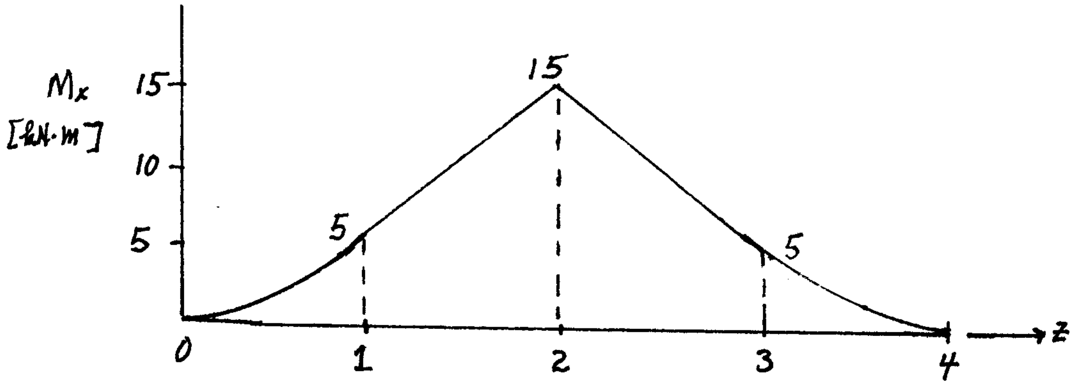


Figure b

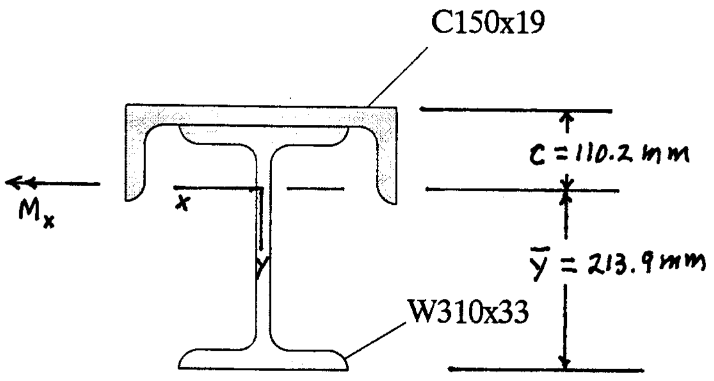


Figure c

The cross section of the beam is shown in Fig. c, where by the results in Problem 7.14,

$$\bar{y} = 213.9 \text{ mm}, \; I_x = 1.025 \times 10^{-4} \text{ m}^4$$

(cont.)

7.16 cont. at 1.5 m from the left end of the beam, $M_x = 10\,\text{kN}\cdot\text{m}$. Hence, the maximum tensile stress at this section occurs at the bottom ($y = 0.2139$ m, Fig. c). It is

$$\sigma_{zz} = \frac{M_x y}{I_x} = \frac{(10\times10^3)(0.2139)}{1.025\times10^{-4}} = 20.87 \text{ MPa}$$

and the maximum compressive stress at the top ($y = -0.1102$ m) is

$$\sigma_{zz} = \frac{M_x y}{I_x} = \frac{(10\times10^3)(-0.1102)}{1.025\times10^{-4}} = -10.75 \text{ MPa}$$

Likewise at the midspan the maximum tensile stress is, since $M_x = 15\,\text{kN}\cdot\text{m}$,

$$\sigma_{zz} = \frac{M_x y}{I_x} = \frac{(15\times10^3)(0.2139)}{1.025\times10^{-4}} = 31.30 \text{ MPa}$$

and the maximum compressive stress is

$$\sigma_{zz} = \frac{M_x y}{I_x} = \frac{(15\times10^3)(-0.1102)}{1.025\times10^{-4}} = -16.13 \text{ MPa}$$

7.17 Since the areas of the cross sections (Fig. a) are each equal to 3020 mm², the thickness t of the box beam is determined as follows:

$$\text{Area} = 2(80)t + 2(111 - 2t)t = 3020 \text{ mm}^2$$

or $4t^2 - 382t + 3020 = 0$. Therefore, $t = 8.698$ mm

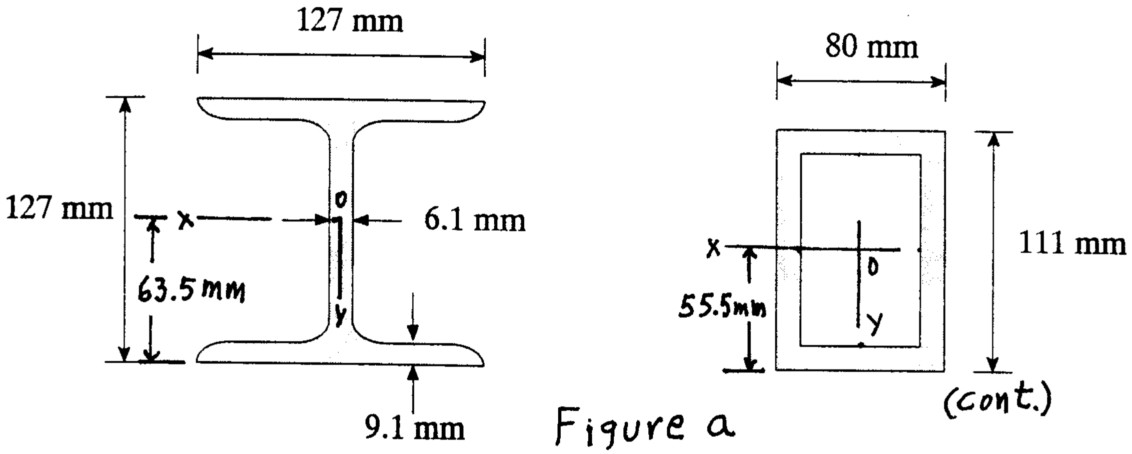


Figure a

(cont.)

7.17 Cont. For the I-beam (using the dimensions shown in Fig. a, and assuming that the flanges are rectangular in cross section)

$$I_x = 8.70 \times 10^6 \text{ mm}^4 = 8.70 \times 10^{-6} \text{ m}^4$$

$$I_y = 3.11 \times 10^6 \text{ mm}^4 = 3.11 \times 10^{-6} \text{ m}^4 \qquad (a)$$

From tables for I properties, $I_x = 8.83 \times 10^6 \text{ mm}^4$ and $I_{yy} = 3.12 \times 10^6 \text{ mm}^4$, which includes the contribution of the fillets.

For the box beam, with $t = 8.698$ mm,

$$I_x = 4.839 \times 10^6 \text{ mm}^4 = 4.839 \times 10^{-6} \text{ m}^4$$

$$I_y = 2.822 \times 10^6 \text{ mm}^4 = 2.822 \times 10^{-6} \text{ m}^4 \qquad (b)$$

There are two extreme positions, one with the load-cable vertically down and the other with the cable swung 15° from the vertical (Fig. b).

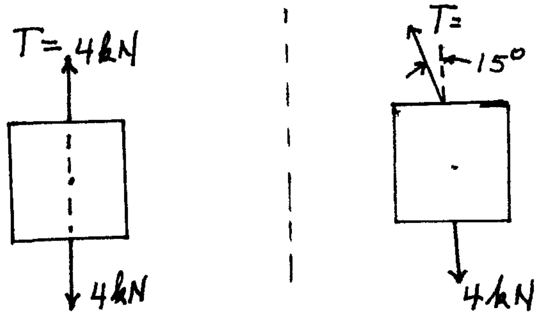


Figure b

Consider first the vertical position. For this case, the maximum flexural stress in the I-beam is, with L= 2m,

$$\sigma_{max} = \frac{M_x y}{I_x} = \frac{(4000)(2)(63.5)}{8.70 \times 10^{-6}}$$

$$= 58.39 \text{ MPa}$$

and in the box beam, $\sigma_{max} = \frac{M_x y}{I_x} = \frac{(4000)(2)(55.5)}{4.839 \times 10^{-6}} = 91.75 \text{ MPa}$

So, in the vertical position, the I-beam has a safety factor of

$$SF = \frac{250}{58.39} = 4.28$$

and the box beam has a safety factor of

$$SF = \frac{250}{91.75} = 2.72$$

(cont.)

7.17 cont. For the case of a swing of 15°, assume that there is no angular velocity at the top of the swing (Fig. b). Hence, by Fig. b,

$$\Sigma F_T = T - 4\cos 15^\circ = 0; \quad T = 4\cos 15^\circ = 3.864 \text{ kN}$$

Then, by Fig. c.

$$M_x = -(3.864\cos 15^\circ)(2) = -7.465 \text{ kN}\cdot\text{m}$$

$$M_y = -(3.864\sin 15^\circ)(2) = -2.000 \text{ kN}\cdot\text{m}$$

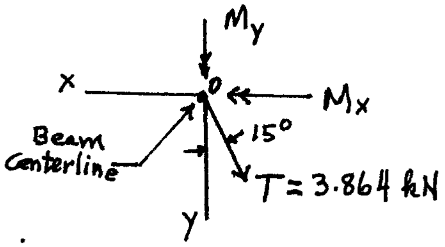


Figure c

Hence, for a swing of 15°, the maximum flexure tensile stress in the I-beam is (at the upper left hand extreme point in the flange; the maximum compressive stress occurs at the lower right hand extreme point)

$$\sigma_{max} = \frac{M_x y}{I_x} - \frac{M_y x}{I_y} = \frac{(-7465)(-63.5)}{8.70\times10^{-6}} - \frac{(-2000)(63.5)}{3.11\times10^{-6}}$$

or

$$\sigma_{max} = 95.32 \text{ MPa}.$$

Similarly for the box beam

$$\sigma_{max} = \frac{M_x y}{I_x} - \frac{M_y x}{I_y} = \frac{(-7465)(-55.5)}{4.839\times10^{-6}} - \frac{(-2000)(0.040)}{2.822\times10^{-6}}$$

or

$$\sigma_{max} = 113.97 \text{ MPa}$$

Then, for the I-beam, the safety factor is

$$SF = \frac{250}{95.32} = 2.62$$

and for the box-beam

$$SF = \frac{250}{113.97} = 2.19$$

In both cases, the I-beam has the larger safety factor, and the maximum flexure stress is larger in the swung position for both beams.

7.18

$$I_x = \frac{250(300)^3}{12} = 562{,}500{,}000 \text{ mm}^4$$

$$I_y = \frac{300(250)^3}{12} = 390{,}600{,}000 \text{ mm}^4$$

$$\tan\alpha = -\frac{I_x \cot\phi}{I_y} = -\frac{562{,}500{,}000(-0.1763)}{390{,}600{,}000} = 0.2539$$

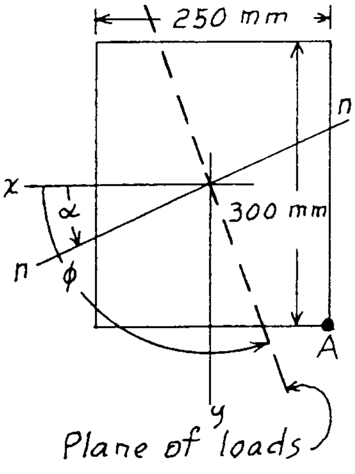
$$\alpha = \underline{0.2486 \text{ rad}}$$

$$M = \frac{PL}{4} = 1000P \text{ (N.mm)}$$

$$M_x = M\sin\phi = 1000P(0.9848) = 985P$$

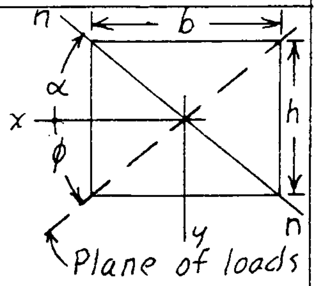
$$\sigma_A = \frac{Y}{SF} = \frac{M_x(y_A - x_A\tan\alpha)}{I_x}$$

$$P = \frac{Y I_x}{SF(985)(y_A - x_A\tan\alpha)} = \frac{25.0(562{,}500{,}000)}{2.50(985)[150-(-125)(0.2539)]} = \underline{31.4 \text{ kN}}$$



7.19 $I_x = bh^3/12\,;\ I_y = hb^3/12$

$$\tan\alpha = -\frac{I_x\cot\phi}{I_y} = -\frac{bh^3(12)}{12\,hb^3}\,\frac{b}{h} = \underline{-\frac{h}{b}}$$



7.20 $A = 300(25) + 275(25) = 14{,}375 \text{ mm}^2$

$$x_0 = y_0 = \frac{300(25)(150) + 275(25)(12.5)}{14{,}375} = 84.24 \text{ mm}$$

$$I_x = I_y = \frac{25(300)^3}{12} + \frac{275(25)^3}{12} + 300(25)(65.76)^2 + 275(25)(71.74)^2$$

$$= 124{,}400{,}000 \text{ mm}^4$$

$$I_{xy} = 300(25)(-71.74)(-65.76) + 275(25)(78.26)(71.74) = 73{,}981{,}000 \text{ mm}^4$$

$$\phi = \frac{\pi}{2}\,;\ \cot\phi = 0$$

$$\tan\alpha = \frac{I_{xy}}{I_y} = \frac{73{,}981{,}000}{124{,}400{,}000} = 0.5947$$

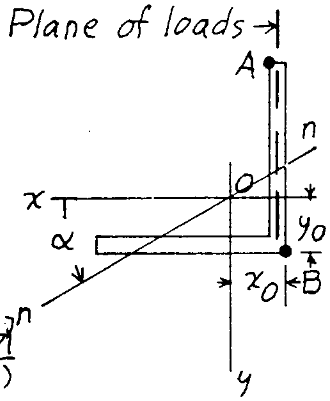
$$\alpha = \underline{0.5365} \text{ RAD}$$

$$M = -PL = -16{,}000(2500) = -40.0 \text{ kN.m} = M_x$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{-40{,}000{,}000[-215.76-(-59.24)(0.5947)]}{124{,}400{,}000-(73{,}981{,}000)(0.5947)}$$

$$= \underline{89.81 \text{ MPa}}$$

$$\sigma_B = \frac{-40{,}000{,}000[84.24-(-84.24)(0.5947)]}{124{,}400{,}000 - 73{,}981{,}000(0.5947)} = \underline{-66.83 \text{ MPa}}$$



7.21 $A = 300 \times 25 + 175 \times 25 = 11{,}875\ mm^2$

$$X_0 = \frac{300(25)(12.5) + 175(25)112.5}{11{,}875} = 49.34\ mm$$

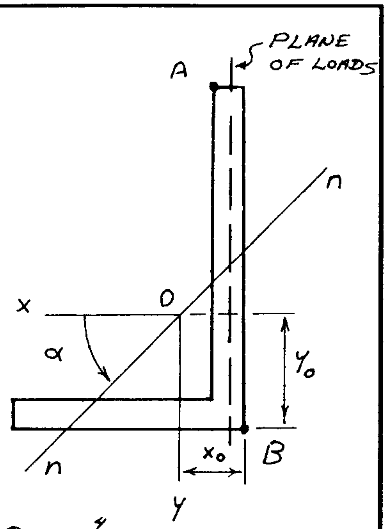
$$Y_0 = \frac{300(25)150 + 175(25)12.5}{11{,}875} = 99.34\ mm$$

$$I_x = \frac{25(300^3)}{12} + \frac{175(25^3)}{12} + 300(25)(50.66^2) + 175(25)(86.84^2)$$

$$I_x = 108{,}700{,}000\ mm^4$$

$$I_y = \frac{300(25^3)}{12} + \frac{25(175^3)}{12} + 300(25)36.84^3 + 175(25)63.16^2$$

$$I_y = 39{,}200{,}000\ mm^4$$



$$I_{xy} = 300(25)(-50.66)(-36.84) + 175(25)(86.84)(63.16) = 38{,}000{,}000\ mm^4$$

$$\phi = \pi/2\ ;\ \cot\phi = 0\ ;\ \tan\alpha = \frac{I_{xy}}{I_y} = \frac{38{,}000{,}000}{39{,}200{,}000} = 0.9694$$

$$\alpha = \underline{\underline{0.7699\ RAD}}$$

$$M_x = M = -PL = -16{,}000(2500) = -40.0\ kN\cdot m$$

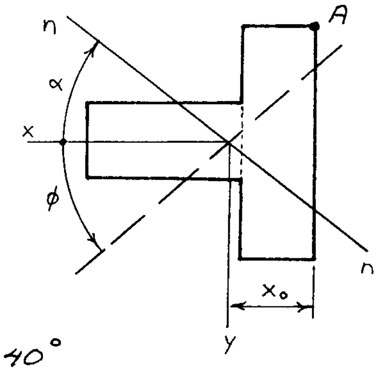
$$\sigma_A = \frac{M_x(Y_A - X_A \tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{-40{,}000{,}000[-200.66 - (-24.34)(0.9694)]}{108{,}700{,}000 - 38{,}000{,}000(0.9694)} = \underline{\underline{98.56\ MPa}}.$$

$$\sigma_B = \frac{-40{,}000{,}000[99.34 - (-49.34)(0.9694)]}{108{,}700{,}000 - 38{,}000{,}000(0.9694)} = \underline{\underline{-81.92\ MPa}}.$$

7.22 $A = 150 \times 50 + 100 \times 50 = 12{,}500\ mm^2$

$$x_0 = \frac{150(50)25 + 100(50)100}{12{,}500} = 55\ mm$$

$$I_x = \frac{50(150^3)}{12} + \frac{100(50^3)}{12} = 15{,}104{,}000\ mm^4$$



$$I_y = \frac{150(50^3)}{12} + \frac{50(100^3)}{12} + 150(50)(55-25)^2 + 100(50)(100-55)^2$$

$$I_y = 22{,}604{,}000\ mm^4\ ,\quad I_{xy} = 0\ ,\quad \phi = \frac{2\pi}{9}\ RAD = 40^\circ$$

$$\tan\alpha = \frac{-I_x \cot\phi}{I_y} = \frac{-15{,}104{,}000(1.1918)}{22{,}604{,}000} = -0.7963$$

$$\alpha = -0.6725\ RAD = -38.53^\circ$$

$$M_x = -2000P \sin\phi = -1286P$$

$$\sigma_A = Y = \frac{SF(M_x)(Y_A - X_A \tan\alpha)}{I_x}$$

$$M_x = -1286P = \frac{Y I_x}{SF(Y_A - X_A \tan\alpha)} = \frac{240(15{,}104{,}000)}{2.0(-75 - (-55)(-0.7963))} = -15{,}257{,}000$$

$$\therefore\ \underline{\underline{P = 11.86\ kN}}$$

7.23

$$I_x = 2\left[\frac{50(150)^3}{12} + 150(50)(50)^2\right] = 65,630,000 \text{ mm}^4$$

$$I_y = \frac{150(100)^3}{12} = 12,500,000 \text{ mm}^4$$

$$I_{xy} = 2(150)(50)(25)(-50) = -18,750,000 \text{ mm}^4$$

$$\phi = \frac{\pi}{2} + \tan^{-1}\frac{50}{100} = 2.0344 \text{ rad}$$

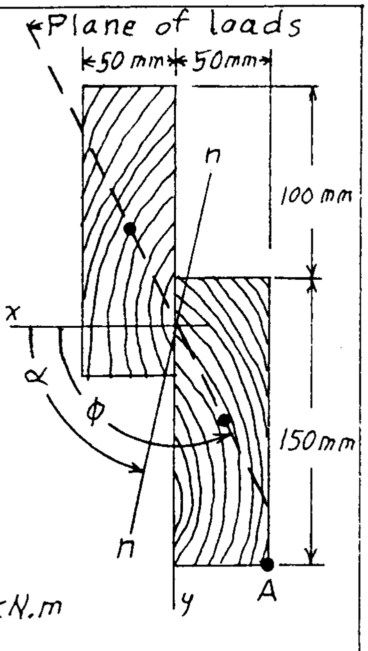
$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} = \frac{-18,750,000 - 65,630,000(-0.5)}{12,500,000-(-18,750,000)(-0.5)}$$

$$= 4.5008$$

$$\alpha = \underline{1.3522 \text{ rad}}$$

$$M = \frac{PL}{4} = \frac{4000(2000)}{4} = 2.00 \text{ kN.m} \;;\; M_x = M\sin\phi = 1.789 \text{ kN.m}$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{1,789,000[125-(-50)(4.5008)]}{65,630,000-(-18,750,000)(4.5008)} = \underline{4.17 \text{ MPa}}$$



7.24

$$\tan\alpha = \frac{-18,750,000 - 65,630,000(-0.3416)}{12,500,000-(-18,750,000)(-0.3416)} = 0.6020 \;;\quad \alpha = \underline{0.5419}$$

$$M_x = M\sin\phi = 2.00(0.9463) = 1.893 \text{ kN.m}$$

$$\sigma_A = \frac{1,893,000[125-(-50)(0.6020)]}{65,630,000-(-18,750,000)(0.6020)} = \underline{3.82 \text{ MPa}}$$

7.25

$$\phi = \tan^{-1}2 = 1.1071 \text{ rad}$$

$$\tan\alpha = -\frac{I_x\cot\phi}{I_y} = -\frac{8,870,000(0.5)}{403,000} = -11.0050$$

$$\alpha = \underline{-1.4802 \text{ rad}}$$

$$M = \frac{wL^2}{8} = \frac{1.00(4000)^2}{8} = 2.00 \text{ kN.m}$$

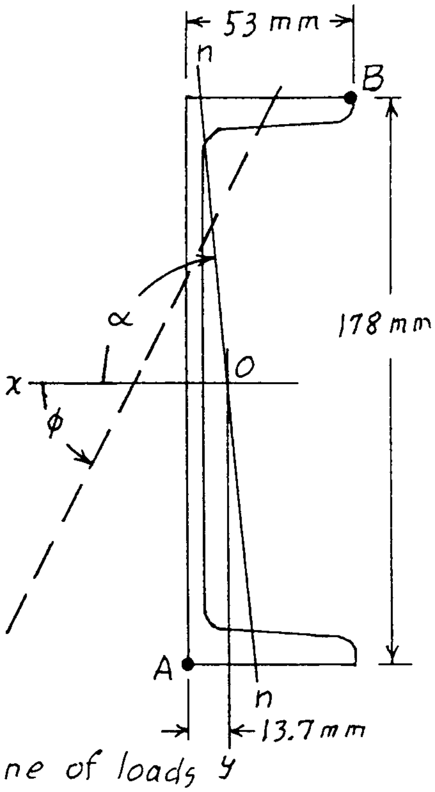
$$M_x = M\sin\phi = 2.00(0.8944) = 1.789 \text{ kN.m}$$

$$\sigma_A = \frac{M_x(y_A - x\tan\alpha)}{I_x} = \frac{1,789,000[89-13.7(-11.0050)]}{8,870,000}$$

$$= \underline{48.36 \text{ MPa}}$$

$$\sigma_B = \frac{1,789,000[-89-(-39.3)(-11.0050)]}{8,870,000}$$

$$= \underline{-105.18 \text{ MPa}}$$



7.26

$$I_x = \frac{10(200)^3}{12} + 2(391{,}000) + 2(1148)(83.8)^2$$

$$= 23{,}570{,}000\ mm^4$$

$$I_y = \frac{200(10)^3}{12} + 2(912{,}000) + 2(1148)(34)^2$$

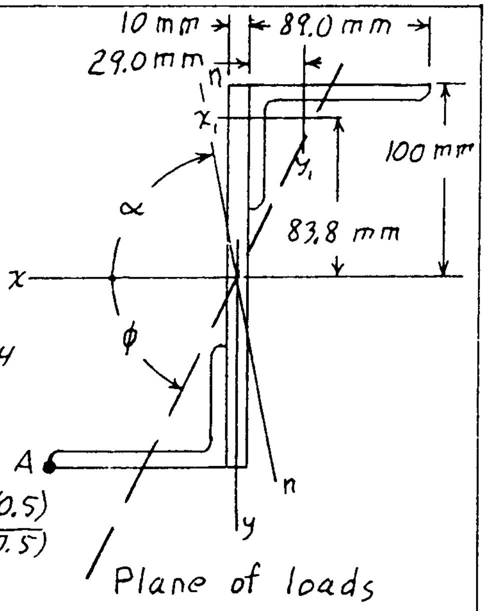
$$= 4{,}490{,}000\ mm^4$$

$$I_{xy} = 2[349{,}000 + 1148(83.8)(34)] = 7{,}240{,}000\ mm^4$$

$$\phi = \tan^{-1} 2 = 1.1071\ rad$$

$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi} = \frac{7{,}240{,}000 - 23{,}570{,}000(0.5)}{4{,}490{,}000 - 7{,}240{,}000(0.5)}$$

$$= -5.2241$$



$$\alpha = \underline{-1.3817\ rad}$$

$$M = \frac{wL^2}{8} = \frac{w(4000)^2}{8} = 2.00\,w\,(kN.m);\quad M_x = M\sin\phi = 1.789\,w\,(kN.m)$$

$$\sigma_A = Y = \frac{SF(M_x)(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha}$$

$$w = \frac{300[23{,}570{,}000 - 7{,}240{,}000(-5.2241)]}{2.50(1{,}789{,}000)[100 - 94(-5.2241)]} = 6.97\ N/mm = \underline{6.97\ kN/m}$$

7.27

$$I_x = \frac{100(200)^3}{12} - \frac{80(160)^3}{12} = 39{,}360{,}000\ mm^4$$

$$I_y = \frac{20(180)^3}{12} + \frac{180(20)^3}{12} = 9{,}840{,}000\ mm$$

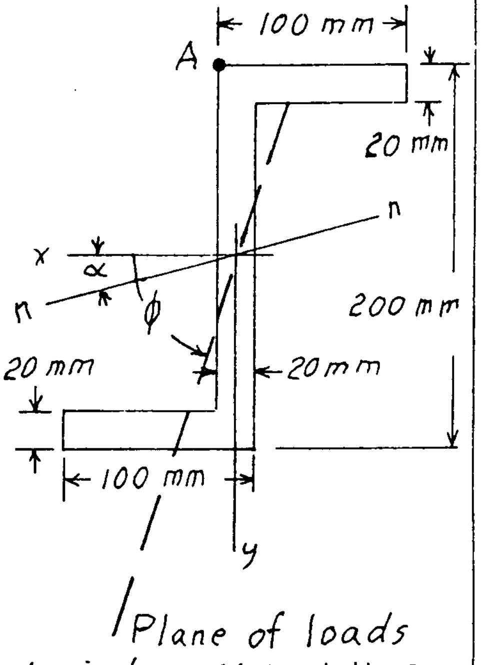
$$I_{xy} = 2(80)(20)(50)(90) = 14{,}400{,}000\ mm$$

$$\phi = 1.25\ rad$$

$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi}$$

$$= \frac{14{,}400{,}000 - 39{,}360{,}000(0.3323)}{9{,}840{,}000 - 14{,}400{,}000(0.3323)}$$

$$= 0.2613$$



$$\alpha = \underline{0.2557\ rad}$$

$$M = -PL = -14{,}000(2000) = -28.0\ kN.m;\quad M_x = M\sin\phi = -26.57\ kN.m$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{-26{,}570{,}000[-100 - 10(0.2613)]}{39{,}360{,}000 - 14{,}400{,}000(0.2613)} = \underline{76.60\ MPa}$$

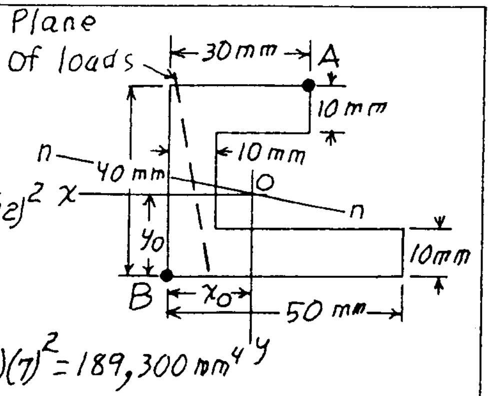
7.28

$$x_0 = \frac{30(10)(15)+20(10)(5)+50(10)(25)}{1000} = 18.0\ mm$$

$$y_0 = \frac{30(10)(35)+(20)(10)(20)+50(10)(5)}{1000} = 17.0\ mm$$

$$I_x = \frac{30(10)^3}{12}+\frac{10(20)^3}{12}+\frac{50(10)^3}{12}+30(10)(18)^2+20(10)(3)^2+50(10)(12)^2$$

$$= 184,300\ mm^4$$



$$I_y = \frac{10(30)^3}{12}+\frac{20(10)^3}{12}+\frac{10(50)^3}{12}+30(10)(3)^2+20(10)(13)^2+50(10)(7)^2 = 189,300\ mm^4$$

$$I_{xy} = 300(3)(-18)+200(13)(-3)+500(-7)(12) = -66,000\ mm^4$$

$$\tan\alpha = \frac{I_{xy}-I_x\cot\phi}{I_y-I_{xy}\cot\phi} = \frac{-66,000-184,300(-0.1763)}{189,300-(-66,000)(-0.1763)} = -0.1886;\ \alpha = \underline{-0.1864\ rad}$$

$$M = -PL = -1150(1000) = -1.15\ kN.m;\ M_x = M\sin\phi = -1.15(0.9848) = -1.133\ kN.m$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{-1,133,000[-23-(-12)(-0.1886)]}{184,300-(-66,000)(-0.1886)} = \underline{166.5\ MPa}$$

$$\sigma_B = \frac{-1,133,000[18-17(-0.1886)]}{184,300-(-66,000)(-0.1886)} = \underline{-139.8\ MPa}$$

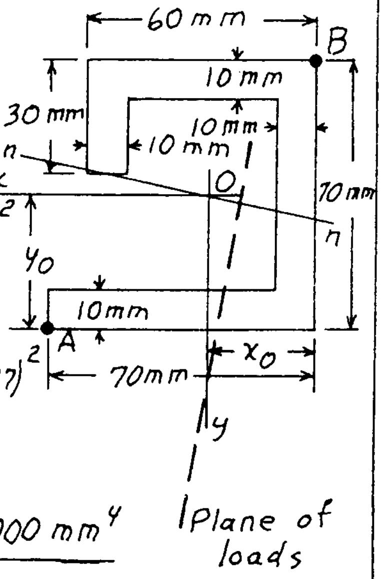
7.29

$$x_0 = \frac{60(10)(30)+60(10)(5)+60(10)(40)+20(10)(55)}{2000} = \underline{28.0\ mm}$$

$$y_0 = \frac{60(10)(65)+60(10)(30)+60(10)(5)+20(10)(50)}{2000} = \underline{35.0\ mm}$$

$$I_x = \frac{60(10)^3}{12}+\frac{10(60)^3}{12}+\frac{60(10)^3}{12}+600(-30)^2+600(5)^2+600(30)^2+200(-15)^2$$

$$= \underline{1,330,000\ mm^4}$$



$$I_y = \frac{10(60)^3}{12}+\frac{60(10)^3}{12}+\frac{10(60)^3}{12}+600(2)^2+600(-23)^2+600(12)^2+200(27)^2$$

$$= \underline{917,000\ mm^4}$$

$$I_{xy} = 600(-30)(2)+600(5)(-23)+600(30)(12)+200(-15)(27) = \underline{30,000\ mm^4}$$

$$\tan\alpha = \frac{I_{xy}-I_x\cot\phi}{I_y-I_{xy}\cot\phi} = \frac{30,000-1,330,000(0.1725)}{917,000-30,000(0.1725)} = -0.2187$$

$$\alpha = \underline{-0.2153\ rad}$$

$$M = \frac{PL}{4} = \frac{5000(2000)}{4} = 2.50\ kN.m;\ M_x = M\sin\phi = 2.50(0.9854) = 2.464\ kN.m$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{2,464,000[35-42(-0.2187)]}{1,330,000-30,000(-0.2187)} = \underline{81.5\ MPa}$$

$$\sigma_B = \frac{2,464,000[-35-(-28)(-0.2187)]}{1,330,000-30,000(-0.2187)} = \underline{-75.8\ MPa}$$

7.30

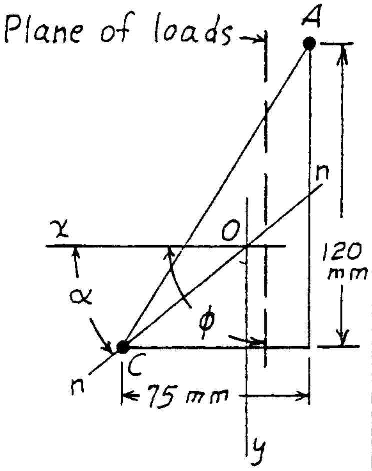
$$I_x = \frac{bh^3}{36} = \frac{75(120)^3}{36} = 3,600,000 \text{ mm}^4$$

$$I_y = \frac{hb^3}{36} = \frac{120(75)^3}{36} = 1,406,000 \text{ mm}^4$$

$$I_{xy} = \frac{b^2h^2}{72} = \frac{(75)^2(120)^2}{72} = 1,125,000 \text{ mm}^4$$

$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} = \frac{I_{xy}}{I_y} = \frac{1,125,000}{1,406,000} = 0.8001$$

$$\alpha = \underline{0.6748 \text{ rad}}$$



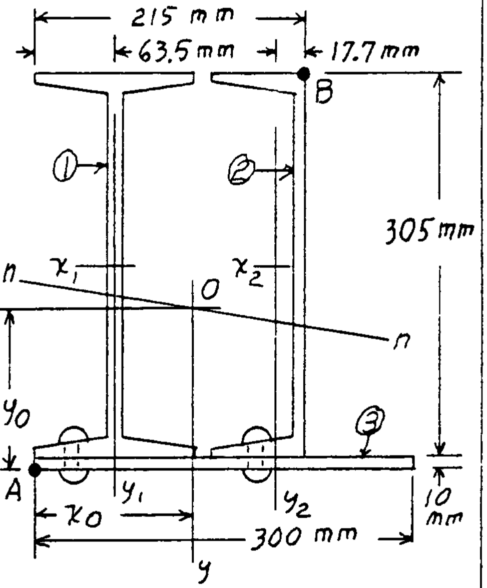
$$M = M_x = -PL = -4000(1250) = -5.00 \text{ kN.m}$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{-5,000,000[-80-(-25)(0.8001)]}{3,600,000 - 1,125,000(0.8001)} = \underline{111.1 \text{ MPa}}$$

$$\sigma_C = \frac{-5,000,000[40-50(0.8001)]}{3,600,000 - 1,250,000(0.8001)} = \underline{0}$$

7.31

$$A_1 = 6030 \text{ mm}^2;\ A_2 = 3930 \text{ mm}^2;\ A_3 = 3000 \text{ mm}^2$$



$$x_0 = \frac{6030(63.5) + 3930(197.3) + 3000(150)}{12,960} = 124.1 \text{ mm}$$

$$y_0 = \frac{6030(162.5) + 3930(162.5) + 3000(5)}{12,960} = 126.0 \text{ mm}$$

$$I_x = 90,300,000 + 53,700,000 + \frac{300(10)^3}{12} + 6030(-36.5)^2$$
$$+ 3930(-36.5)^2 + 3000(121.0)^2 = 201,220,000 \text{ mm}^4$$

$$I_y = 3,900,000 + 1,610,000 + \frac{10(300)^3}{12} + 6030(60.6)^2$$
$$+ 3930(-73.2)^2 + 3000(-25.9)^2 = 73,225,000 \text{ mm}^4$$

$$I_{xy} = 6030(-36.5)(60.5) + 3930(-36.5)(-73.2) + 3000(121.0)(-25.9)$$
$$= -12,217,000 \text{ mm}^4$$

$$\phi = \underline{1,571 \text{ rad}}$$

$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} = \frac{I_{xy}}{I_y} = \frac{-12,217,000}{73,225,000} = -0.1668; \quad \alpha = -0.1653 \text{ rad}$$

$$M = M_x = \frac{wL^2}{8} = \frac{20(6000)^2}{8} = 90.0 \text{ kN.m}$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{90,000,000[126.0 - 124.1(-0.1668)]}{201,220,000 - (-12,217,000)(-0.1668)} = \underline{66.3 \text{ MPa}}$$

$$\sigma_B = \frac{90,000,000[-189.0-(-90.9)(-0.1668)]}{201,220,000 - (-12,217,000)(-0.1668)} = \underline{-92.3 \text{ MPa}}$$

7.32

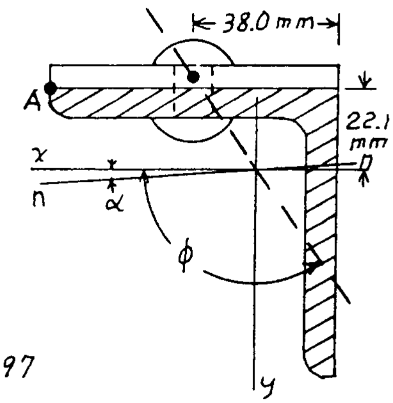
$A = 1148 \text{ mm}^2$

$I_x = I_y = 629{,}000 \text{ mm}^4$

$I_{xy} = -371{,}000 \text{ mm}^4$

$\phi = \frac{\pi}{2} + \tan^{-1}\frac{38.0-22.1}{22.1+3} = 2.1355 \text{ rad}$

$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi} = \frac{-371{,}000-629{,}000(-0.6335)}{629{,}000-(-371{,}000)(-0.6335)} = 0.0697$



$\alpha = 0.0696 \text{ rad}$

$M = -50{,}000\sqrt{(38.0-22.1)^2+(22.1+3)^2} = -1.486 \text{ kN.m}$

$M_x = M\sin\phi = -1.486(0.8447) = -1.255 \text{ kN.m}$

$\sigma_A = \frac{P}{A} + \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{50{,}000}{1148} + \frac{-1{,}255{,}000[-22.1-(53.9)(0.0697)]}{629{,}000-(-371{,}000)(0.0697)} = \underline{93.1 \text{ MPa}}$

7.33

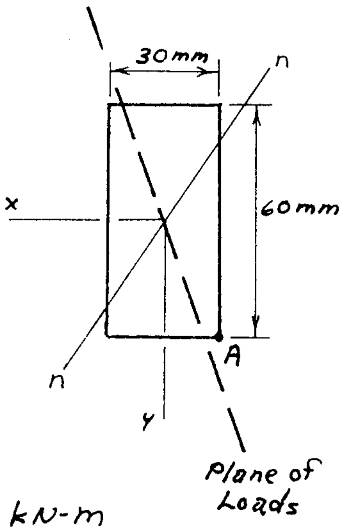
$I_x = \frac{bh^3}{12} = \frac{30(60^3)}{12} = 540{,}000 \text{ mm}^4$

$I_y = \frac{hb^3}{12} = \frac{60(30^3)}{12} = 135{,}000 \text{ mm}^4$

$I_{xy} = 0$, $\phi = 110° = 1.92 \text{ RAD}$

$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} = \frac{-540{,}000(-0.364)}{135{,}000} = 1.4559$

$\therefore \alpha = 0.9689 \text{ RAD}$



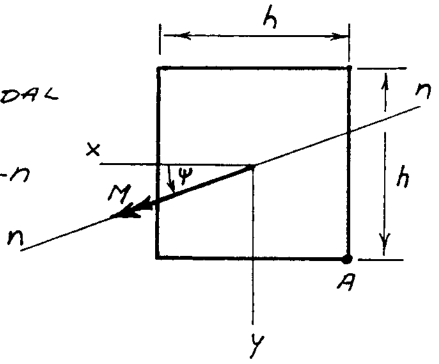
$R_1 = \frac{3.0(700)}{1500} = 1.4 \text{ kN}$; $R_2 = 1.6 \text{ kN}$

$M_{max} = 0.800R_1 = 1.12 \text{ kN-m}$, $M_x = M_{max}\sin\phi = 1.052 \text{ kN-m}$

$\sigma_{max} = \sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{1{,}052{,}000[30+15(1.456)]}{540{,}000} = \underline{\underline{100.99 \text{ MPa}}}$

7.34

$$I_x = I_y = h^4/12 \quad , \quad I_{xy} = 0$$



SINCE $I_x = I_y$, ANY PAIR OF ORTHOGONAL CENTROIDAL AXES IS A SET OF PRINCIPAL AXES. HENCE, n-n COINCIDES WITH THE AXIS OF BENDING.

a) CONSIDER ONLY $0 \le \psi \le 90°$

$$M_x = M\cos\psi \quad , \quad \alpha = \psi$$

$$\sigma_{max} = \sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{M\cos\psi\left(h/2 - (-h/2)\tan\psi\right)}{h^4/12}$$

$$\underline{\underline{\sigma_{max} = \frac{6M}{h^3}(\sin\psi + \cos\psi)}}$$

b) $\psi = 0°$: $\sigma_{max} = \underline{\underline{\frac{6M}{h^3}}}$

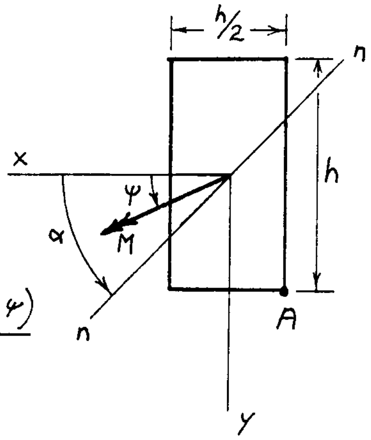
$\psi = 15°$: $\sigma_{max} = \frac{6M}{h^3}(1.2247) = \underline{\underline{\frac{7.348M}{h^3}}}$

$\psi = 45°$: $\sigma_{max} = \frac{6M}{h^3}(1.4141) = \underline{\underline{\frac{8.485M}{h^3}}}$

7.35

$$I_x = h^4/24 \, , \quad I_y = h^4/96 \, , \quad I_{xy} = 0$$

$$\phi = \psi + 90° \quad \therefore \cot\phi = -\tan\psi \, , \quad M_x = M\cos\psi$$



a) CONSIDER ONLY $0 \le \psi \le 90°$

$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} = \frac{-\frac{h^4}{24}(-\tan\psi)}{h^4/96} = 4\tan\psi$$

$$\sigma_{max} = \sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{M\cos\psi\left(h/2 - (-h/4)4\tan\psi\right)}{h^4/24}$$

$$\underline{\underline{\sigma_{max} = \frac{24M}{h^3}\left(\frac{1}{2}\cos\psi + \sin\psi\right)}}$$

b) $\psi = 0°$: $\sigma_{max} = \frac{24M}{h^3}(1/2) = \underline{\underline{\frac{12M}{h^3}}}$

$\psi = 30°$: $\sigma_{max} = \frac{24M}{h^3}(0.9330) = \underline{\underline{\frac{22.392M}{h^3}}}$

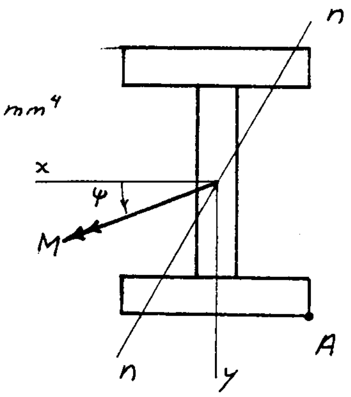
$\psi = 45°$: $\sigma_{max} = \frac{24M}{h^3}\left(\frac{3}{2}\frac{1}{\sqrt{2}}\right) = \underline{\underline{\frac{25.456M}{h^3}}}$

$\psi = 60°$: $\sigma_{max} = \frac{24M}{h^3}(1.1160) = \underline{\underline{\frac{26.785M}{h^3}}}$

$\psi = 90°$: $\sigma_{max} = \frac{24M}{h^3}(1.0) = \underline{\underline{\frac{24M}{h^3}}}$

7.36

$\sigma_{max} = 120 MPa$, FIND M



$I_x = 2\left(\frac{1}{12}(100)(20^3) + 100(20)(60^2)\right) + \frac{1}{12}(20)(100^3) = 16{,}200{,}000\ mm^4$

$I_y = 2\left(\frac{1}{12}(20)(100^3)\right) + \frac{1}{12}(100)(20^3) = 3{,}400{,}000\ mm^4$

$I_{xy} = 0$, $\phi = \psi + 90° = 110°$

$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi} = \frac{-16.2\times10^6(-0.36397)}{3.4\times10^6} = 1.7342$$

$\alpha = 60.03°$

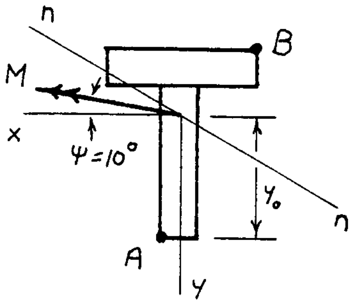
$M_x = M\cos\psi = 0.9397M$

$$\sigma_{max} = \sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{0.9397M(70-(-50)(1.7342))}{16.2\times10^6} = 120\ N/mm^2$$

$\underline{\underline{M = 13{,}200{,}000\ N\text{-}mm = 13.2\ kN\cdot m}}$

7.37

$\sigma_{max} = 150 MPa$, FIND M



$y_0 = \left[50(200)100 + 50(200)(225)\right] / 2(50)200 = 162.5\ mm$

$I_x = \frac{1}{12}(200)(50^3) + 200(50)(62.5)^2 + \frac{1}{12}(50)(200^3) + 50(200)(62.5^2)$

$I_x = 113{,}540{,}000\ mm^4$, $I_{xy} = 0$

$I_y = \frac{1}{12}\left(50(200)^3 + 200(50^3)\right) = 35{,}420{,}000\ mm^4$

$\phi = -10° + 90° = 80°$

$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} = \frac{-113.54\times10^6(0.17633)}{35.42\times10^6} = -0.5652$$

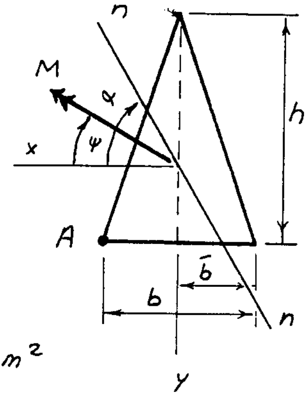
$\alpha = -29.48°$, $M_x = M\cos\psi = 0.9848M$

$$\sigma_{max} = \sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x - I_{xy}\tan\alpha} = \frac{0.9848M(162.5-(25)(-0.5652))}{113.54\times10^6} = 150\ N/mm^2$$

$\underline{\underline{M = 97{,}910{,}000\ N\cdot m = 97.91\ kN\cdot m}}$

NOTE: $\sigma_B = \dfrac{0.9848(97.91\times10^6)(-87.5-(-100)(-0.5626))}{113.54\times10^6} = 122.08\ MPa < \sigma_A$

$\boxed{7.38}$ $\sigma_{max} = 90\,MPa$, FIND M.



$$I_x = \frac{bh^3}{36} = \frac{100(150^3)}{36} = 9{,}375{,}000\,mm^4,\quad I_{xy} = 0$$

$$I_y = 2\left[\frac{h\bar{b}^3}{36} + \frac{h\bar{b}}{2}\left(\frac{\bar{b}}{3}\right)\right] = 2\left[\frac{150(50^3)}{36} + \frac{150(50)}{2}\left(\frac{50}{3}\right)^2\right] = 3{,}125{,}000\,mm^4$$

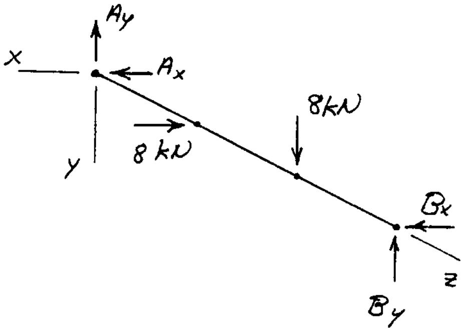
$$\phi = -30^\circ + 90^\circ = 60^\circ,\quad M_x = M\cos\psi = 0.866M$$

$$\tan\alpha = \frac{-I_x\cot\phi}{I_y} = \frac{-9.375\times10^6(0.5773)}{3.125\times10^6} = -1.7321,\ \alpha = -60^\circ$$

$$\sigma_{max} = \sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x} = \frac{0.866M(50-(50)(-1.7321))}{9.375\times10^6} = 90\,N/mm^2$$

$$\underline{\underline{M = 7{,}132{,}000\,N\cdot mm = 7.132\,kN\cdot m}}$$

$\boxed{7.39}$ REACTIONS:



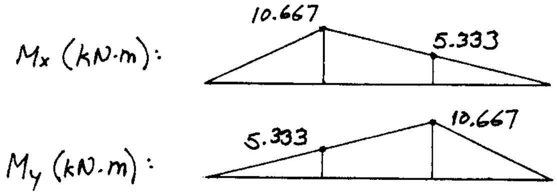
$$\Sigma M_{By} = 0:\ A_x = \frac{4m(8kN)}{6m} = 5.333\,kN$$

$$\Sigma F_x = 0:\ B_x = 8kN - A_x = 2.667\,kN$$

$$\Sigma M_{Bx} = 0:\ A_y = \frac{2m(8kN)}{6m} = 2.667\,kN$$

$$\Sigma F_y = 0:\ B_y = 8kN - A_y = 5.333\,kN$$

MOMENT DIAGRAMS:



BY INSPECTION, σ_{max} OCCURS IN THE MIDDLE THIRD OF THE BEAM SPAN; MORE SPECIFICALLY AT A LOAD POINT OR AT MIDSPAN.

AT $z = 2m$: $M_x = 10.667$, $M_y = 5.333$

$$\therefore M = \sqrt{10.667^2 + 5.333^2} = 11.926\,kN\cdot m.$$

AT $z = 3m$: $M_x = M_y = 8\,kN\cdot m$, $M = 8\sqrt{2} = 11.314\,kN\cdot m$

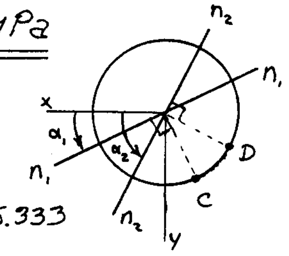
AT $z = 4m$: $M = 11.926\,kN\cdot m$

M_{max}, & THEREFORE σ_{max} OCCURS AT THE LOCATIONS OF THE LOADS.

$$I = \frac{1}{4}\pi r^4 = \frac{\pi}{4}\left(\frac{75}{2}\right)^4 = 1.553\times10^6\,m^4$$

$$\sigma_{max} = \frac{Mr}{I} = \frac{11.926\times10^6(37.5)}{1.553\times10^6}$$

$$\underline{\underline{\sigma_{max} = 288.0\,MPa}}$$



AT $z = 2m$:

$M_x = 10.667$, $M_y = 5.333$

$$\alpha_1 = \tan^{-1}\frac{5.333}{10.667} = 26.56^\circ$$

σ_{max} OCCURS AT $\alpha_1 + 90^\circ = 116.6^\circ$
(POINT C)

AT $z = 4m$: $M_x = 5.333$, $M_y = 10.667$

$$\alpha_2 = \tan^{-1}\frac{10.667}{5.333} = 63.43^\circ$$

σ_{max} OCCURS AT $\alpha_2 + 90^\circ = 153.43^\circ$
(POINT D)

7.40

$$I_x = \tfrac{1}{12}(100)(200^3) = 66.67\times10^6\ \text{mm}^4$$

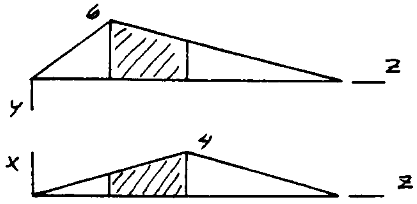
$$I_y = \tfrac{1}{12}(200)(100^3) = 16.67\times10^6\ \text{mm}^4$$

$$I_x/I_y = 4$$

BY INSPECTION, σ_{max} OCCURS AT POINT A WITHIN THE INTERVAL $2\text{m} \le z \le 4\text{m}$

M_x (kN·m)

M_y (kN·m)



$M_x(z) = 8 - z$

$M_y(z) = z$

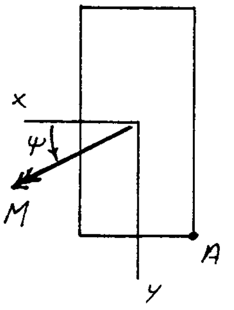
z IN (m)
M IN (kN·m)

$$\tan\psi = M_y/M_x = \frac{z}{8-z},\quad \phi = \psi + 90^\circ,$$

$$\cot\phi = \cot(\psi+90^\circ) = -\tan\psi = \frac{-z}{8-z}$$

$$\tan\alpha = \frac{-I_x\cot\phi}{I_y} = (-4)\left(\frac{-z}{8-z}\right) = \frac{4z}{8-z}$$

$$\sigma_A = \frac{M_x(y_A - x_A\tan\alpha)}{I_x} = \frac{(8-z)(10^6)\left(100-(-50)\left(\frac{4z}{8-z}\right)\right)}{66.67(10^6)} = 12 + 1.5z$$



$$\sigma_{max} = \sigma_A\Big|_{z=4\text{m}} = 12 + 1.5(4) = \underline{\underline{18.0\ \text{MPa}}}$$

7.41

$$v = \frac{P_y L^3}{48 E I_x} = \frac{P\sin\phi\, L^3}{48 E I_x} = \frac{31{,}400(0.9848)(4000)^3}{48(12{,}000)(562{,}500{,}000)} = \underline{6.11\ \text{mm}}$$

$$u = -v\tan\alpha = -6.11(0.2539) = \underline{-1.55\ \text{mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{(-1.55)^2 + 6.11^2} = \underline{6.30\ \text{mm}}$$

7.42

$$v = \frac{P_y L^3}{3E(I_x - I_{xy}\tan\alpha)} = \frac{16{,}000(2500)^3}{3(71{,}700)[124{,}400{,}000 - 70{,}760{,}000(0.5688)]} = \underline{13.81\ \text{mm}}$$

$$u = -v\tan\alpha = -13.81(0.5688) = \underline{-7.86\ \text{mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{(-7.86)^2 + 13.81^2} = \underline{15.89\ \text{mm}}$$

7.43

$$v = \frac{P_y L^3}{3E(I_x - I_{xy}\tan\alpha)} = \frac{16{,}000(2500)^3}{3(71{,}700)[108{,}700{,}000 - 38{,}000{,}000(0.9694)]} = \underline{16.17\ \text{mm}}$$

$$u = -v\tan\alpha = -16.17(0.9694) = \underline{-15.68\ \text{mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{(-15.68)^2 + 16.17^2} = \underline{22.52\ \text{mm}}$$

7.44

$$v = \frac{P \sin \phi L^3}{48E(I_x - I_{xy} \tan \alpha)} = \frac{4000(0.8944)(2000)^3}{48(12,000)[65,630,000-(-18,750,000)(4.5008)]} = \underline{0.33 \text{ mm}}$$

$$u = -v \tan \alpha = -0.33(4.5008) = \underline{-1.49 \text{ mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{(-1.49)^2 + (0.33)^2} = \underline{1.53 \text{ mm}}$$

7.45

$$v = \frac{5 w \sin \phi L^4}{384 E I_x} = \frac{5(1.00)(0.8944)(4000)^4}{384(200,000)(8,870,000)} = \underline{1.68 \text{ mm}}$$

$$u = -v \tan \alpha = -1.68(-11.0050) = \underline{18.49 \text{ mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{18.49^2 + 1.68^2} = \underline{18.57 \text{ mm}}$$

7.46

$$v = \frac{5 w \sin \phi L^4}{384 E(I_x - I_{xy} \tan \alpha)} = \frac{5(6.5)(0.8944)(4000)^4}{384(200,000)[23,570,000-7,240,000(-5.2241)]} = \underline{1.58 \text{ mm}}$$

$$u = -v \tan \alpha = -1.58(-5.2241) = \underline{8.25 \text{ mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{8.25^2 + 1.58^2} = \underline{8.40 \text{ mm}}$$

7.47

$$v = \frac{P \sin \phi L^3}{3E(I_x - I_{xy} \tan \alpha)} = \frac{14,000(0.9490)(2000)^3}{3(200,000)[39,360,000-14,400,000)(0.2613)]} = \underline{4.98 \text{ mm}}$$

$$u = -v \tan \alpha = -4.98 \ (0.2613) = \underline{-1.30 \text{ mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{(-1.30)^2 + 4.98^2} = \underline{5.15 \text{ mm}}$$

7.48

$$v = \frac{P \sin \phi L^3}{3E(I_x - I_{xy} \tan \alpha)} = \frac{1250(0.9848)(1000)^3}{3(72,000)[184,300-(-66,000)(-0.1886)]} = \underline{33.16 \text{ mm}}$$

$$u = -v \tan \alpha = -33.16(-0.1886) = \underline{6.25 \text{ mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{6.25^2 + 33.16^2} = \underline{33.74 \text{ mm}}$$

7.49

$$v = \frac{P \sin \phi L^3}{48E(I_x - I_{xy} \tan \alpha)} = \frac{5000(0.9854)(2000)^3}{48(72,000)[1,330,000-30,000(-0.2187)]} = \underline{8.53 \text{ mm}}$$

$$u = -v \tan \alpha = -8.53(-0.2187) = \underline{1.87 \text{ mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{1.87^2 + 8.53^2} = \underline{8.73 \text{ mm}}$$

7.50

$$v = \frac{PL^3}{3E(I_x - I_{xy} \tan \alpha)} = \frac{4000(1250)^3}{3(200,000)[3,600,000-1,125,000(0.8001)]} = \underline{4.82 \text{ mm}}$$

$$u = -v \tan \alpha = -4.82(0.8001) = \underline{-3.86 \text{ mm}}$$

$$\delta = \sqrt{u^2 + v^2} = \sqrt{(-3.86)^2 + 4.82^2} = \underline{6.18 \text{ mm}}$$

7.51 By Fig. a,

$$\sum F_y = 3 - R_1 - R_2 = 0$$

$$\circlearrowleft + \sum M_0 = 1.5 R_2 - 0.8(3) = 0$$

The solution of these equations is

$$R_1 = 1.4\ \text{kN},\ R_2 = 1.6\ \text{kN}$$

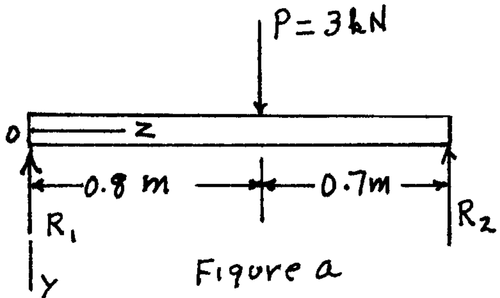


Figure a

By Fig. b,

$$I_x = \frac{1}{12}(0.03)(0.06)^3 = 5.4\times10^{-7}\ \text{m}^4$$

$$I_y = \frac{1}{12}(0.06)(0.03)^3 = 1.35\times10^{-7}\ \text{m}^4$$

$$I_{xy} = 0, \quad \phi = 110^\circ$$

Therefore,

$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi} = 1.456$$

or

$$\alpha = 0.96898\ \text{rad} = 55.518^\circ\ (\text{Fig. b})$$

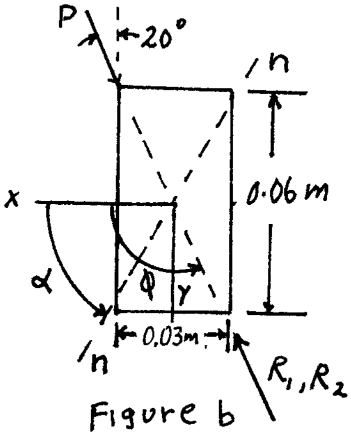


Figure b

For $0 \le z \le 0.8$ m, by Figs. a and b,

$$M_x = (R_1 \cos 20^\circ) z = 1315.6 z \quad [\text{N}\cdot\text{m}];\ z \text{ in meters}$$

and

$$\frac{d^2 v_1}{dz^2} = -\frac{M_x}{E(I_x - I_{xy}\tan\alpha)} = -\frac{1315.6 z}{(200\times10^9)(5.4\times10^{-7})} = (1.218\times10^{-2}) z \qquad (a)$$

since $E = 200$ GPa and v_1 is the displacement in the y direction. Integration of Eq. (a) yields

$$v_1 = -(2.0302\times10^{-3}) z^3 + C_1 z + C_2 \qquad (b)$$

For $z = 0$, $v_1 = 0$. Therefore, $C_2 = 0$.

For $0.8 \le z \le 1.50$m, Figs. (a) and (b) yield

$$M_x = (R_1 \cos 20^\circ) z - (P\cos 30^\circ)(z - 0.8) = -1503.5 z + 2255.3 \quad [\text{N}\cdot\text{m}]$$

(cont.)

7.51 cont. For $0.8 \le z \le 1.50$ m, the y displacement is determined from

$$\frac{d^2 v_2}{dz^2} = -\frac{M_x}{E(I_x - I_{xy}\tan\alpha)} = (1.3921 \times 10^{-2})z - 2.0882 \times 10^{-2}$$

Integration yields

$$v_2 = (2.3202 \times 10^{-3})z^3 - 1.0441 \times 10^{-2} z^2 + C_3 z + C_4$$

at $z = 1.5$ m, $v_2 = 0$; therefore, $C_4 = 1.5661 \times 10^{-2} - 1.50\, C_3$.

Hence, $v_2 = (2.3202 \times 10^{-3})z^3 - (1.0441 \times 10^{-2})z^2 - (1.50 - z)C_3 + 1.5661 \times 10^{-2}$ (c)

at $z = 0.8$ m, $v_1 = v_2$ and $dv_1/dz = dv_2/dz$. Therefore by Eqs. (b) and (c), we have for $v_1 = v_2$ at $z = 0.8$ m

$$C_3 = 1.6009 \times 10^{-2} - 1.1428 C_1 \qquad (d)$$

Similarly by differentiation of Eqs. (b) and (c), for $dv_1/dz = dv_2/dz$ at $z = 0.8$ m, we have

$$C_3 = 8.3528 \times 10^{-3} + C_1 \qquad (e)$$

The solution of Eqs. (d) and (e) is

$$C_1 = 3.5730 \times 10^{-3} \text{ m/m}, \quad C_3 = 1.1926 \times 10^{-2} \text{ m/m}$$

Therefore, $C_4 = -2.2280 \times 10^{-3}$ m. So, by Eqs. (b) and (c),

$$v_1 = -2.0302 \times 10^{-3} z^3 + 3.5730 \times 10^{-3} z \quad [\text{m}] \qquad (f)$$

$$v_2 = 2.3202 \times 10^{-3} z^3 - 1.0441 \times 10^{-2} z^2 + 1.1926 \times 10^{-2} z - 2.2280 \times 10^{-3} \quad [\text{m}] \qquad (g)$$

At the midpoint of the beam $z = 0.75$ and by Eq. (f), $v_1 = 1.823$ mm. Therefore, the displacement u in the

(cont.)

7.51 cont. x direction (Fig. b) is

$$u_1 = -v_1 \tan\alpha = -1.823(1.456) = -2.655 \text{ mm}$$

So, the displacement perpendicular to the neutral axis n-n (Fig. b) is

$$\delta = \sqrt{u_1^2 + v_1^2} = 3.22 \text{ mm}$$

7.52 For the square cross section of Fig. a,

$$I_x = I_y = \tfrac{1}{12}h^4, \quad I_{xy} = 0 \qquad (a)$$

With Eq. (a),

$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi} = -\frac{I_x}{I_y}\cot\phi$$

or

$$\tan\alpha = -\cot\phi \qquad (b)$$

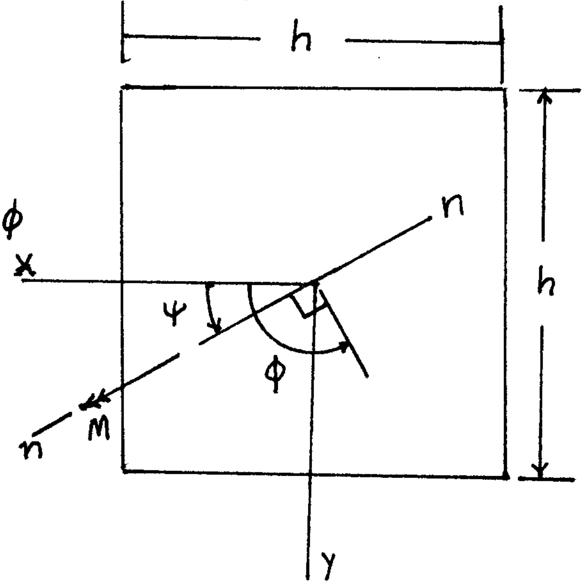


Figure a

By Fig. a, $\phi = \psi + 90^\circ$. Hence,

$$\cot\phi = \cot(\psi + 90^\circ) = \frac{\cos(\psi + 90^\circ)}{\sin(\psi + 90^\circ)} = \frac{-\sin\psi}{\cos\psi} = -\tan\psi$$

or by Eq. (b),

$$\tan\alpha = \tan\psi \qquad (c)$$

Therefore, $\alpha = \psi$, and the neutral axis n-n coincides with the axis of bending (Fig. a). Also, by Fig. a, $M_x = M\cos\psi$ and by Eq. (7.20), the y component v of the displacement may be found from

$$\frac{d^2 v}{dz^2} = -\frac{M_x}{EI_x} = -\frac{12 M \cos\psi}{E h^4}$$

Integration of this equation yields

(cont.)

7.52 cont.

$$\frac{dv}{dz} = -\frac{12M\cos\psi}{Eh^4}z + C_1$$

$$v = -\frac{6M\cos\psi}{Eh^4}z^2 + C_1 z + C_2$$

Since $v=0$ at $z=0$, $C_2=0$. Also since the slope $dv/dz=0$ at $z=0$, $C_1=0$. Consequently, at $z=L$, the free end of the beam, the y displacement is

$$v = -\frac{6M\cos\psi}{Eh^4}L^2 \qquad (d)$$

With Eqs. (c) and (d), the x displacement component u is

$$u = -v\tan\alpha = -v\tan\psi = \frac{6M\sin\psi}{Eh^4}L^2 \qquad (e)$$

Then, with Eqs. (d) and (e), the displacement δ perpendicular to the neutral axis n-n (Fig. a) is

$$\delta = \sqrt{u^2+v^2} = \frac{6ML^2}{Eh^4}$$

Alternatively note that the area moment of inertia relative to axis n-n is [see Eq. (B.10)]

$$I_n = I_x\cos^2\psi + I_y\sin^2\psi - I_{xy}\sin 2\psi = \frac{1}{12}h^4$$

Then, by elementary mechanics of materials

$$\delta = \frac{1}{2}\frac{ML^2}{EI_n} = \frac{6ML^2}{Eh^4}$$

7.53 By Fig. a,

$$I_x = \frac{1}{12}\left[100(140)^3 - 80(100)^3\right] = 16.2\times10^6 \text{ mm}^4$$

$$I_y = 2\left[\frac{1}{12}(20)(100)^3\right] + \frac{1}{12}(100)(20)^3 = 3.4\times10^6 \text{ mm}^4 \quad (a)$$

$$I_{xy} = 0$$

$$\phi = \psi + 90^\circ = 110^\circ \qquad (b)$$

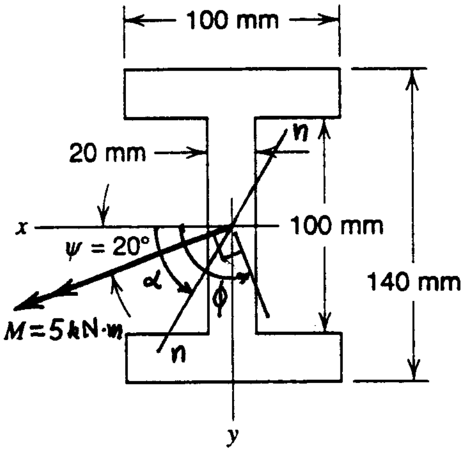


Figure a

With Eqs. (a) and (b), the location of the neutral axis n-n is obtained from

$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} = -\frac{I_x}{I_y}\cot\phi = 1.7342 \qquad (c)$$

or

$$\alpha = 60.03^\circ$$

By Fig. a, $M_x = M\cos 20^\circ = 4698.5$ N·m. So by Eqs. (7.20) and (a)

$$\frac{d^2v}{dz^2} = -\frac{M_x}{EI_x} = -\frac{4698.5}{(200\times10^9)(16.2\times10^{-6})} = -1.4501\times10^{-3}$$

where v is the y component of the displacement. Integration yields

$$\frac{dv}{dz} = -1.4501\times10^{-3}z + C_1$$

$$v = -7.2507\times10^{-4}z^2 + C_1 z + C_2$$

Since the slope $dv/dz = 0$ at $z = 0$ (the fixed end), $C_1 = 0$ and since $v = 0$ at $z = 0$, $C_2 = 0$. Therefore, at the free end ($z = L = 3.0$ m),

$$v = -7.2507\times10^{-4}(3.0)^2 = -6.5256\times10^{-3} \text{ m} \qquad (d)$$

(cont.)

7.53 cont. The x component u of the displacement is, with Eqs. (c) and (d)

$$u = -v \tan \alpha = 6.5256 \times 10^{-3}(1.7342)$$

or

$$u = 11.3167 \times 10^{-3} m \qquad (e)$$

The total displacement δ is, with Eqs. (d) and (e),

$$\delta = \sqrt{u^2+v^2} = 13.06 \times 10^{-3} m = 13.06 \text{ mm}$$

Alternatively, by elementary mechanics of materials,

$$\delta = \frac{1}{2}\frac{M_n L^2}{EI_n} \qquad (f)$$

where by Fig. a,

$$M_n = M \cos(\alpha - \psi) = 5000 \cos 40.03^\circ = 3825.54 \text{ N}\cdot\text{m}$$

and by Eq. (B.10),

$$I_n = I_x \cos^2\alpha + I_y \sin^2\alpha - I_{xy} \sin 2\alpha$$
$$= 16.2 \times 10^{-6} \cos^2(60.03^\circ) + 3.4 \times 10^{-6} \sin^2(60.03^\circ)$$

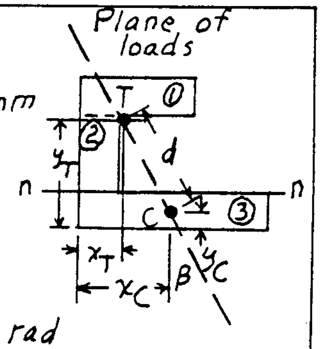
or

$$I_n = 6.5942 \times 10^{-6} \text{ m}4$$

Then, by Eq. (f),

$$\delta = \frac{1}{2}\frac{(3825.54)(3.0)^2}{(200 \times 10^9)(6.5942 \times 10^{-6})} = 13.06 \text{ mm}$$

7.54 $A_T = A_C = 500\ \text{mm}^2;\ x_C = 25\ \text{mm};\ y_C = 5\ \text{mm}$



(a)

$$x_T = \frac{30(10)(15)+20(10)(5)}{500} = 11\ \text{mm};\quad y_T = \frac{30(10)(35)+20(10)(20)}{500} = 29\ \text{mm}$$

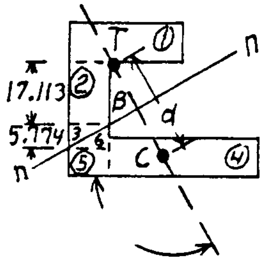
$$d = \sqrt{(x_T - x_C)^2 + (y_T - y_C)^2} = \sqrt{(11-25)^2 + (29-5)^2} = 27.78\ \text{mm}$$

$$P_P = \frac{A_T Y d}{L} = \frac{500(200)(27.78)}{1000} = \underline{2.778\ \text{kN}}$$

$$\tan\beta = \frac{25-11}{29-5} = 0.5833;\quad \beta = 0.5281\ \text{rad};\quad \phi = \beta + \frac{\pi}{2} = 2.0989\ \text{rad}$$

(b)

$$x_T = \frac{30(10)(15) + 17.113(10)(5) + \frac{1}{2}(5.774)(10)(\frac{10}{3})}{500} = 10.904\ \text{mm}$$



$$y_T = \frac{30(10)(35)+17.113(10)(21.4435)+\frac{1}{2}(5.774)(10)(10.9623)}{500} = 28.972\ \text{mm}$$

$$x_C = \frac{40(10)(30) + 7.113(10)(5) + \frac{1}{2}(5.774)(10)(\frac{2}{3}10)}{500} = 25.096\ \text{mm}$$

$$y_C = \frac{40(10)(5) + 7.113(10)(3.5565) + \frac{1}{2}(5.774)(10)(9.0377)}{500} = 5.028\ \text{mm}$$

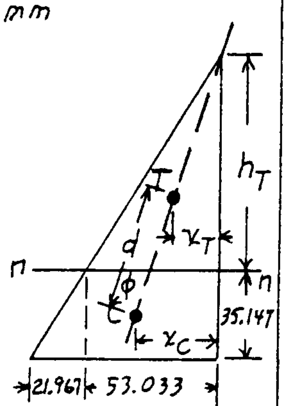
$$d = \sqrt{(x_T - x_C)^2 + (y_T - y_C)^2} = \sqrt{(10.904-25.096)^2 + (28.972-5.028)^2} = 27.83\ \text{mm}$$

$$P_P = \frac{A_T Y d}{L} = \frac{500(200)(27.83)}{1000} = \underline{2.783\ \text{kN}}$$

$$\tan\beta = \frac{25.096-10.904}{28.972-5.028} = 0.5927;\quad \beta = 0.5350\ \text{rad};\quad \phi = \beta + \frac{\pi}{2} = 2.1058\ \text{rad}$$

7.55 $A_T = A_C = \frac{120(75)}{4} = 2250\ \text{mm}^2 = \frac{1}{2} h_T \frac{75 h_T}{120};\ h_T = 84.853\ \text{mm}$

$$x_T = \frac{1}{3}\,\frac{75(84.853)}{120} = 17.678\ \text{mm}$$



$$y_T = 35.147 + \frac{1}{3}\,84.853 = 63.431\ \text{mm}$$

$$x_C = \frac{53.033(35.147)\frac{53.033}{2} + \frac{1}{2}(21.967)(35.147)(53.033 + \frac{21.967}{3})}{2250}$$

$$= 32.322\ \text{mm}$$

$$y_C = \frac{53.033(35.147)\frac{35.147}{2} + \frac{1}{2}(21.967)(35.147)(\frac{35.147}{3})}{2250} = 16.568\ \text{mm}$$

$$d = \sqrt{(x_T - x_C)^2 + (y_T - y_C)^2} = \sqrt{(17.678-32.322)^2 + (63.431-16.568)^2} = 49.10\ \text{mm}$$

$$P_P = \frac{A_T Y d}{L} = \frac{2250(240)(49.10)}{1250} = \underline{21.21\ \text{kN}}$$

$$\tan\phi = \frac{63.431-16.568}{32.322-17.678} = 3.2002$$

$$\phi = \underline{1.2679\ \text{rad}}$$

8.1 Let V have magnitude of I.

$$I = \frac{t(h+2b_1)^3}{12} + 2bt\left(\frac{h}{2}\right)^2 = \frac{tbh^2}{4}\left[2 + \frac{h}{3b} + \frac{2b_1}{b}\left(1 + \frac{2b_1}{h} + \frac{4b_1^2}{3h^2}\right)\right]$$

$$q_A = b_1 t\left(\frac{h}{2} + \frac{b_1}{2}\right) = \frac{b_1 t h}{2} + \frac{b_1^2 t}{2}$$

$$q_B = q_A + bt\frac{h}{2} = \frac{b_1 t h}{2} + \frac{b_1 t}{2} + \frac{bth}{2}$$

At location ℓ, $q = t\ell\left(\frac{h}{2} + b_1 - \frac{\ell}{2}\right)$

$$F_1 = \int q\, d\ell = t\int_0^{b_1}\left(\frac{h\ell}{2} + b_1\ell - \frac{\ell^2}{2}\right)d\ell = \frac{thb_1^2}{4} + \frac{tb_1^3}{3}$$

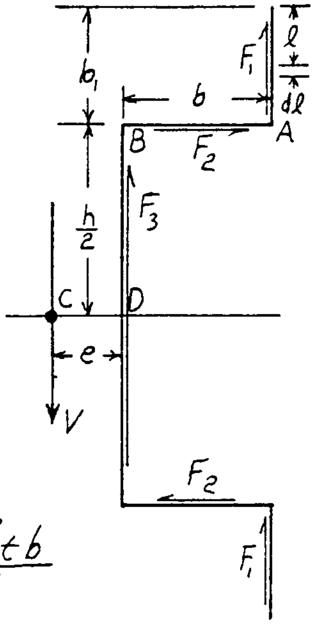
$$F_2 = \frac{q_A+q_B}{2}b = \frac{b_1 thb}{2} + \frac{b_1^2 tb}{2} + \frac{b^2 th}{4}$$

Take moments about D.

$$Ve = F_2 h - 2F_1 b = \frac{b_1 th^2 b}{2} + \frac{b_1^2 thb}{2} + \frac{b^2 th^2}{4} - \frac{2b_1^2 thb}{4} - \frac{2b_1^3 tb}{3}$$

$$= \frac{tb^2h^2}{4}\left[1 + \frac{2b_1}{b}\left(1 - \frac{4b_1^2}{3h^2}\right)\right]$$

$$e = \frac{b\left[1 + \frac{2b_1}{b}\left(1 - \frac{4b_1^2}{3h^2}\right)\right]}{2 + \frac{h}{3b} + \frac{2b_1}{b}\left(1 + \frac{2b_1}{h} + \frac{4b_1^2}{3h^2}\right)}$$



8.2 Let V have magnitude of I.

$$I = \frac{2th^3}{12} - \frac{t(h-2b_1)^3}{12} + 2tb\left(\frac{h}{2}\right)^2$$

$$= \frac{tbh^2}{4}\left[2 + \frac{h}{2b} + \frac{2b_1}{b}\left(1 - \frac{2b_1}{h} + \frac{4b_1^2}{3h^2}\right)\right]$$

$$q_A = tb_1\left(\frac{h}{2} - \frac{b_1}{2}\right) = \frac{tb_1 h}{2} - \frac{tb_1^2}{2}$$

$$q_B = q_A + tb\frac{h}{2} = \frac{tb_1 h}{2} - \frac{tb_1^2}{2} + \frac{tbh}{2}$$

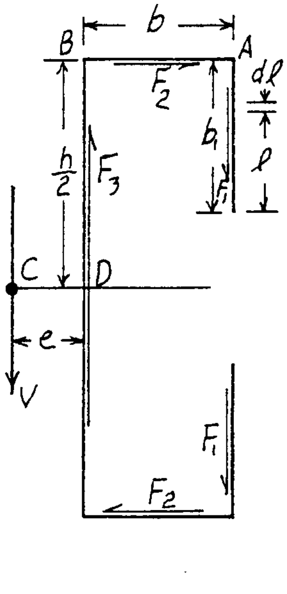
At location ℓ, $q = t\ell\left(\frac{h}{2} - b_1 + \frac{\ell}{2}\right)$

$$F_1 = \int q\, d\ell = t\int_0^{b_1}\left(\frac{h\ell}{2} - b_1\ell + \frac{\ell^2}{2}\right)d\ell = \frac{tb_1^2 h}{4} - \frac{tb_1^3}{3}$$

$$F_2 = \frac{q_A+q_B}{2}b = \frac{tb_1 bh}{2} - \frac{tb_1^2 b}{2} + \frac{tb^2 h}{4}$$

$$Ve = F_2 h + 2F_1 b = \frac{tb^2h^2}{4}\left[1 + \frac{2b_1}{b}\left(1 - \frac{4b_1^2}{3h^2}\right)\right]$$

$$e = \frac{b\left[1 + \frac{2b_1}{b}\left(1 - \frac{4b_1^2}{3h^2}\right)\right]}{2 + \frac{h}{3b} + \frac{2b_1}{b}\left(1 - \frac{2b_1}{h} + \frac{4b_1^2}{3h^2}\right)}$$



8.3 Let V have magnitude of I

$$I = \frac{t_w h^3}{12} + 2t_f(b_1+b)\left(\frac{h}{2}\right)^2 = \frac{t_f b h^2}{4}\left[2 + \frac{2b_1}{b} + \frac{t_w h}{3t_f b}\right]$$

$$q_A = t_f b \frac{h}{2} = \frac{t_f b h}{2}$$

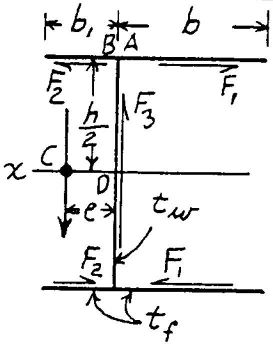
$$q_B = t_f b_1 \frac{h}{2} = \frac{t_f b_1 h}{2}$$

$$F_1 = \frac{1}{2} q_A b = \frac{t_f b^2 h}{4}$$

$$F_2 = \frac{1}{2} q_B b_1 = \frac{t_f b_1^2 h}{4}$$

$$Ve = (F_1 - F_2)h = \frac{t_f b^2 h^2}{4}\left(1 - \frac{b_1^2}{b^2}\right)$$

$$e = \frac{b\left(1 - \frac{b_1^2}{b^2}\right)}{2 + \frac{2b_1}{b} + \frac{t_w h}{3t_f b}}$$



8.4 Let V have magnitude of I.

$$I = \frac{2\sqrt{2}\,t(2b/\sqrt{2})^3}{12} - \frac{\sqrt{2}\,t(2b/\sqrt{2} - 2b_1/\sqrt{2})^3}{12}$$

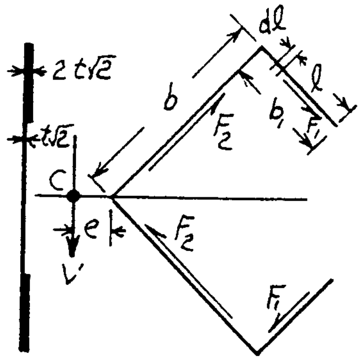
$$= \frac{tb^3}{3}\left(1 + \frac{3b_1}{b} - \frac{3b_1^2}{b^2} + \frac{b_1^3}{b^3}\right)$$

$$q = t\ell\left(\frac{b}{\sqrt{2}} - \frac{b_1}{\sqrt{2}} + \frac{\ell}{2\sqrt{2}}\right)$$

$$F_1 = \frac{t}{\sqrt{2}}\int_0^{b_1}\left(b\ell - b_1\ell + \frac{\ell^2}{2}\right)d\ell = \frac{tb\,b_1^2}{2\sqrt{2}} - \frac{tb_1^3}{3\sqrt{2}}$$

$$Ve = 2F_1 b = \frac{tb^2 b_1^2}{3\sqrt{2}}\left(3 - \frac{2b_1}{b}\right)$$

$$e = \frac{\frac{b_1^2}{\sqrt{2}\,b^2}\left(3 - \frac{2b_1}{b}\right)}{1 + \frac{3b_1}{b} - \frac{3b_1^2}{b^2} + \frac{b_1^3}{b^3}}$$



8.5 Let V have magnitude of I.

$$I = 2\int_0^\theta t(R\sin\phi)^2 R\,d\phi = tR^3(\theta - \sin\theta\cos\theta)$$

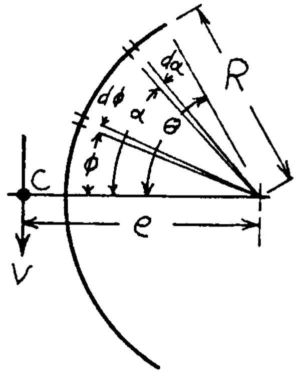
$$q = \int_\phi^\theta t(R\sin\alpha)R\,d\alpha = tR^2(\cos\phi - \cos\theta)$$

$$dM = R\,dF = R\,q\,R\,d\phi$$

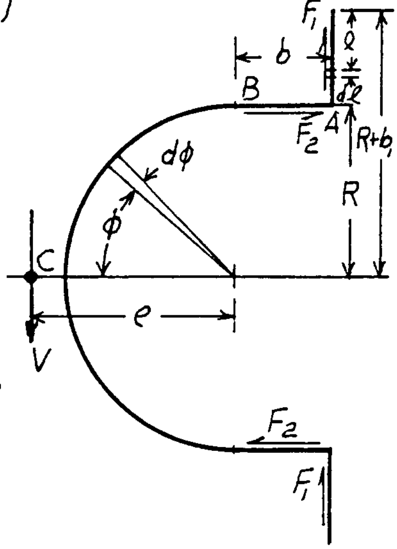
$$M = 2R^4 t\int_0^\theta(\cos\phi - \cos\theta)\,d\phi = 2R^4 t(\sin\theta - \theta\cos\theta)$$

$$Ve = M$$

$$e = \frac{2R(\sin\theta - \theta\cos\theta)}{\theta - \sin\theta\cos\theta}$$



8.6 Let V have magnitude of I.



$$I=\frac{t(2R+2b_1)^3}{12}-\frac{t(2R)^3}{12}+2tbR^2+tR^3\left(\frac{\pi}{2}-\sin\frac{\pi}{2}\cos\frac{\pi}{2}\right)$$

$$=\frac{tR^3}{6}\left[3\pi+\frac{12}{R}(b+b_1)+4\frac{b_1^2}{R^2}\left(3+\frac{b_1}{R}\right)\right.$$

$$q_\ell=t\ell\left(R+b_1-\frac{\ell}{2}\right)$$

$$q_A=tb_1\left(R+\frac{b_1}{2}\right)=tb_1R+\frac{tb_1^2}{2}$$

$$q_B=q_A+tbR=tb_1R+\frac{tb_1^2}{2}+tbR$$

$$q_\phi=q_B+tR^2\left(\cos\phi-\cos\frac{\pi}{2}\right)=tb_1R+\frac{tb_1^2}{2}+tbR+tR^2\cos\phi$$

$$dM=R\,dF=q_\phi R^2 d\phi$$

$$M=2tR^2\int_0^{\pi/2}\left(Rb_1+Rb+\frac{b_1^2}{2}+R^2\cos\phi\right)d\phi$$

$$=2tR^2\left(\frac{Rb_1\pi}{2}+\frac{Rb\pi}{2}+\frac{b_1^2\pi}{4}+R^2\right)$$

$$F_1=\int q_\ell\,d\ell=t\int_0^{b_1}\left(R\ell+b_1\ell-\frac{\ell^2}{2}\right)d\ell$$

$$=\frac{tRb_1^2}{2}+\frac{tb_1^3}{3}$$

$$F_2=b\,\frac{q_A+q_B}{2}=tb\left(Rb_1+\frac{b_1^2}{2}+\frac{Rb}{2}\right)$$

$$Ve=M+2RF_2-2bF_1=\frac{tR^4}{6}\left[12+\frac{6\pi}{R}(b_1+b)+6\frac{b^2}{R^2}+12\frac{bb_1}{R^2}+3\pi\frac{b_1^2}{R^2}-4\frac{b_1^3b}{R^4}\right]$$

$$e=\frac{R\left[12+\frac{6\pi}{R}(b_1+b)+6\frac{b^2}{R^2}+12\frac{bb_1}{R^2}+3\pi\frac{b_1^2}{R^2}-4\frac{b_1^3b}{R^4}\right]}{3\pi+\frac{12}{R}(b_1+b)+4\frac{b_1^2}{R^2}\left(3+\frac{b_1}{R}\right)}$$

For $b_1=0$

$$e=\frac{R\left[4+2\pi\frac{b}{R}+2\frac{b^2}{R^2}\right]}{\pi+4\frac{b}{R}}$$

For $b=0$

$$e=\frac{3R\left[4+2\pi\frac{b_1}{R}+\pi\frac{b_1^2}{R^2}\right]}{3\pi+4\frac{b_1^3}{R^3}+12\frac{b_1}{R}+12\frac{b_1^2}{R^2}}$$

8.7 Let V have magnitude of I.

$$I = \frac{41.5(83)^3}{12} - \frac{38.5(77)^3}{12} + \frac{38.5(63)^3}{12} - \frac{38.5(50)^3}{12} = 720{,}800 \text{ mm}^4$$

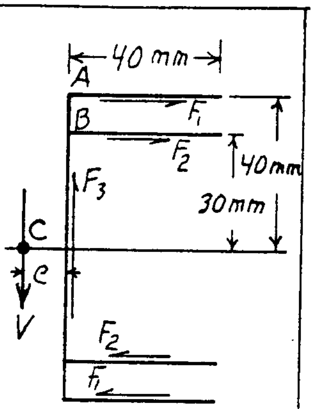
$$q_A = 40(3)(40) = 4800 \text{ N/mm}; \quad q_B = 40(3)(30) = 3600 \text{ N/mm}$$

$$F_1 = \frac{q_A}{2}(40) = 2400(40) = 96{,}000 \text{ N}$$

$$F_2 = \frac{q_B}{2}(40) = 1800(40) = 72{,}000 \text{ N}$$

$$Ve = 80F_1 + 60F_2$$

$$e = \frac{80(96{,}000) + 60(72{,}000)}{720{,}800} = \underline{16.65 \text{ mm}}$$



8.8 Let V have magnitude of I.

$$I = \frac{52.5(102.5)^3}{12} - \frac{47.5(97.5)^3}{12} + \frac{23.75(52.5)^3}{12} - \frac{26.25(47.5)^3}{12}$$

$$= 1{,}094{,}500 \text{ mm}^4$$

$$q_A = 25(2.5)(25) = 1563 \text{ N/mm}; \quad q_B = q_A + 25(2.5)(37.5) = 3906 \text{ N/mm}$$

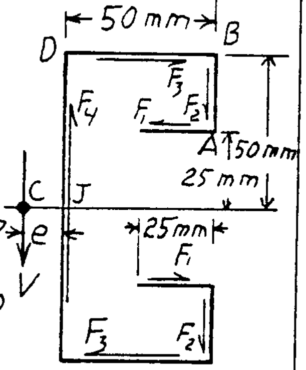
$$q_D = q_B + 50(2.5)(50) = 10{,}156 \text{ N/mm}; \quad q_J = q_D + 50(2.5)(25) = 13{,}281 \text{ N/mm}$$

$$F_1 = \frac{q_A}{2}(25) = \frac{1563(25)}{2} = 19{,}540 \text{N}; \quad F_3 = \frac{q_B + q_D}{2}(50) = 351{,}550 \text{ N}$$

$$F_4 = \left[q_D + \frac{2}{3}(q_J - q_D)\right](100) = \left[10{,}156 + \frac{2}{3}(13{,}281 - 10{,}156)\right](100) = 1{,}223{,}930 \text{ N}$$

$$V(50 + e) = 100\,F_3 + 50\,F_4 - 50\,F_1$$

$$e = \frac{100(351{,}550) + 50(1{,}223{,}930) - 50(19{,}540)}{1{,}094{,}500} - 50 = \underline{37.14 \text{ mm}}$$



8.9 Let V have magnitude of I.

$$I = 197{,}100{,}000 \text{ mm}^4; \quad \tan\theta = 0.1000$$

$$q_\ell = \frac{13.53 + (13.53 + \ell\tan\theta)}{2}(\ell)\left(193.2 - \frac{\ell}{2}\tan\frac{\theta}{2}\right)$$

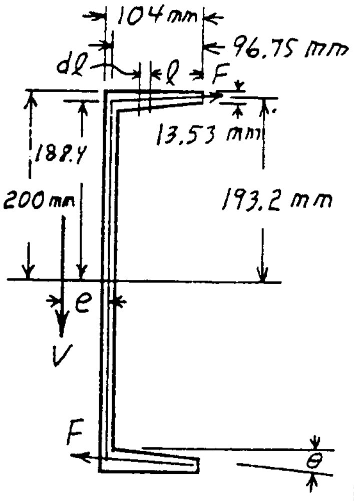
$$F = \int_0^{96.75} q_\ell \, d\ell = 15{,}021{,}000 \text{ N}$$

$$Ve = F\cos\frac{\theta}{2}(2)(188.4)$$

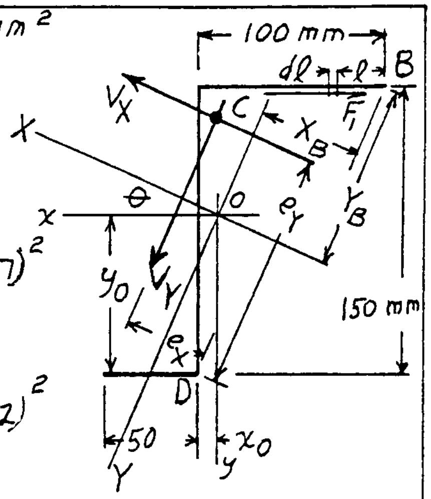
$$e = \frac{15{,}021{,}000(0.9988)(2)(188.4)}{197{,}100{,}000} = 28.68 \text{ mm}$$

The mean flange thickness is 18 mm. Use Eq. (8.9)

$$e = \frac{96.75}{2 + \dfrac{14.5(382)}{3(18)(96.75)}} = \underline{31.62} \text{ mm}$$



$\boxed{8.10}$ $A_1 = 100(4) = 400\ mm^2;\ A_2 = 750\ mm^2;\ A_3 = 300\ mm^2$



$$y_0 = \frac{400(150) + 750(75)}{1550} = 80.17\ mm$$

$$x_0 = \frac{400(50) - 300(25)}{1550} = 8.62\ mm$$

$$I_x = \frac{100(4)^3}{12} + 400(69.83)^2 + \frac{5(150)^3}{12} + 750(5.17)^2 + \frac{50(6)^3}{12} + 300(80.17)^2$$
$$= 5,306,390\ mm^4$$

$$I_y = \frac{4(100)^3}{12} + 400(41.38)^2 + \frac{150(5)^3}{12} + 750(8.62)^2 + \frac{6(50)^3}{12} + 300(33.62)^2$$
$$= 1,477,137\ mm^4$$

$$I_{xy} = 400(-69.83)(-41.38) + 750(5.17)(8.62) + 300(80.17)(33.62) = 1,997,845\ mm^4$$

$$\tan 2\theta = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(1,997,845)}{5,306,390 - 1,477,137} = -1.0435$$

$$\underline{\theta = -0.4023\ rad}$$

$$I_X = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} = 6,158,924\ mm^4;\ I_Y = I_x + I_y - I_X = 624,603\ mm^4$$

$$X_B = x_B \cos\theta + y_B \sin\theta = -91.38(0.91977) - 69.83(-0.39246) = -56.643\ mm$$

$$Y_B = y_B \cos\theta - x_B \sin\theta = -69.83(0.91977) - (-91.38)(-0.39246) = -100.1\ mm$$

At distance ℓ from B for load V_Y parallel to Y-axis,

$$q = \frac{V_Y}{I_X} A'\bar{Y} = \frac{V_Y}{I_X} t\ell\left[|Y_B| - \frac{1}{2}\ell|\sin\theta|\right]$$

$$F_1 = \int_0^{100} q\,d\ell = \frac{V_Y t}{I_X}\left[100.1\frac{\ell^2}{2} - \frac{0.39246}{2}\frac{\ell^3}{3}\right]_0^{100} = 435,040\frac{V_Y t}{I_X}$$

$$\Sigma M_D = 0 = V_Y e_X - 150 F_1$$

$$e_X = \frac{150(435,040)(4)}{6,158,924} = \underline{42.38\ mm}$$

At distance ℓ from B for load V_X parallel to X-axis,

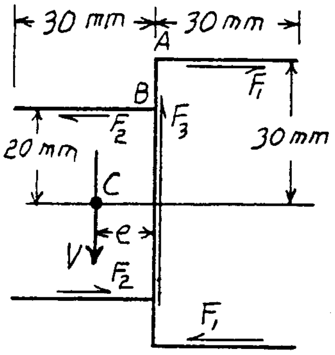
$$q = \frac{V_X A'\bar{X}}{I_Y} = \frac{V_X}{I_Y} t\ell\left[|X_B| - \frac{\ell}{2}\cos\theta\right]$$

$$F_1 = \int_0^{100} q\,d\ell = \frac{V_X t}{I_Y}\left[56.643\frac{\ell^2}{2} - \frac{0.91977}{2}\frac{\ell^3}{3}\right]_0^{100} = 129,920\frac{V_X t}{I_Y}$$

$$\Sigma M_D = 0 = V_X e_Y - 150 F_1$$

$$e_Y = \frac{150(129,920)(4)}{624,603} = \underline{124.80\ mm}$$

8.11 Let V have magnitude of I.



$$I = \frac{31.5(63)^3}{12} - \frac{28.5(57)^3}{12} + \frac{28.5(43)^3}{12} - \frac{28.5(37)^3}{12} = 285,070 \text{ mm}^4$$

$$q_A = 30(3)(30) = 2700 \text{ N/mm}; \quad q_B = 30(3)(20) = 1800 \text{ N/mm}$$

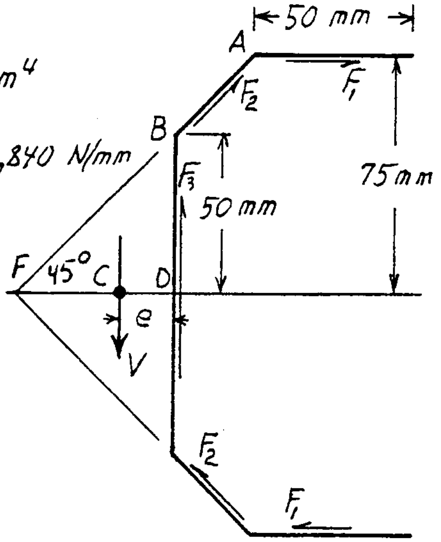
$$F_1 = \frac{q_A}{2}(30) = \frac{2700(30)}{2} = 40,500 \text{ N}$$

$$F_2 = \frac{q_B}{2}(30) = \frac{1800(30)}{2} = 27,000 \text{ N}$$

$$Ve = 60F_1 - 40F_2$$

$$e = \frac{60(40,500) - 40(27,000)}{285,070} = \underline{4.74 \text{ mm}}$$

8.12 Let V have magnitude of I.



$$I = \frac{4\sqrt{2}(150)^3}{12} - \frac{(4\sqrt{2}-4)(100)^3}{12} + 2(50)(4)(75^2) = 3,703,000 \text{ mm}^4$$

$$q_A = 50(4)(75) = 15,000 \text{ N/mm}; \quad q_B = q_A + 25\sqrt{2}(4)(62.5) = 23,840 \text{ N/mm}$$

$$q_D = q_B + 50(4)(25) = 28,840 \text{ N/mm}$$

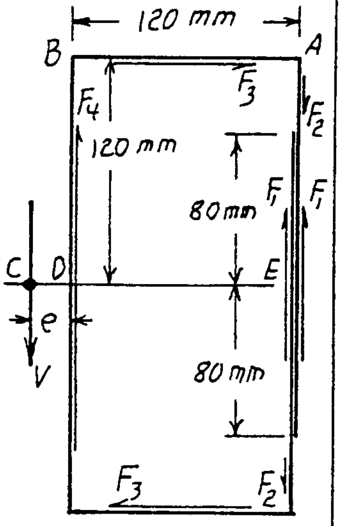
$$F_1 = \frac{q_A}{2}(50) = \frac{15,000(50)}{2} = 375,000 \text{ N}$$

$$F_3 = \left[q_B + \tfrac{2}{3}(q_D - q_B)\right](100) = 2,717,300 \text{ N}$$

$$\Sigma M_F = 0 = V(50-e) + 150F_1 - 50F_3$$

$$e = 50 - \frac{50(2,717,300) - 150(375,000)}{3,703,000} = \underline{28.50 \text{ mm}}$$

8.13 Let V have magnitude of I.



$$I = \frac{2(5)(240)^3}{12} + \frac{5(160)^3}{12} + 2(120)(5)(120)^2 = 30,507,000 \text{ mm}^4$$

$$q_A = 200(5)(20) = 20,000 \text{ N/mm}; \quad q_B = q_A + 120(5)(120) = 92,000 \text{ N/mm}$$

$$q_D = q_B + 120(5)(60) = 128,000 \text{ N/mm}$$

$$F_3 = \frac{q_A + q_B}{2}(120) = \frac{(20,000 + 92,000)(120)}{2} = 6,720,000 \text{ N}$$

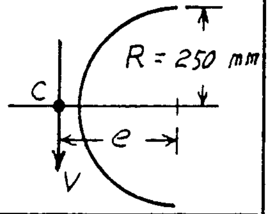
$$F_4 = \left[q_B + \tfrac{2}{3}(q_D - q_B)\right](240) = 27,840,000 \text{ N}$$

$$\Sigma M_E = 0 = V(e+120) - 120F_4 - 240F_3$$

$$e = \frac{120(27,840,000 + 240(6,720,000)}{30,507,000} - 120 = \underline{42.28 \text{ mm}}$$

8.14 From Table 8.1, Fig. E,

$$e = \frac{4R}{\pi} = \frac{4(250)}{\pi} = \underline{318.31\ \text{mm}}$$



8.15 Let V have magnitude of I.

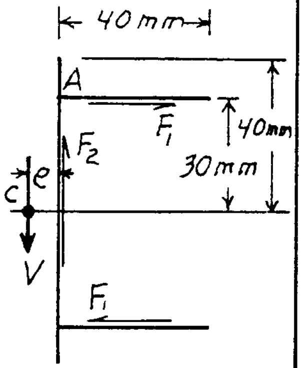
$$I = \frac{3(80)^3}{12} + 2(38.5)(3)(30)^2 = 335{,}900\ \text{mm}^4$$

$$q_A = 40(3)(30) = 3600\ \text{N/mm}$$

$$F_1 = \frac{q_A}{2}(40) = \frac{3600(40)}{2} = 72{,}000\ \text{N}$$

$$Ve = 60\,F_1$$

$$e = \frac{60(72{,}000)}{335{,}900} = \underline{12.86\ \text{mm}}$$



8.16

$$I_x = \frac{1}{12}(80)(250)^3 - \frac{1}{12}(60)(230)^3 - \frac{1}{12}(10)(160)^3$$
$$+ (2)\left(\frac{1}{12}\right)(45)(10)^3 + (2)(45\times10)(80)^2$$
$$= 45.6858\times10^6\ \text{mm}^4. \quad \text{Let } V = I_x.$$

$$F_1 = \int_{80}^{120} q\,d\ell = \int_{80}^{120} \bar{A}'\bar{y}'\,d\ell = \int_{80}^{120} t\ell\left(80+\frac{\ell}{2}\right)d\ell$$
$$= \frac{80t\ell^2}{2} + \frac{t\ell^3}{6}\Bigg|_{80}^{120} = 10\left[40(120^2-80^2)+\frac{(120^3-80^3)}{6}\right]$$
$$= 5.2266\times10^6$$

$$q_A = (40\times10)\left(80+\frac{40}{2}\right) = 40{,}000$$

$$q_B = q_A + (70\times10)\times120 = 40{,}000 + 84{,}000 = 124{,}000$$

$$F_2 = \frac{q_A+q_B}{2}(70) = \frac{40{,}000+124{,}000}{2}(70) = 5.7400\times10^6$$

$$F_3 = \int_0^{50} q\,d\ell = \int_0^{50} \bar{A}'\bar{y}'\,d\ell = \int_0^{50}(50\times10)(80)\,d\ell = 2.000\times10^6$$

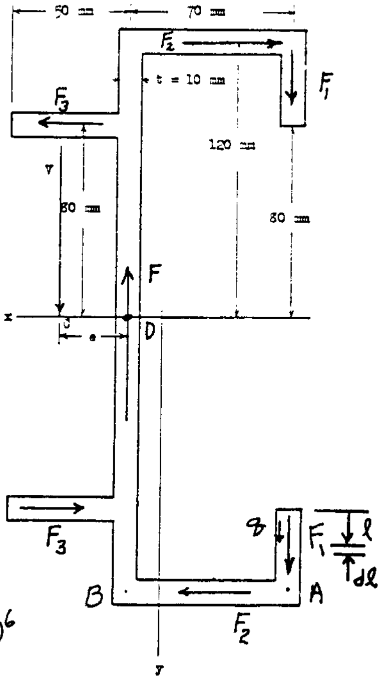


Fig. P8.16

$$\Sigma M_D = Ve - 240F_2 - 2F_1\times70 + 160F_3 = 0$$

$$e = \frac{140F_1 + 240F_2 - 160F_3}{V}$$

$$= \frac{140(5.2266\times10^6) + 240(5.7400\times10^6) - 160(2.000\times10^6)}{45.6858\times10^6}$$

or $e = 39.17$ mm

8.17

By Fig. C, Table 8.1 and Fig. P8.17, $b_1 = 25$ mm, $b = 50$ mm, $t_w = t_f = 3.00$ mm, and $h = 100$ mm. (See also Prob. 8.3).

For $b_1 = 25 \leq b = 50$,

$$e = 50\left[\frac{1-\left(\frac{25}{50}\right)^2}{2+\frac{2(25)}{50}+\frac{(3)(100)}{3(3)(50)}}\right] = 10.23 \text{ mm}$$

8.18

By Fig. B, Table 8.1 and Fig. P8.18, $b_1 = 25$ mm, $b = 50$ mm, $h = 100$ mm and $t = 2.50$ mm. (See also Prob. 8.2). Therefore,

$$e = 50\left[\frac{1+\frac{2(25)}{50}\left[1-\frac{4}{3}\left(\frac{25}{100}\right)^2\right]}{2+\frac{100}{3(50)}+\frac{2(25)}{50}\left[1-2\left(\frac{25}{100}\right)+\frac{4}{3}\left(\frac{25}{100}\right)^2\right]}\right] = 29.49 \text{ mm}.$$

8.19

Let $V = I$. (Fig. A)

$$I = \frac{1}{12}(26)(52)^3 - \frac{1}{12}(24)(48)^3 + \frac{1}{12}(24)(27)^3 - \frac{1}{12}(24)(23)^3 = 98498.7 \text{ mm}^4$$

$$q_A = (25)(2)(25) = 1250$$

$$q_B = (25)(2)(12.5) = 625$$

$$F_1 = \frac{1}{2} q_A (25) = \frac{1}{2}(1250)(25) = 15625$$

$$F_2 = \frac{1}{2} q_B (25) = \frac{1}{2}(625)(25) = 7812.5$$

$$Ve = 50F_1 + 25F_2$$

$$\therefore\ e = \frac{(50)(15625)+(25)(7812.5)}{98498.7} = 9.91 \text{ mm}$$

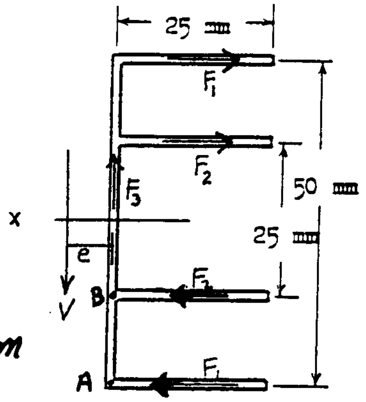


Fig. A

8.20

By Fig. A, Table 8.1 and Fig. P8.20, we have $b = 25$ mm, $b_1 = 12.5$ mm, $h = 25$ mm, and $t = 2.00$ mm.

Therefore,

$$e = 25\left[\frac{1+\frac{2(21.5)}{25}\left[1-\frac{4}{3}\left(\frac{12.5}{25}\right)^2\right]}{2+\frac{25}{3(25)}+\frac{2(12.5)}{25}\left[1+\frac{2(12.5)}{25}+\frac{4}{3}\left(\frac{12.5}{25}\right)^2\right]}\right]$$

$$= 10.85 \text{ mm}$$

8.21 Let $V = I$, Fig. A, where

$$I = 2\int_0^{50/\sqrt{3}} \left(\frac{l}{2}\right)^2 (2\,dl) + 2\left(\frac{1}{12}\right)(2)(10.566)^3$$

$$+ 2(10.566 \times 2)\left(14.434 + \frac{10.566}{2}\right)^2$$

$$\therefore\ I = 24{,}842.5 \text{ mm}^4$$

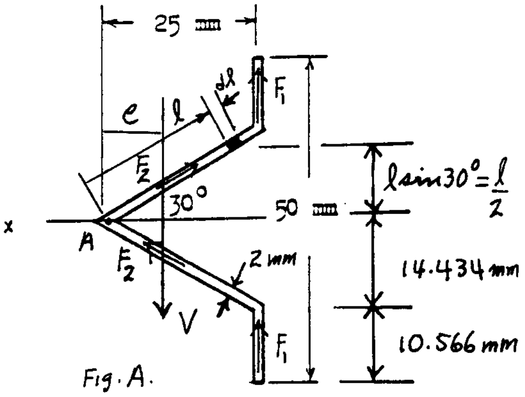


Fig. A.

$$F_1 = \int_0^{10.566} q\,dl = \int_0^{10.566} A'\bar{y}'\,dl = \int_0^{10.566} (lt)\left(25 - \frac{l}{2}\right) dl$$

$$= t\left[\frac{25 l^2}{2} - \frac{l^3}{6}\right]_0^{10.566} = 2397.81$$

$$\Sigma M_A = Ve - 2F_1(25) = 0$$

$$\therefore\ e = \frac{2F_1(25)}{V} = \frac{(50)(2397.81)}{24{,}842.5} = 4.83 \text{ mm}$$

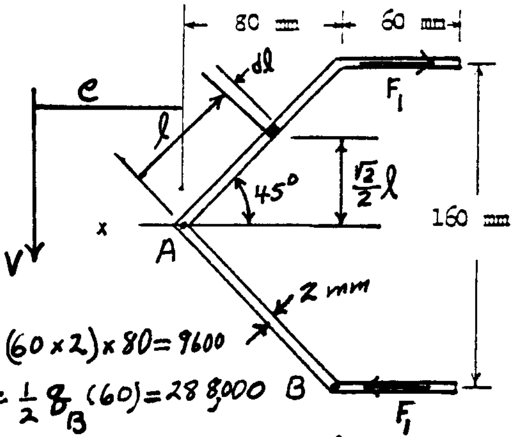
8.22 Let $V = I$, where (Fig A)

$$I = 2\int_0^{160/\sqrt{2}} \left(l\frac{\sqrt{2}}{2}\right)^2 (2\,dl) + \frac{2}{12}(60)(2)^3$$

$$+ 2(60 \times 2)(80)^2$$

$$= 2\left(\frac{l^3}{3}\right)\Big|_0^{160/\sqrt{2}} + 80 + 1{,}536{,}000$$

$$= 965436 + 80 + 1{,}536{,}000$$



or $I = 2{,}501{,}516 \text{ mm}^4$; $q_B = (60 \times 2) \times 80 = 9600$

$F_1 = \frac{1}{2} q_B(60) = 288{,}000$

$$\Sigma M_A = Ve - 160F_1 = 0$$

Fig. A

or $$e = \frac{160F_1}{V} = \frac{160(288{,}000)}{2{,}501{,}516} = 18.42 \text{ mm}$$

8.23 Let $V = I$, Fig. A, where

$$I = 2\int_0^{80/\sqrt{2}} \left(40 + \frac{\sqrt{2}}{2} l\right)^2 (2.5\,dl) + \frac{2}{12}(60)(2.5)^3 + 2(60 \times 2.5)(40)^2$$

$$+ \frac{1}{12}(2.5)(80)^3 = 5\left[1600 l + 20\sqrt{2} l^2 + \frac{1}{6} l^3\right]_0^{80/\sqrt{2}} + 156.25 + 480{,}000 + 106{,}666.7$$

or $I = 1{,}642{,}769 \text{ mm}^4$

(Cont.)

8.23 cont.

$$q_A = t\left(80l - \frac{\sqrt{2}}{4}l^2\right)\Bigg|_{\substack{l=\frac{80}{\sqrt{2}} \\ t=2.5}} = 8485.28$$

$$q_B = q_A + (60\times2.5)(40) = 14,485.28$$

$$q_E = q_B + (40\times2.5)(20) = 16,485.28$$

$$q_{ave} = q_B + \frac{2}{3}(q_E - q_B) = 15,818.61$$

$$\therefore\ F_2 = \frac{1}{2}(q_A + q_B)(60) = 689,116.8$$

$$F_3 = (q_{ave})(80) = 1,265,488.8$$

$$\Sigma M_D = Ve - 80F_2 - 20F_3 = 0$$

$$\therefore e = \frac{80F_2 + 20F_3}{V} = 48.97\ mm$$

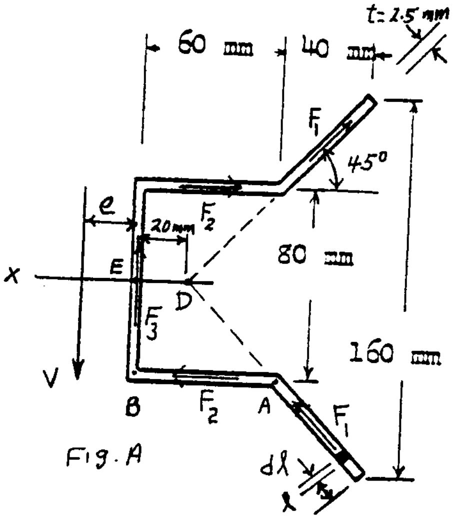


Fig. A

8.24 Let $V = I$, Fig. A, where

$$I = 2\int_0^{\sqrt{2}b}\left(s\frac{\sqrt{2}}{2}\right)^2(t\,ds) + (2)\left(\frac{1}{12}tb^3\right) + (2)(tb)\left(\frac{3b}{2}\right)^2$$

$$= \frac{ts^3}{3}\Big|_{s=\sqrt{2}b} + \frac{1}{6}tb^3 + \frac{9}{2}tb^3 = 5.609tb^3$$

$$q = (tl)\left(2b - \frac{l}{2}\right) = t\left(2bl - \frac{l^2}{2}\right)$$

$$\therefore\ F_1 = \int_0^b q\,dl = t\int_0^b\left(2bl - \frac{l^2}{2}\right)dl$$

$$= t\left(bl^2 - \frac{l^3}{6}\right)\Big|_{l=b} = \frac{5}{6}tb^3$$

$$\Sigma M_D = Ve + 2F_1 b = 0$$

$$\therefore\ e = -\frac{2F_1 b}{V} = -\frac{\left(\frac{5}{3}tb^3\right)b}{5.609tb^3} = -0.297b$$

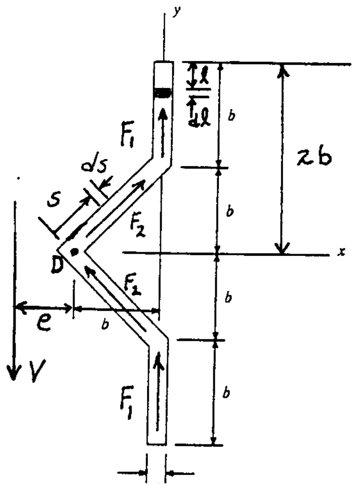


Fig. A

8.25 By symmetry, the centroid 0 and the shear center C are on the x axis. The location of the centroid is (Fig. A)

$$\bar{x}_0 = \frac{(2)(50)(2.5)(25)}{(2)(50)(2.5)+(100)(2.5)} = 12.50 \text{ mm from center of the web.}$$

Hence,
$$I_x \approx \frac{1}{2}(2.5)(100)^3 + (2)(50)(2.5)(50)^2$$
$$= 0.833(10^6)\ \text{mm}^4$$

$$I_y \approx (100)(2.5)(12.5)^2 + (2)(\tfrac{1}{12})(2.5)(50)^3 + (2)(50)(2.5)(12.5)^2$$
$$= 0.130(10^6)\ \text{mm}^4$$

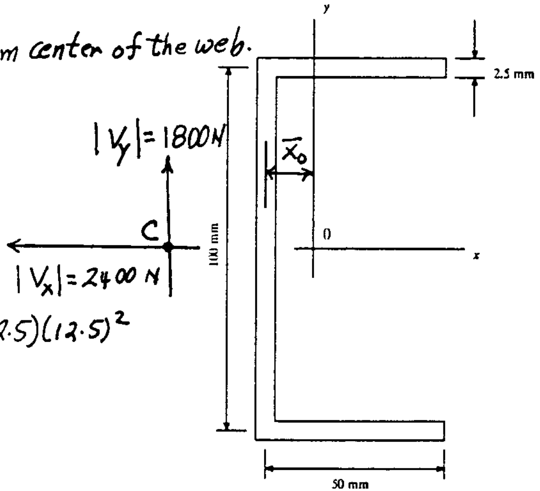


Fig. A

First, we compute the shear stress τ due to V_x. For section AB, Fig. B, we have

$$\tau_{AB(V_x)} = \frac{q}{t} = \frac{(2.5)(l)(37.5-\frac{l}{2})}{2.5}\frac{V_x}{I_y}$$
$$= l\left(37.5-\frac{l}{2}\right)\frac{2400}{0.130\times10^6} = 0.6923 l - 0.009231 l^2 \quad (a)$$

Thus, $\tau_{AB(V_x)}$ is parabolic in the flanges.

For section BC, we have

$$\tau_{BC(V_x)} = \tau_B + \frac{(2.5l)(-12.5)}{(2.5)}\left(\frac{2400}{0.130\times10^6}\right)$$
$$= 11.5375 - 0.23077 l \quad (b)$$

Thus, $\tau_{BC(V_x)}$ is linear in BC.

The shear stress distribution in the cross section due to V_x is shown in Fig. C.

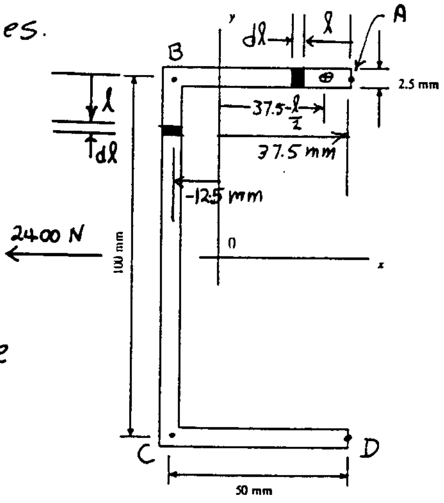


Fig. B

(Cont.)

8.25 cont.

Next, we compute the shear stress due to V_y. For section AB, Fig. D, we have $q = (lt)(50)\frac{V_y}{I_x}$

$$\tau_{AB(V_y)} = \frac{q}{t} = 50l\frac{V_y}{I_x} = 0.108045l \quad (c)$$

For section BC, we have

$$\tau_{BC(V_y)} = \tau_B + l\left(50 - \frac{l}{2}\right)\frac{V_y}{I_x} \quad (d)$$

$$= 5.402 + 0.108045l - 0.00108045l^2$$

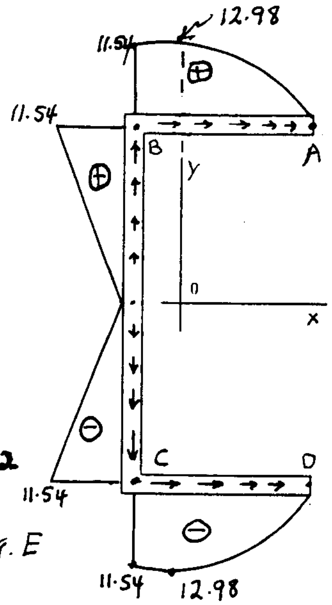


Fig. C

The shear distribution due to V_y is shown in Fig. E

Hence, by Eqs. (a) and (c),

$$\tau_{AB} = \tau_{AB(V_x)} + \tau_{AB(V_y)}$$

or

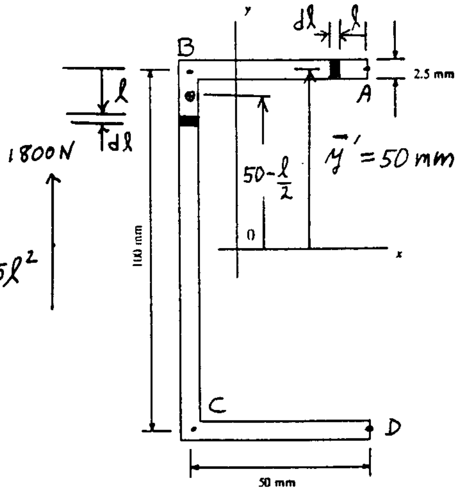
$$\tau_{AB} = 0.8003l - 0.009231l^2 \quad (e)$$

and by Eqs. (b) and (d),

$$\tau_{BC} = \tau_{BC(V_x)} + \tau_{BC(V_y)}$$

or

$$\tau_{BC} = 16.9404 - 0.12273l - 0.00108045l^2 \quad (f)$$



For section CD, by Eqs. (a) and (c)

$$\tau_{CD} = -\tau_{AB(V_x)} + \tau_{AB(V_y)}$$

or

$$\tau_{CD} = -0.58421l + 0.009231l^2 \quad (g)$$

Fig. D

The total shear distribution is shown in Fig. F. Note the locations of the maximum shear stress in each section.

(cont.)

8.25 Cont.

By Eq.(e), the maximum value of τ_{AB} occurs for $\frac{d\tau_{AB}}{d\ell} = 0.8003 - 2(0.009231)\ell = 0$

or at $\ell = 43.3485$ mm.

For this value of ℓ, $\tau_{AB(max)} = 17.346$ MPa

For τ_{BC}, the maximum value occurs at B and it is $\tau_{BC(max)} = 16.940$ MPa, by Eq.(f), with $\ell = 0$.

For τ_{CD}, Eq(g) yields

$$\frac{d\tau_{CD}}{d\ell} = -0.58421 + 2(0.009231)\ell = 0$$

or $\ell = 31.6439$. For this value of ℓ, by Eq.(g), $\tau_{CD(max)} = -9.243$ MPa

For $\ell = 80.70$, $\tau_{BC} = 0$ (Fig. F).

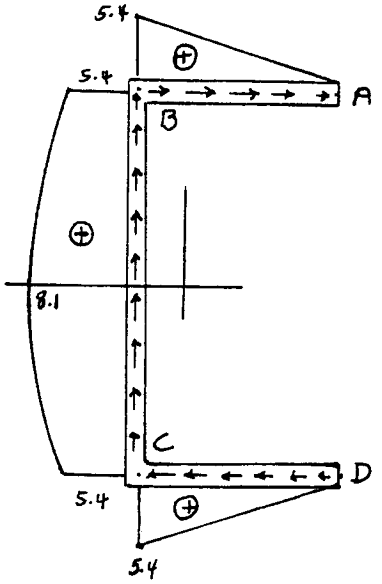


Fig. E.

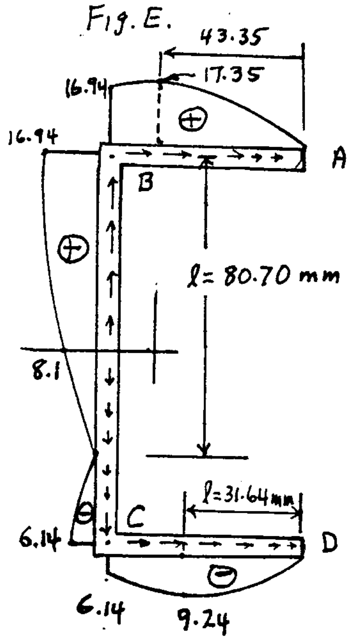
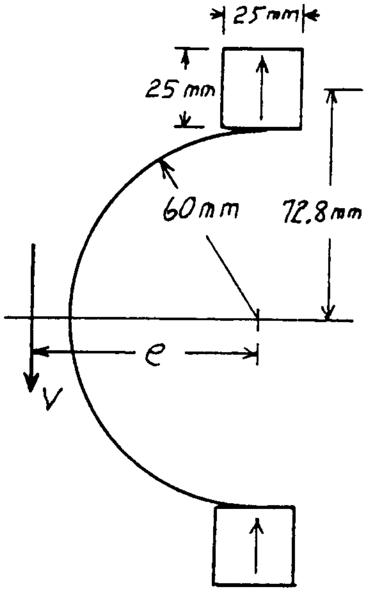


Fig. F

8.26



Let $V = I$.

$$I = 2\left[\frac{25^4}{12} + (25)(25)(72.8)^2\right]$$
$$= 6,690,000 \text{ mm}^4$$

$$q = A'\bar{y}' = (25)(25)(72.8)$$
$$= 45,500 \text{ mm}$$

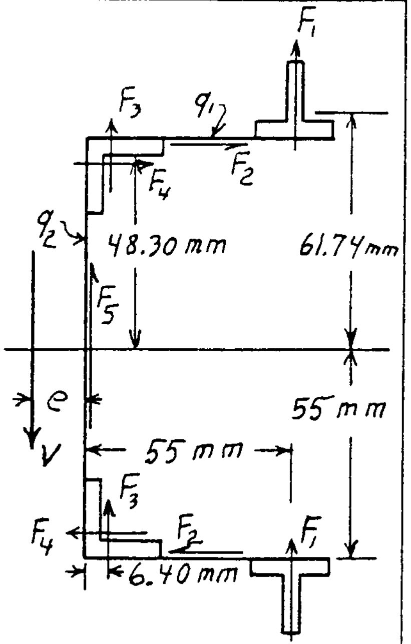
$$M = 2\int_0^{\pi/2} q R^2 d\theta = 2qR^2\frac{\pi}{2} = 45,500(60)^2\pi$$
$$= 514,590,000 \text{ N}\cdot\text{mm}$$

$$Ve = M \quad \therefore \quad e = \frac{M}{V} = \frac{514,590,000}{6,690,000}$$

or

$$e = 76.92 \text{ mm}$$

8.27 Let V have magnitude of I. The area of the T-section is 144 mm^2 and its centroid is located 6.44 mm from its base. The area of the angle section is 146 mm^2 and its centroid is located 6.40 mm from its base.



$$I = 2(146)(48.30)^2 + 2(144)(61.74)^2 = 1{,}779{,}000\ mm^4$$

$$q_1 = 144(61.74) = 8891\ N/mm$$

$$q_2 = q_1 + 146(48.30) = 15{,}942\ N/mm$$

$$F_1 = (6.44 + 0.3)q_1 = 6.74(8891) = 59{,}925\ N$$

$$F_2 = (55 - 6.7)q_1 = 429{,}435\ N$$

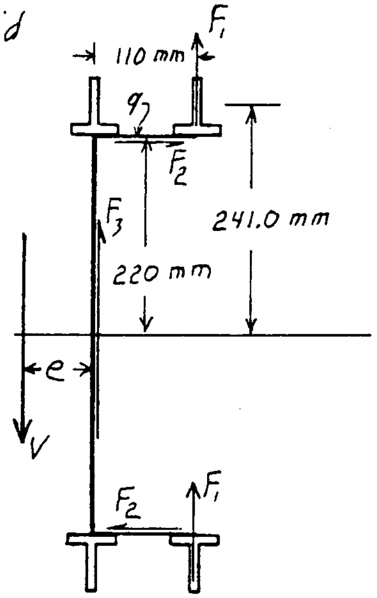
$$F_3 = 6.7\,q_1 = 59{,}570\ N$$

$$F_4 = 6.7\,q_2 = 106{,}811\ N$$

$$Ve = 2F_2(55) + 2F_4(48.30) - 2F_1(55) - 2F_3(6.7)$$

$$e = \frac{2(429{,}435)(55) + 2(106{,}811)(48.30) - 2(59{,}925)(55) - 2(59{,}570)(6.7)}{1{,}779{,}000} = \underline{28.20\ mm}$$

8.28 Let V have magnitude of I. The area of each T-section is 1000 mm^2 and the centroid is located 20.5 mm from its base.



$$I = 4(1000)(241.0)^2 = 232{,}324{,}000\ mm^4$$

$$q = 1000(241.0) = 241{,}000\ N/mm$$

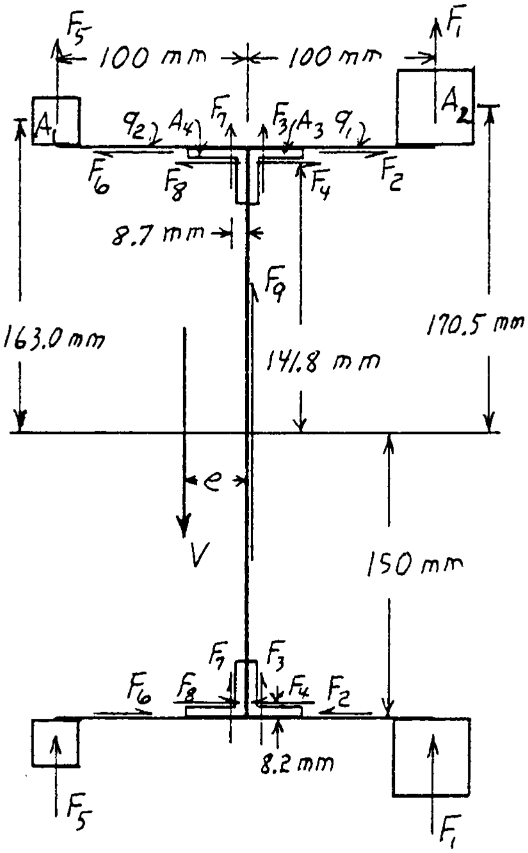
$$F_1 = (20.5 + 0.5)q = 21.0(241{,}000) = 5{,}061{,}000\ N$$

$$F_2 = 110\,q = 26{,}510{,}000\ N$$

$$Ve = 440F_2 - 2(110)F_1$$

$$e = \frac{440(26{,}510{,}000) - 2(110)(5{,}061{,}000)}{232{,}324{,}000} = \underline{45.41\ mm}$$

8.29 Let V have magnitude of I. The area of each angle section is $278\ mm^2$ and the centroid is located 7.7 mm from its base.



$$I = 2(40)(40)(170.5)^2 + 2(25)(25)(163.0)^2 + 4(278)(141.8)^2$$

$$= 148{,}595{,}000\ mm^4$$

$$q_1 = (40)(40)(170.5) = 272{,}800\ N/mm$$

$$q_2 = (25)(25)(163.0) = 101{,}875\ N/mm$$

Let q_3 be the shear flow between the vertical web and A_3.

$$q_3 = q_1 + 278(141.8) = 312{,}220\ N/mm$$

Let q_4 be the shear flow between the vertical web and A_4.

$$q_4 = q_2 + 278(141.8) = 141{,}295\ N/mm$$

$$F_1 = 20.5\,q_1 = 20.5(272{,}800) = 5{,}592{,}000\ N$$

$$F_2 = (100-8.7)q_1 = 24{,}907{,}000\ N$$

$$F_3 = 8.2\,q_1 = 2{,}237{,}000\ N$$

$$F_4 = 8.7\,q_3 = 2{,}716{,}000\ N$$

$$F_5 = 13.0\,q_2 = 1{,}324{,}000\ N$$

$$F_6 = (100-8.7)q_2 = 9{,}301{,}000\ N$$

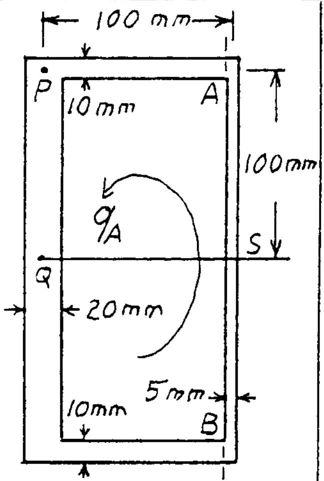
$$F_7 = 8.2\,q_2 = 835{,}000\ N$$

$$F_8 = 8.7\,q_4 = 1{,}229{,}000\ N$$

$$Ve = 300(F_2 - F_6) + 283.6(F_4 - F_8) - 100(2F_1 - 2F_5) - 8.7(2F_3 - 2F_7)$$

$$e = \frac{300(15{,}606{,}000) + 283.6(1{,}487{,}000) - 100(8{,}536{,}000) - 8.7(2{,}804{,}000)}{148{,}595{,}000} = \underline{28.44\ mm}$$

8.30 Make saw cuts at A and B to make two beams (beam 1 at left and beam 2). The shear load V has two parts; let $V_1 = I_1$ and $V_2 = I_2$ in magnitudes. $I = \frac{112.5(210)^3}{12} - \frac{87.5(190)^3}{12} = 36,808,000 \text{ mm}^4$



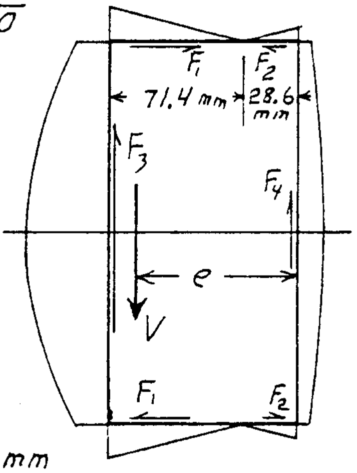
$q_S = 100(5)(50) = 25,000$ N/mm; $q_P = 100(10)(100) = 100,000$ N/mm

$q_Q = q_P + 100(20)(50) = 200,000$ N/mm

Superimpose shear flow q_A counterclockwise as shown. Satisfy Eq.(8.11) starting at P.

$$0 = \left[q_A - 100,000 - \tfrac{2}{3}(200,000 - 100,000)\right]\frac{200}{20} + 2\left[q_A - \frac{100,000}{2}\right]\frac{100}{10}$$
$$+ \left[q_A + \tfrac{2}{3}(25,000)\right]\frac{200}{5}$$

$q_A = 28,571$ N/mm



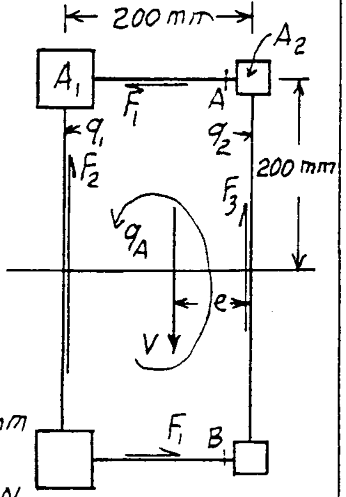
$F_1 = \frac{71,430}{2}(71.4) = 2,550,000$ N; $F_2 = \frac{28,571}{2}(28.6) = 409,000$ N

$F_3 = \left[71,430 + \tfrac{2}{3}(171,430 - 71,430)\right]200 = 27,619,000$ N

$Ve = 200(F_1 - F_2) + 100 F_3$

$$e = \frac{(2,550,000 - 409,000)(200) + 27,619,000(100)}{36,808,000} = \underline{86.67 \text{ mm}}$$

8.31 Make saw cuts at A and B to make two beams. The shear load V has two parts; let $V_1 = I_1$ and $V_2 = I_2$ in magnitudes.



$I = 2(900 + 300)(200)^2 = 96,000,000 \text{ mm}^4$

$q_1 = 900(200) = 180,000$ N/mm; $q_2 = 300(200) = 60,000$ N/mm

Superimpose shear flow q_A counterclockwise as shown. Satisfy Eq.(8.11) starting at A_1.

$0 = (q_A - 180,000)\frac{400}{1} + 2q_A\frac{200}{1} + (q_A + 60,000)\frac{400}{1}$; $q_A = 40,000$ N/mm

$F_1 = 200(40,000) = 8,000,000$ N; $F_2 = 400(180,000 - 40,000) = 56,000,000$ N

$Ve = 200 F_2 - 400 F_1$

$$e = \frac{200(56,000,000) - 400(8,000,000)}{96,000,000} = \underline{83.33 \text{ mm}}$$

8.32 For Problem 8.31 let the web thickness between the two areas A_1 be increased to 2 mm. Satisfy Eq. (8.11) starting at A_1

$$0=(q_A-180,000)\frac{400}{2}+2q_A\frac{200}{1}+(q_A+60,000)\frac{400}{1}$$

$$q_A=12,000\ N/mm$$

$$F_1=200(12,000)=2,400,000\ N$$

$$F_2=400(180,000-12,000)=67,200,000\ N$$

$$Ve=200F_2-400F_1$$

$$e=\frac{200(67,200,000)-400(2,400,000)}{96,000,000}=\underline{130.00\ mm}$$

8.33 By inspection of the axes of symmetry (Fig. a), the shear center coincides with the centroid of the section. Therefore, the shear center is located a distance $e=\frac{1}{3}(0.866025L)=0.28868L$ from the vertical member of the section.

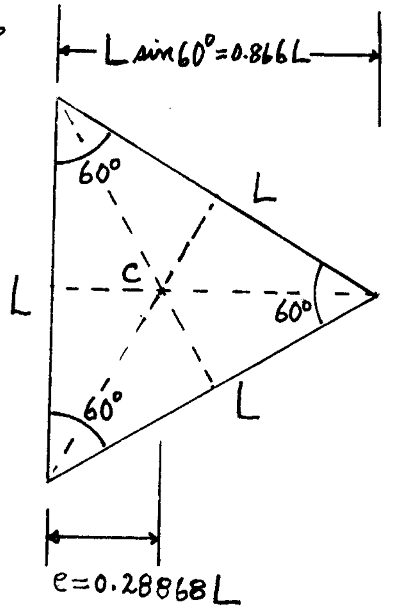


Figure a

8.34 By Fig. a, the moment of inertia about the x axis is

$$I_x = I_{xAC} + I_{xBC} + I_{xAB} \qquad (a)$$

Also, by Fig. a, $\sin\theta = \frac{1}{3}$ and $\theta = 19.47°$. Hence,

$$y = s\sin\theta = \frac{1}{3}s \qquad (b)$$

and

$$I_{xAC} \approx \frac{1}{12}L^3t$$

$$I_{xBC} = \int y^2 dA \approx \int_0^{1.5L} \left(\frac{s}{3}\right)^2\left(\frac{3t}{4}\right)ds = \frac{3}{32}L^3t \qquad (c)$$

$$I_{xAB} = I_{xBC} \approx \frac{3}{32}L^3t$$

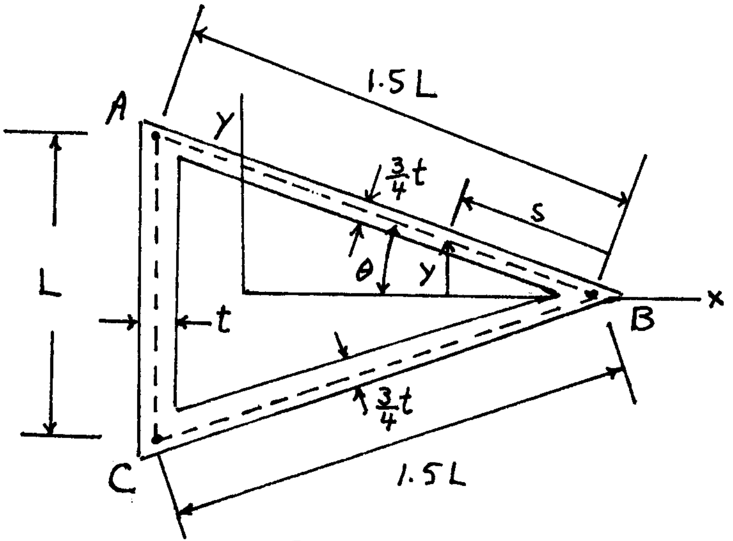


Figure a

Hence, by Eqs. (a) and (c),

$$I_x \approx \frac{13}{48}L^3t \qquad (d)$$

Take cuts at A and C (Fig. b). Then, the shear flows in sections AC and ABC are shown in Fig. b, where with Eqs. (c), we take

$$V_1 = I_{xAC} = \frac{1}{12}L^3t$$

$$V_2 = I_{xBC} + I_{xAB} = \frac{3}{16}L^3t \qquad (e)$$

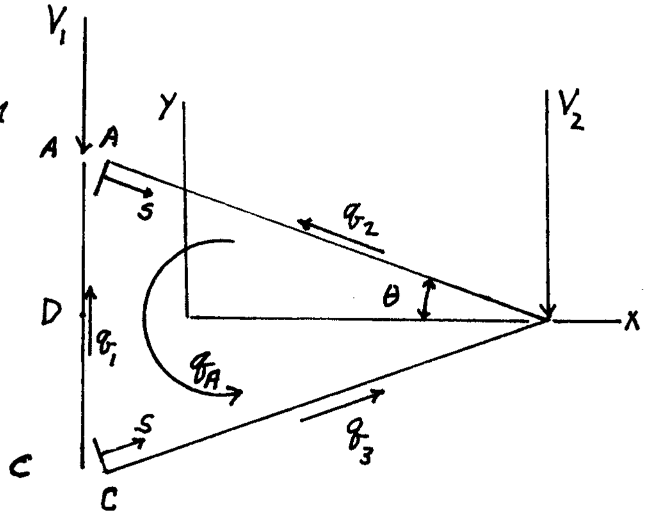


Figure b

Then, by Figs. a and b, with $\sin\theta = \frac{1}{3}$,

$$q_D = \frac{V_1}{I_{xAC}}A_1'\bar{y}_1' = \left(\frac{1}{2}Lt\right)\left(\frac{L}{4}\right) = \frac{1}{8}L^2t$$

(f)

$$q_2 = q_3 = \left(\frac{3}{4}t\right)(s)\bar{y}_2 = \left(\frac{3}{4}st\right)\left[\left(\frac{3L}{2} - \frac{s}{2}\right)\sin\theta\right] = \frac{1}{8}(3Ls - s^2)t$$

(cont.)

8.34 cont. and

$$q_B = q_2\Big|_{s=1.5L} = q_3\Big|_{s=1.5L} = \frac{9}{32}L^2 t \qquad (g)$$

adding q_A (Fig. b), we obtain the shear flow diagram shown in Fig. d. Then by Eq. (8.11) and Fig. d, we obtain

$$\oint \frac{q}{t}\,d\ell = \left(q_A - \frac{2}{3}q_D\right)\frac{L}{t} + \left(q_A + \frac{2}{3}q_B\right)(2)\left(\frac{3L}{2}\right)\left(\frac{4}{3t}\right) = 0 \qquad (h)$$

or by Eqs. (f), (g), and (h),

$$q_A = \frac{2}{15}(q_D - 4q_B) = -\frac{2}{15}L^2 t \qquad (i)$$

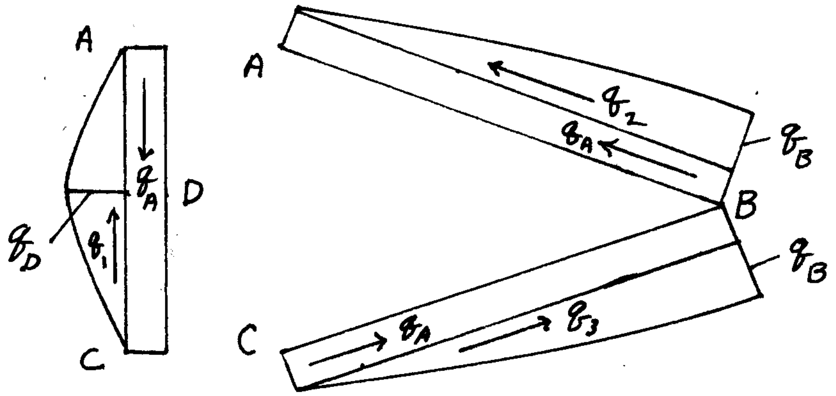


Figure d

Hence, the moment equilibrium equation for moments about point B yields (Fig. e)

$$Ve - F_1(1.5L \sin 19.47°) = 0 \qquad (j)$$

where by Eqs (e) and Fig. (d)

$$V = V_1 + V_2 = \frac{13}{48}L^3 t\,, \quad F_1 = \left(\frac{2}{3}q_D - q_A\right)L \qquad (k)$$

(cont.)

8.34 Cont.

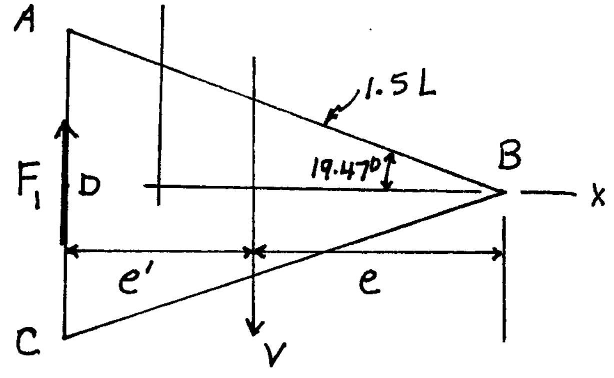


Figure e

By Eqs.(j) and(k), with Eqs.(f) and(i), we obtain

$$\frac{13}{48}L^3te = \frac{13}{180}L^4t$$

or

$$e = \frac{4}{15}L = 0.2667L$$

Therefore, $e' = 1.5L\sin 19.47^\circ - e = 0.2333L$

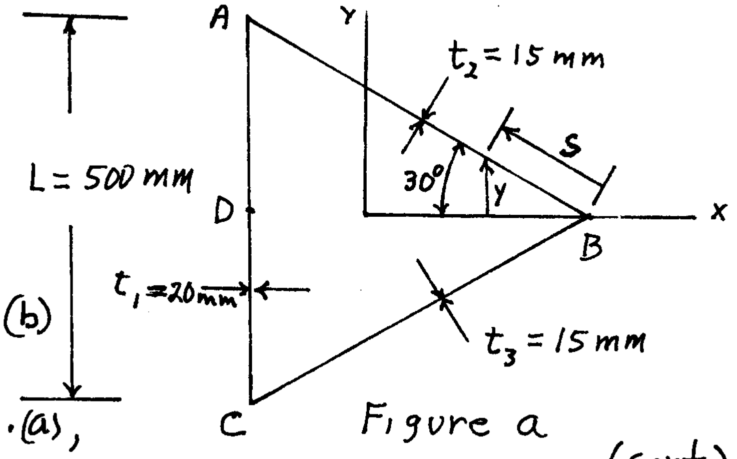
8.35 Let $L_1 = L_2 = L_3 = 0.5\,m$ and $t_1 = 20\,mm$, $t_2 = t_3 = 15\,mm$ (Fig. P8.33). The solution for the location of the shear center follows: Consider the centerline of the box beam (Fig a).

By Fig. a,

$$y = s\sin 30^\circ = \frac{1}{2}s \quad (a)$$

The moment of inertia of the cross section about the x axis is

$$I_x = I_{xAC} + I_{xBC} + I_{xAB} \quad (b)$$



where with Fig.a and Eq.(a),

Figure a

(cont.)

8.35 cont.

$$I_{xAC} \approx \frac{1}{12}(20)(500)^3 = 208.33 \times 10^6 \text{ mm}^4$$

$$I_{xBC} = \int y^2 dA \approx \frac{15}{4}\int_0^{500} s^2 ds = 156.25 \times 10^6 \text{ mm}^4 \qquad (c)$$

$$I_{xAB} = I_{xBC} \approx 156.25 \times 10^6 \text{ mm}^4$$

Hence, by Eqs. (b) and (c),

$$I_x \approx 520.83 \times 10^6 \text{ mm}^4 \qquad (d)$$

Take cuts at A and C (Fig. b). The shear flows in sections AC and ABC are shown in Fig. b, where we take the magnitudes of the shear loads as

$$V_1 = I_{xAC}, \quad V_2 = I_{xBC} + I_{xAB} \qquad (e)$$

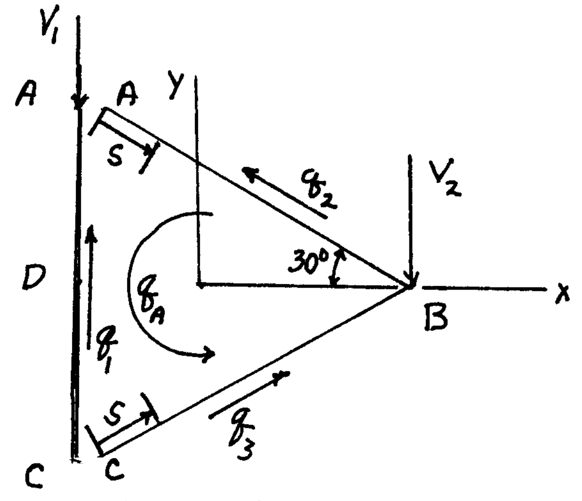


Figure b

also by Eqs. (e) and Figs. a and b,

$$q_D = \frac{V_1}{I_{xAC}} A_1' \bar{y}_1' = \frac{1}{2} L t_1 \left(\frac{L}{4}\right) = \frac{1}{2}(500)(20)\left(\frac{500}{4}\right) = 625 \text{ kN/mm}$$

$$q_2 = A_2' \bar{y}_2 = 15s\left[\left(L - \frac{s}{2}\right)\sin 30^\circ\right] = 3.75s(1000 - s) \quad [\text{kN/mm}] \qquad (f)$$

$$q_3 = q_2 = 3.75s(1000 - s) \quad [\text{kN/mm}]$$

and

(cont.)

8.35 cont.

$$q_B = q_2\big|_{s=500} = q_3\big|_{s=500} = 937.5 \text{ kN/mm} \qquad (g)$$

Adding q_A, we obtain the shear flow diagram shown in Fig. c. Then, by Eq. (8.11) and Fig. c, we obtain

$$\oint \frac{q}{t}\,dl = \left(q_A - \frac{2}{3}q_D\right)\frac{L}{t_1} + \left(q_A + \frac{2}{3}q_B\right)\frac{L}{t_2} + \left(q_A + \frac{2}{3}q_B\right)\frac{L}{t_3} = 0 \qquad (h)$$

So, by Eqs. (f), (g), and (h), we obtain q_A as

$$q_A = -340.91 \text{ kN/mm} \qquad (i)$$

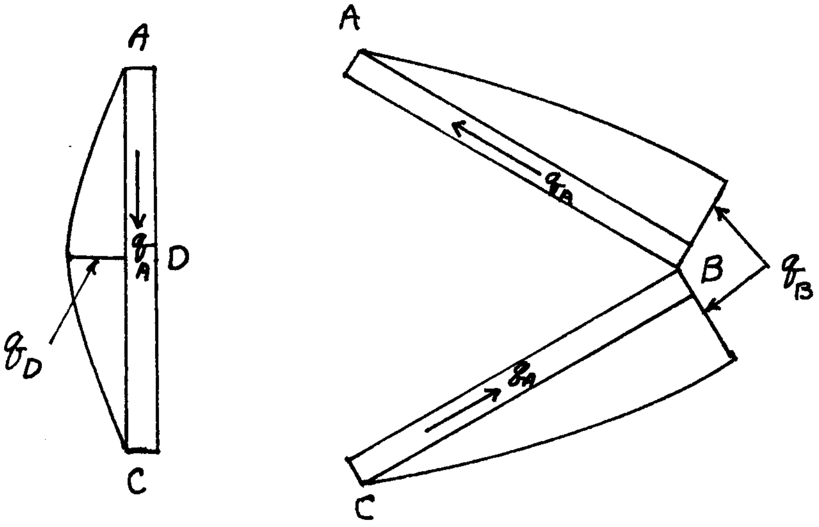


Figure c

Hence, the moment equilibrium equation for moments about point B (Fig. d) yields

$$Ve - F_1 L \sin 30^\circ = 0 \qquad (j)$$

Where by Eqs. (c) and (e), and Figs. c and d,

(cont.)

8.35 cont.

$$V = V_1 + V_2 = 520.83 \times 10^6 \text{ N}$$

$$F_1 = \left(\tfrac{2}{3} q_D - q_A\right)L = \left[\tfrac{2}{3}(625) + 340.91\right](500) = 378.79 \times 10^6 \text{ N} \quad (k)$$

By Eqs. (j) and (k), with Fig. d, $e = 181.8$ mm, or

$$e' = L \sin 30^\circ - e = 68.2 \text{ mm}$$

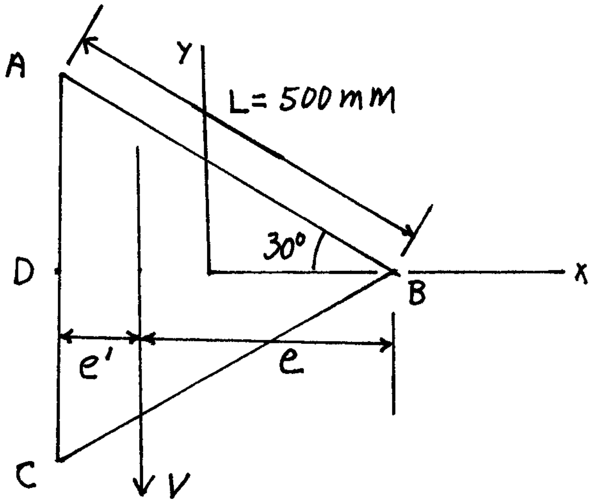


Figure d

8.36 By Fig. a, for the vertical section ABC,

$$I_{x1} = \frac{1}{12}(5)(500)^3 = 5.2083 \times 10^7 \text{ mm}^4$$

For the semicircular section,

$$I_{x2} = \int y^2 dA = \int_0^{\pi} (250 \cos\theta)^2 (5) 250 \, d\theta$$

or

$$I_{x2} = 1.2272 \times 10^8 \text{ mm}^4$$

Therefore for the cross section

$$I_x = I_{x1} + I_{x2} = 1.7480 \times 10^8 \text{ mm}^4 \quad (a)$$

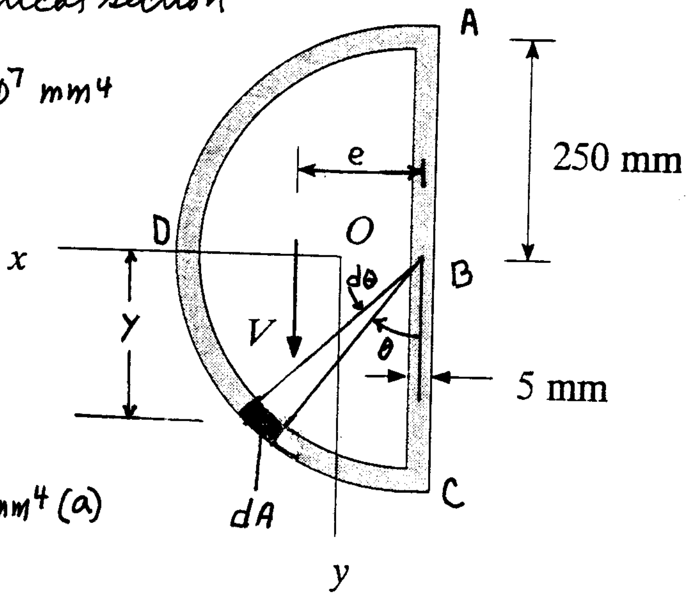


Figure a

(Cont.)

8.36 cont. Make vertical cuts at A and C (Fig. b). Let the shear loads $V_1' = I_{x1}$ and $V_2' = I_{x2}$. For the vertical section, the shear flow at B is, by Fig. b,

$$q_B = \frac{V_1}{I_{x1}} A'\bar{y}' = 5(250)(125) = 156.25\ \text{kN/mm} \qquad (b)$$

The shear flow in ABC is parabolic.

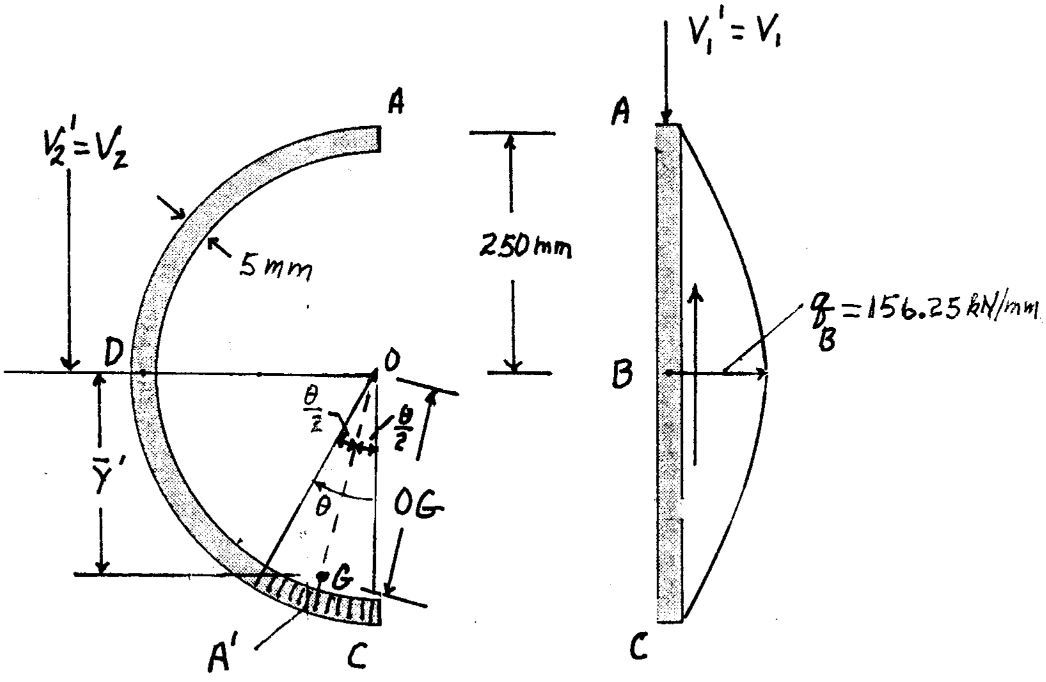


Figure b

For the semicircular section ADC, consider an arc of θ radians, and area $A' = 5(250\theta) = 1250\theta$. The centroid G of the arc is located at $OG = (250 \sin \theta/2)/(\theta/2)$. Hence,

$$\bar{y}' = OG \cos\frac{\theta}{2} = \frac{250\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{\theta} = \frac{250\sin\theta}{\theta}$$

Therefore, in the semicircle,

$$q(\theta) = A'\bar{y}' = 312500\sin\theta = 312.5\sin\theta\ \text{kN/mm} \qquad (C)$$

(cont.)

8.36 cont. For zero rotation (Fig. c),

$$\oint \frac{q(\theta)}{t}\,dl - q_A\left(\frac{250\pi}{5}\right) - \left(q_A + \frac{2}{3}q_B\right)\frac{500}{5} = 0 \qquad \text{(d)}$$

Where

$$\oint \frac{q(\theta)}{t}\,dl = \int_0^{\pi} (312.5 \sin\theta)(250\,d\theta) = 31250 \qquad \text{(e)}$$

Then, by Eqs. (b), (d), and (e),

$q_A = 81.038$ kN/mm (f)

Also, by Fig. c, we have with Eqs. (c) and (f),

$$\circlearrowleft + \sum M_B = Ve - \int [(q(\theta) - q_A)\,r\,d\theta]\,r = 0$$

or with $r = 250$ mm and

$$V = I_x = 1.748 \times 10^8 \text{ N}$$
$$= 1.748 \times 10^5 \text{ kN},$$

$$e = \frac{250^2}{1.748 \times 10^5}\int_0^{\pi} (312.50 \sin\theta - 81.038)\,d\theta$$

or

$$e = 132.4 \text{ mm}.$$

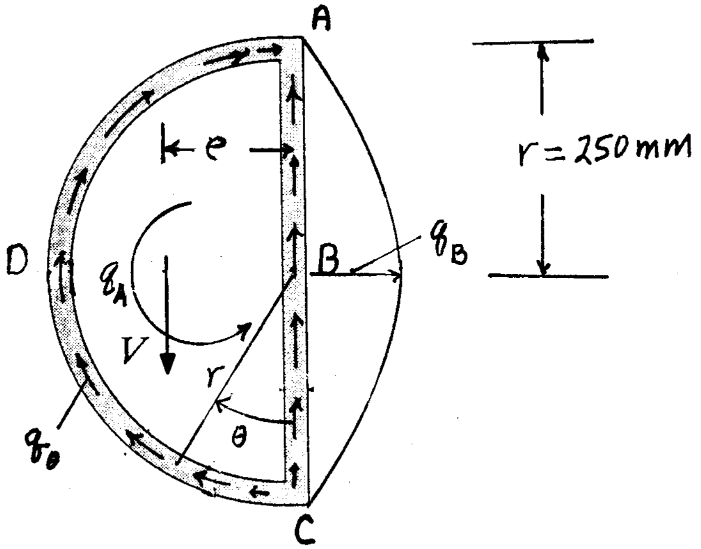


Figure c

Note that as a check, by Fig. C and equilibrium of vertical forces,

$$V = \left(q_A + \frac{2}{3}q_B\right)500 + \int_0^{\pi}\left[(q(\theta) - q_A)\sin\theta\right]250\,d\theta$$

or with Eqs. (b), (c), and (f),

$$V = 92{,}605 \text{ kN} + 82{,}198 \text{ kN} = 1.748 \times 10^8 \text{ N} = |I_x|.$$

Also, since by Eqs. (c) and (f), $q_\theta - q_A = 0$ for $\theta = 15.03°$ and for $\theta = 164.97°$, the shear stress flow

(cont.)

8.36 Cont. distribution is shown in Fig. d.

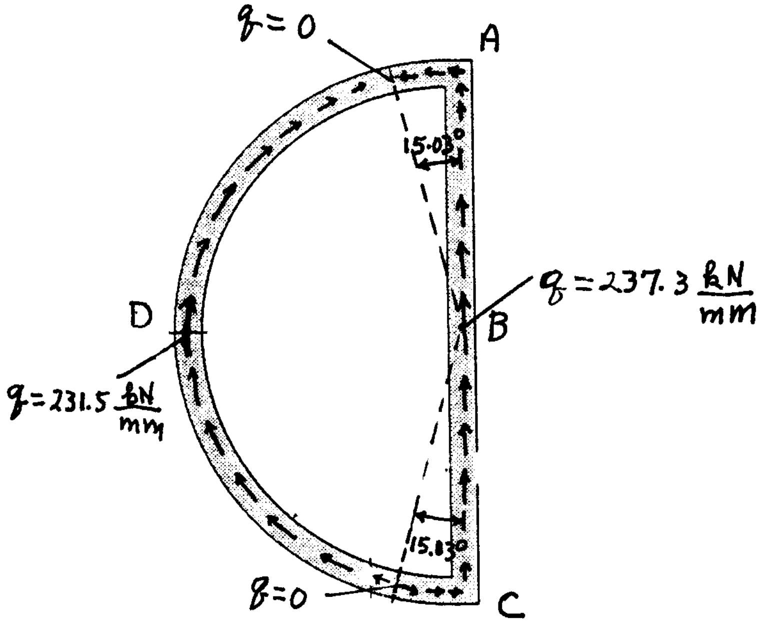


Figure d

8.37 For the rectangular section ABQDE (Fig. a)

$$I_{x_1} = 2\left[\tfrac{1}{12}(300)(5)^3 + 300(5)(250)^2\right] + \tfrac{1}{12}(5)(500)^3$$

or

$$I_{x_1} = 2.3959 \times 10^8\ mm^4 = 2.3959 \times 10^{-4}\ m^4$$

For the semicircular section,

$$I_{x2} = \int y^2 dA = 2\int_0^{\pi/2} (250\cos\theta)^2 (5)(250\,d\theta)$$

or

$$I_{x2} = 1.2272 \times 10^8\ mm^4 = 1.2272 \times 10^{-4}\ m^4$$

Therefore, for the cross section,

$$I_x = I_{x_1} + I_{x2} = 3.6231 \times 10^8 mm^4 = 3.6231 \times 10^{-4}\ m^4 \qquad (a)$$

(cont.)

8.37 cont. Make vertical cuts at A and E (Fig. b).

Let $V_1' = I_{x1}$ and $V_2' = I_{x2}$. Then the shear flow for the parts of the cut beam is determined as follows (with $V_1' = V_1$ and $V_2' = V_2$:

For the rectangular section, the shear flows at points A, B, Q, D, and E are (Fig. b),

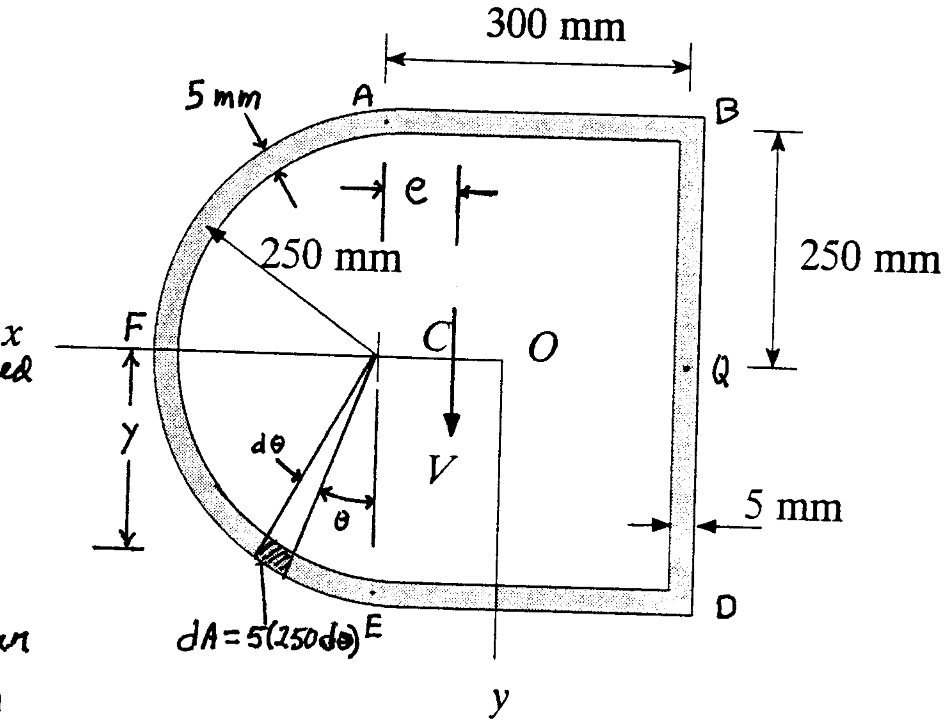


Figure a

$$q_B = q_D = \frac{V_1 A'\bar{Y}_1'}{I_{x1}} = 300(5)(250) = 375 \text{ kN/mm}$$

$$q_Q = q_B + 250(5)(125) = 531.25 \text{ kN/mm} \qquad \text{(b)}$$

$$q_A = q_E = 0$$

For the semicircular section AFE, consider an arc of θ rad and area $A' = 5(250\theta) = 1250\theta$. The centroid G of the arc is located at $OG = \frac{250 \sin(\theta/2)}{(\theta/2)}$. Hence, by Fig. b,

$$\bar{y}' = OG \cos\frac{\theta}{2} = \frac{250\,(2 \sin\frac{\theta}{2} \cos\frac{\theta}{2})}{\theta} = 250(\sin\theta)/\theta$$

Therefore, in the semicircle, the shear flow is

$$q(\theta) = A'\bar{y}' = 312500 \sin\theta = 312.5 \sin\theta \quad [\text{kN/mm}] \quad \text{(c)}$$

(Cont.)

8.37 cont. Then, for zero rotation of the cross section (Fig. c),

$$\oint \frac{q(\theta)}{t}dl - q_A\frac{250\pi}{5} - \left(q_A+\frac{1}{2}q_B\right)\left(\frac{300}{5}\right) - \left[q_A+q_D+\frac{2}{3}(q_Q-q_B)\right]\left(\frac{500}{3}\right) - \left(q_A+\frac{1}{2}q_D\right)\left(\frac{300}{5}\right) = 0 \qquad (d)$$

$$\oint \frac{q(\theta)}{t}dl = \int_0^{\pi}\frac{(312.5\sin\theta)(250d\theta)}{5} = 31250 \text{ kN/mm} \qquad (e)$$

By Eqs. (b), (d), and (e),

$$q_A = -103.87 \text{ kN/mm} \qquad (f)$$

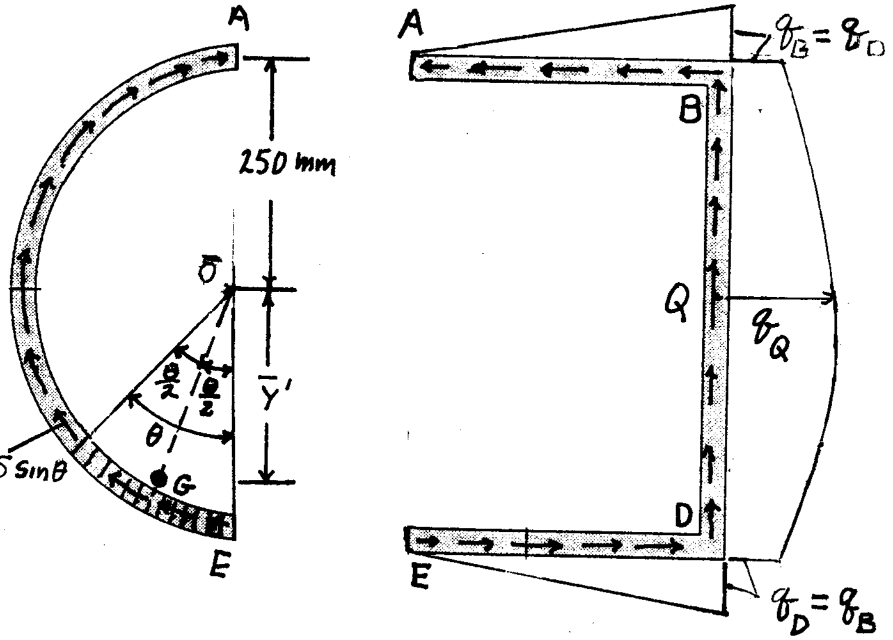


Figure b

Hence, by Fig. C, we have with Eqs. (b), (c), and (f),

$$\circlearrowleft + \sum M_{\bar{O}} = Ve + \oint_0^{\pi}[(q(\theta)-q_A)(250d\theta)(250) - \left(q_A+\frac{1}{2}q_B\right)(300)(250)$$
$$- \left[q_A+q_B+\frac{2}{3}(q_Q-q_B)\right](500)(300) - \left(q_A+\frac{1}{2}q_D\right)(300)(250) = 0 \qquad (g)$$

where

$$\oint_0^{\pi}[(q(\theta)-q_A)(250d\theta)](250) = (250)^2\int_0^{\pi}(312.5\sin\theta+103.87)d\theta$$
$$= 5.9457\times10^{10} \text{ N}\cdot\text{mm} \qquad (h)$$

Therefore, $Ve = -5.9457\times10^{10} + 6.2722\times10^{6} + 5.6294\times10^{10}$
$+ 6.2722\times10^{6}$

or

$$e = -8.695 \text{ mm}$$

(cont.)

8.37 cont.

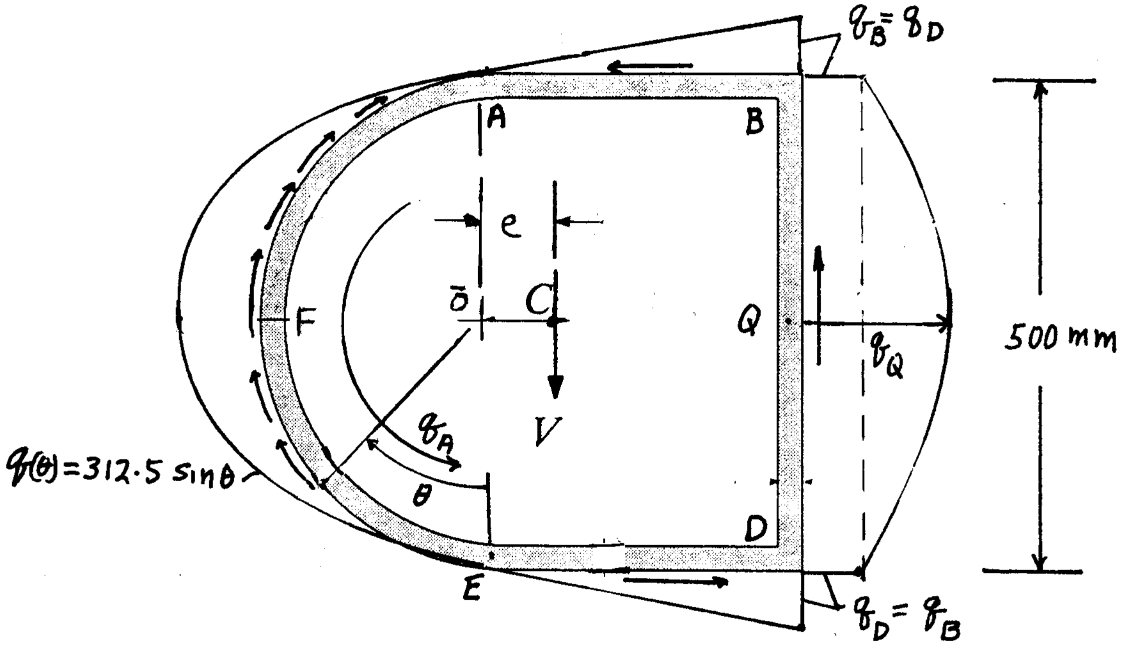


Figure c

Note as a check by Fig. c and Eqs. (b) and (f), equilibrium of vertical forces yields

$$V = \left[q_A + q_D + \tfrac{2}{3}(q_Q - q_D)\right](500)$$

$$- \int_0^{\pi} \left[(q(\theta) - q_A)\sin\theta\right] 250\, d\theta$$

Hence,

$$V = 187648.33 + 174653.46$$

or

$$V = 3.623 \times 10^5\ \text{kN} = 3.623 \times 10^8\ \text{N} = |I_x|$$

8.38 For the sloping section AFE (Fig. a), $y = l\sqrt{2}/2$ and $dl = \frac{2}{\sqrt{2}}dy$. Therefore, $dA = t\,dl = 3\sqrt{2}\,dy$. Hence,

$$I_{x1} = \int y^2 dA = 2\int_0^{80} y^2 (3\sqrt{2}\,dy) = 1.448 \times 10^6 \text{ mm}^4$$

For the rectangular section,

$$I_{x2} = 2\left[\tfrac{1}{12}(80)(3)^3 + (80)(3)(80)^2\right] + \tfrac{1}{12}(3)(160)^3 = 4.096 \times 10^6 \text{ mm}^4$$

So, for the cross section,

$$I_x = I_{x1} + I_{x2} = 5.544 \times 10^6 \text{ mm}^4 \qquad (a)$$

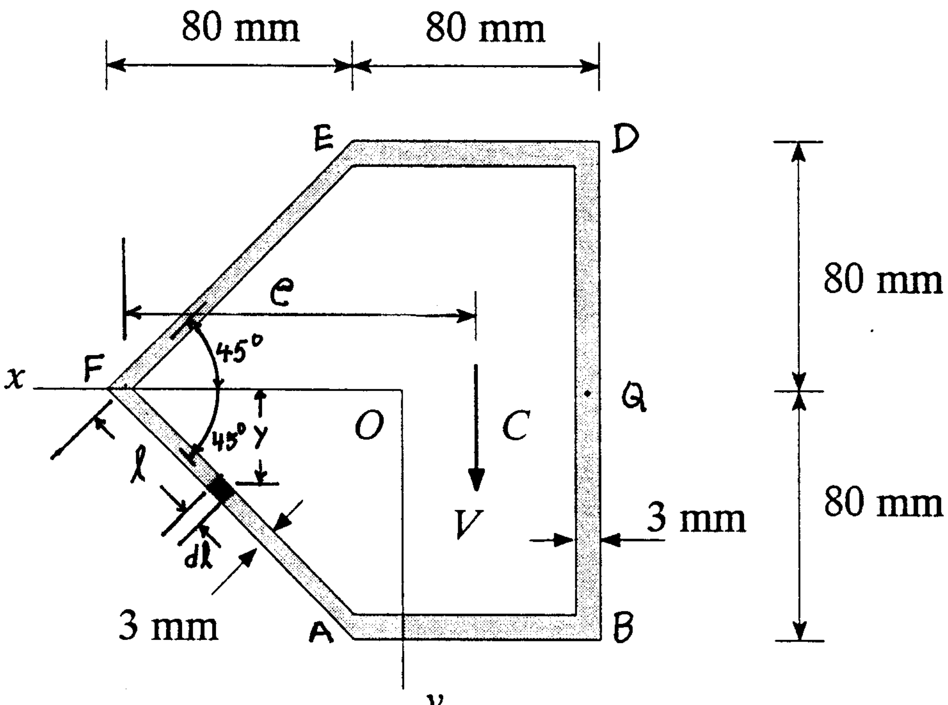


Figure a

Take vertical cuts at A and E and let the shear load $V' = |I_x|$, so that $V_1' = I_{x1}$ and $V_2' = I_{x2}$. For section AFE, the shear flow is (with $V_1' = V_1$).

$$q(l) = \frac{V_1}{I_{x1}} A_1' \bar{y}_1' = (80\sqrt{2} - l)(3)\left(\frac{80\sqrt{2} + l}{2}\right)\frac{\sqrt{2}}{2}$$

or

$$q(l) = 9600\sqrt{2} - \frac{3\sqrt{2}}{4}l^2 \qquad (b)$$

(Cont.)

8.38 cont. For the rectangular section, the shear flows at points A, B, Q, D, and F are (Figs. a and b)

$$q_B = q_D = \frac{V_z}{I_{xz}} A'\bar{y}' = 80(3)(80) = 19{,}200 \text{ N/mm}$$

$$q_Q = q_B + 80(3)(40) = 28{,}800 \text{ N/mm} \qquad \text{(c)}$$

$$q_A = q_E = 0$$

For zero rotation of the cross section (Figs. a and c), starting at A and traversing counterclockwise, we have

$$\int \frac{q}{t} dl = \left(q_A + \frac{1}{2} q_B\right)\frac{80}{3} + \left[q_A + q_B + \frac{2}{3}(q_Q - q_B)\right]\frac{160}{3} + \left(q_A + \frac{1}{2} q_D\right)\frac{80}{3} + 2\int_0^{80\sqrt{2}} [q_A - q(l)]\frac{dl}{3} = 0 \qquad \text{(d)}$$

where, by Eq.(b),

$$2\int_0^{80\sqrt{2}} [q_A - q(l)]\frac{dl}{3} = 75.4247\, q_A - 682667.8 \qquad \text{(e)}$$

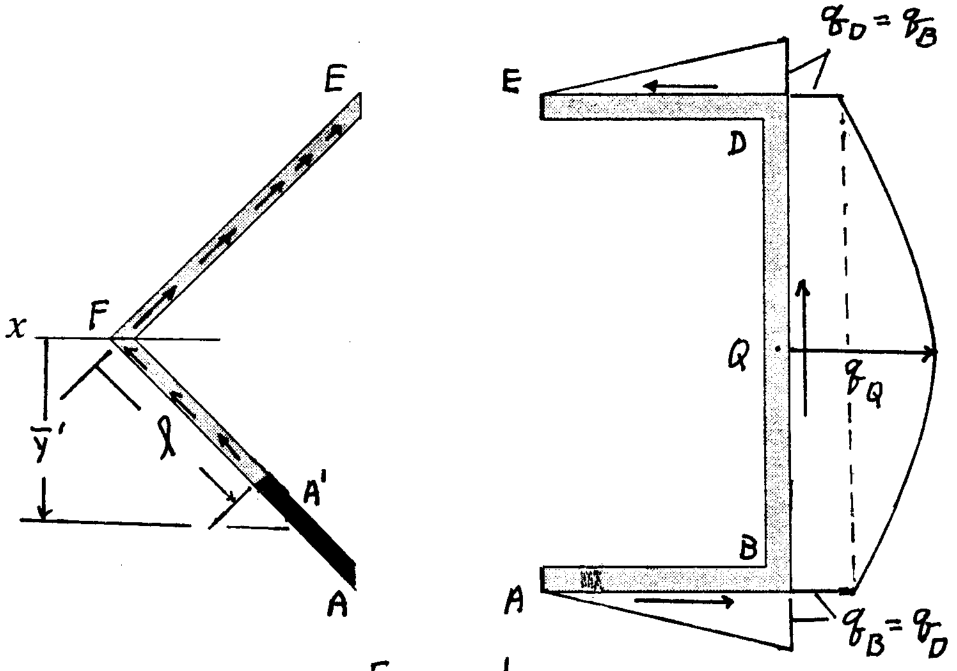


Figure b

(cont.)

8.38 Cont. By Eqs. (c), (d), and (e),

$$q_A = -6560.82 \text{ N/mm} \qquad (f)$$

Hence, by Fig. C, we have, with Eqs. (c) and (f),

$$\circlearrowleft + \sum M_F = Ve - 2(q_A + \tfrac{1}{2} q_B)(80)(80) - [q_A + q_B + \tfrac{2}{3}(q_Q - q_B)](160)(160) = 0 \quad (g)$$

Therefore, with $V = |I_x| = 5.544 \times 10^6$ N, Eq. (g) yields

$$e = \frac{5.263 \times 10^8}{5.544 \times 10^6} = 94.9 \text{ mm}$$

as a check, by Fig. C,

$$\sum F_y = V - 2\int_0^{80\sqrt{2}} [q(l) - q_A]\frac{\sqrt{2}}{2} dl - [q_A + q_B + \tfrac{2}{3}(q_Q - q_B)](160) = 0$$

With Eqs. (b), (c), and (f), we find

$$V = 5.544 \times 10^6 \text{ N} = |I_x|$$

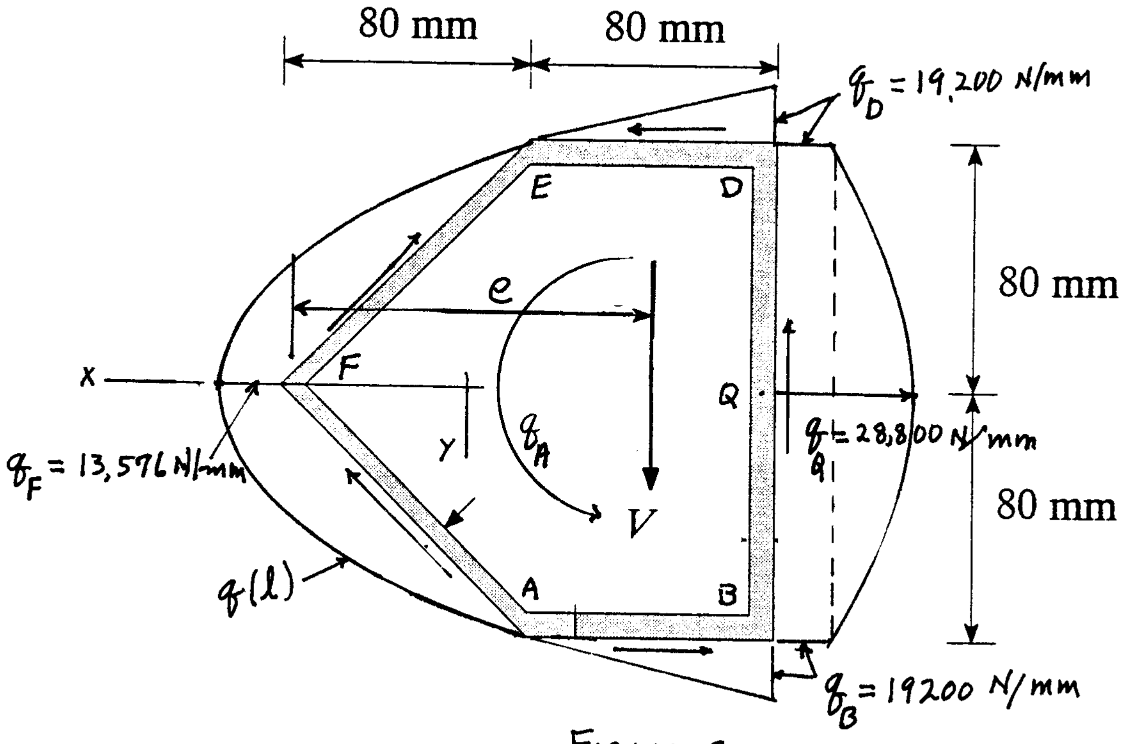


Figure C

8.39 For the rectangular section ABQDE (Fig. a),

$$I_{x_1} = 2\left[\tfrac{1}{12}(50)(4)^3 + 50(4)(75)^2\right] + \tfrac{1}{12}(4)(150)^3 = 3.3755\times10^6 \text{ mm}^4$$

For section EFGHA, with $y = 50 + \frac{\sqrt{2}}{2}\ell$,

$$I_{x_2} = \int y^2 dA = 2\int_0^{25\sqrt{2}}\left(50 + \tfrac{\sqrt{2}}{2}\ell\right)^2(4d\ell) + \tfrac{1}{12}(4)(100)^3 = 1.4529\times10^6 \text{ mm}^4$$

Therefore, for the cross section

$$I_x = I_{x_1} + I_{x_2} = 4.8284\times10^6 \text{ mm}^4 \qquad (a)$$

Take vertical cuts at A and E (Fig. b), and let the magnitude of shear V' be equal to $|I_x|$, so that $V_1' = I_{x_1}$ and $V_2' = I_{x_2}$. Then, the shear flow for the cut sections is determined as follows (with $V_1' = V_1$ and $V_2' = V_2$):

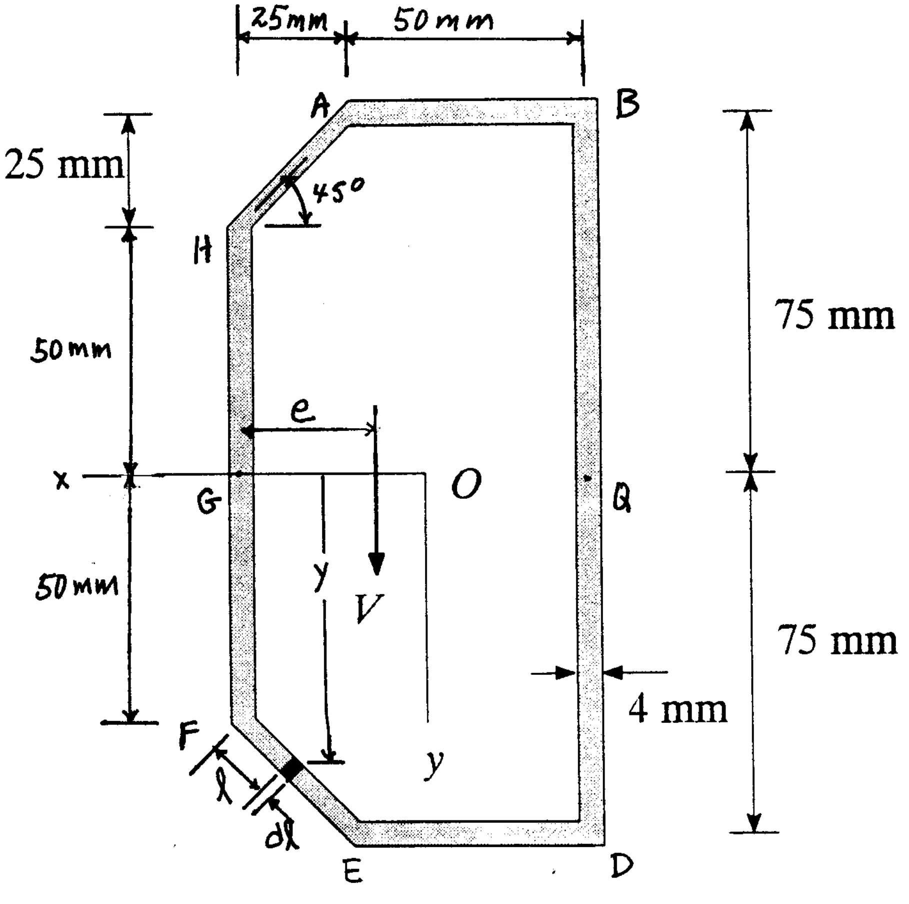


Figure a

(Cont.)

8.39 cont. For the rectangular section, the shear flows at points A, B, Q, D, and E are (Fig. b)

$$q_B = q_D = \frac{V_1}{I_{x1}} A'\bar{y}' = 50(4)(75) = 15{,}000 \text{ N/mm}$$

$$q_Q = q_D + A'\bar{y}' = 15{,}000 + 4(75)(37.5) = 26{,}250 \text{ N/mm} \qquad (b)$$

$$q_A = q_E = 0$$

For part EFGHA, the shear flow in segment EF is

$$q(\ell) = \frac{V_2}{I_{x2}} A'\bar{y}' = (4\ell)(75 - \tfrac{\sqrt{2}}{4}\ell) = 300\ell - \sqrt{2}\ell^2 \qquad (c)$$

By Eq. (c), for $\ell = 25\sqrt{2}$ mm,

$$q_F = q_H = 8838.83 \text{ N/mm}$$

$$(d)$$

$$q_G = q_F + A'\bar{y}' = 8838.83 + 4(50)(25) = 13{,}838.83 \text{ N/mm}$$

The shear flows are shown in Fig. c.

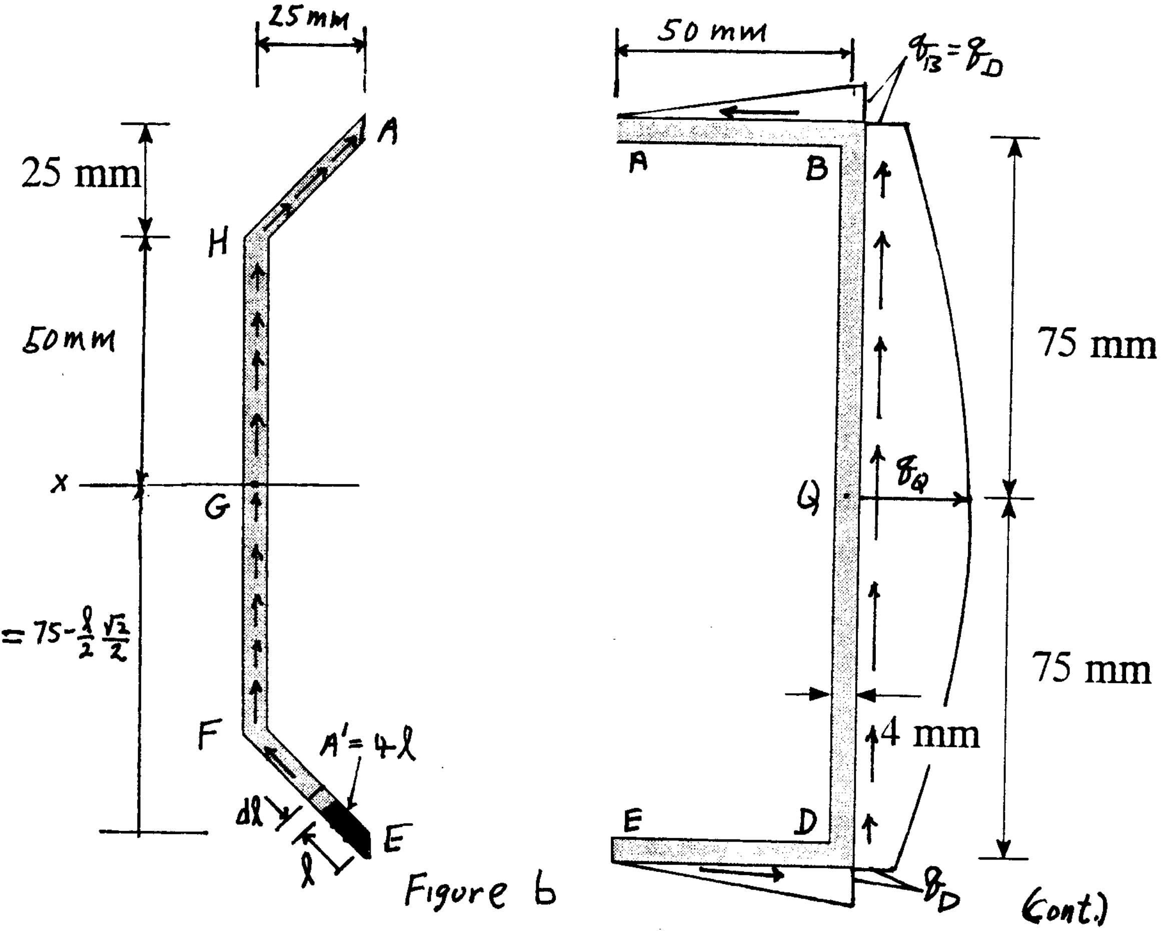


Figure b

(Cont.)

8.39 cont. For zero rotation (Fig. c), starting at E, we get

$$\oint \frac{q}{t}dl = \left(q_A+\frac{1}{2}q_D\right)\frac{50}{4} + \left[q_A+q_D+\frac{2}{3}(q_Q-q_D)\right]\frac{150}{4} + \left(q_A+\frac{1}{2}q_B\right)\frac{50}{4}$$
$$+\int_0^{25\sqrt{2}}[q_A-q(l)]\frac{dl}{4} + \left[q_A-q_H+\frac{2}{3}(q_G-q_H)\right]\frac{100}{4} + \int_0^{25\sqrt{2}}[q_A-q(l)]\frac{dl}{4} = 0 \qquad (e)$$

Where $$\int_0^{25\sqrt{2}}[q_A-q(l)]\frac{dl}{4} = \int_0^{25\sqrt{2}}[q_A-300l+\sqrt{2}l^2]\frac{dl}{4} = 8.8388q-41{,}666.7 \qquad (f)$$

By Eqs. (b), (d), (e) and (f), we find

$$q_A = -6119.15 \text{ N/mm} \qquad (g)$$

Hence, by Fig. c, with Eqs. (b), (d), and (g),

$$\circlearrowleft + \sum M_{\bar{O}} = V(e+50) - \left[q_F-q_A+\frac{2}{3}(q_G-q_F)\right](100)(50-2\left(q_A+\frac{1}{2}q_B\right)(50)(75)$$
$$-\left[q_A+q_D+\frac{2}{3}(q_Q-q_D)\right](150)(125) = 0$$

Therefore, $V(e+50) = 408.954\times10^6$

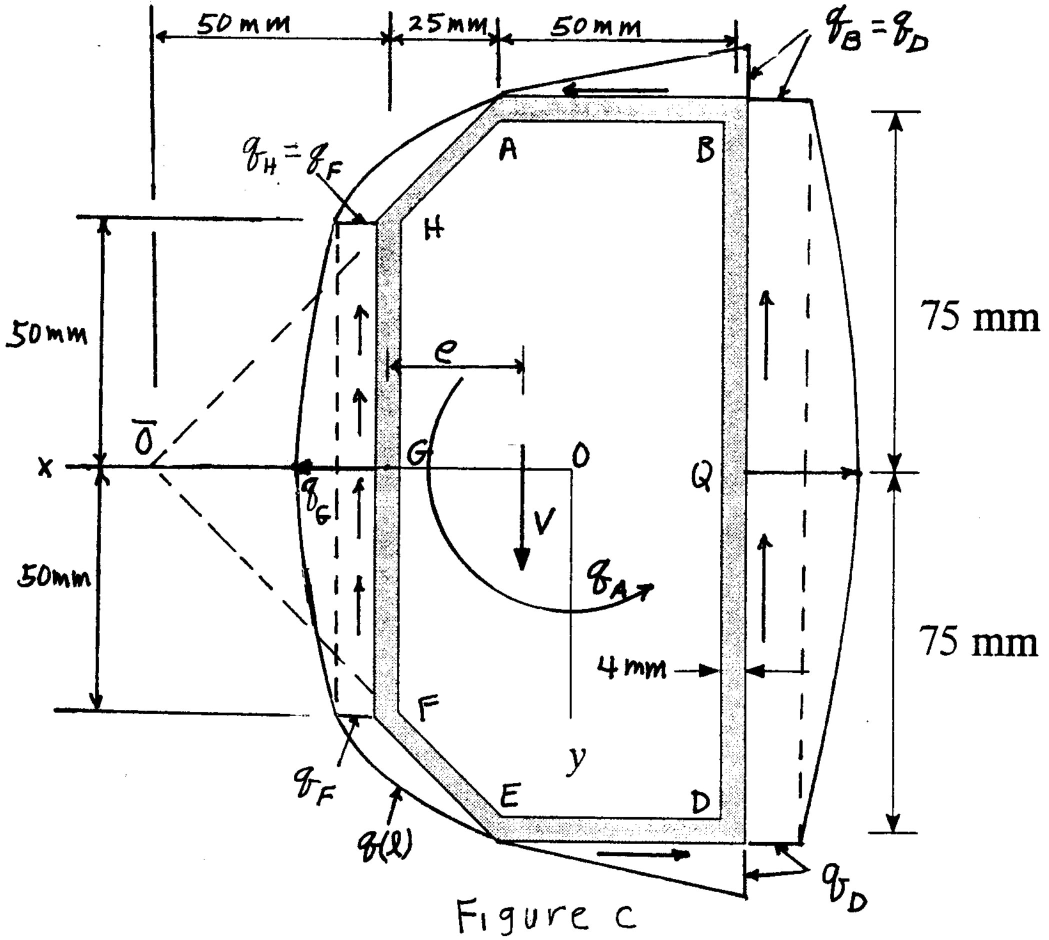


Figure c

(Cont.)

8.39 cont. and with $V = 4.8284 \times 10^6$ N, we obtain

$$e = \frac{408.954 \times 10^6}{4.8284 \times 10^6} - 50 = 34.70 \text{ mm}$$

As a check, by Fig. C,

$$V = 2 \int_0^{25\sqrt{2}} [q(l) - q_A] \frac{\sqrt{2}}{2} dl + [(q_F - q_A) + \tfrac{2}{3}(q_G - q_F)](100)$$
$$+ [q_A + q_D + \tfrac{2}{3}(q_Q - q_D)](150)$$

or

$$V = 4.828 \times 10^6 = |I_x|$$

8.40 The centroid of each square stringer is located 12.5 mm from its base. The distance from the x axis to the centroid is (Fig. a)

$$\bar{y} = 60 + 0.30 + 12.50 = 72.8 \text{ mm}$$

The approximate value of I_x is

$$I_x \approx 2[\tfrac{1}{12}(25)^4 + 25(25)(72.8)^2]$$

or $I_x \approx 6.6899 \text{ mm}^4$ (a)

Take saw cuts at B and D (Fig. b). The shear flow in section DEB is

$$q = A'\bar{y}' = 25(25)(72.8) = 45500 \frac{N}{mm} \quad (b)$$

The shear flow in section BD is zero. Hence, for zero rotation

$$\oint \frac{q}{t} dl = (q_B - q)\frac{60\pi}{0.6} + q_B\left(\frac{121.2}{0.6}\right) = 0$$

or

$$q_B = 27,693.4 \frac{N}{mm} \quad (c)$$

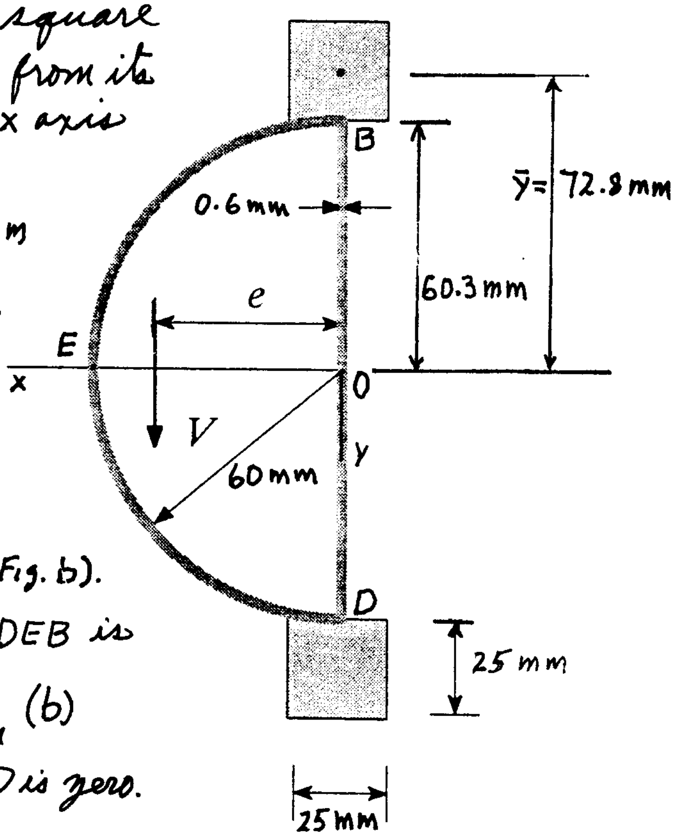


Figure a

(cont.)

8.40 cont. Then, by Fig. c, with Eqs. (b) and (c),

$$\circlearrowleft + \Sigma M_O = Ve - \int_0^{\pi} (q - q_B)(60\,d\theta)60 = 0, \quad \text{or with } V = 6.6899 \times 10^6\ \text{mm}^4 = |I_x|$$

$$e = 201{,}387{,}450 / 6{,}689{,}900 = 30.1\ \text{mm}$$

As a check, by Figs. b and c, with Eqs. (b) and (c),

$$V = \int_0^{\pi} (q - q_B)(\sin\theta)(60\,d\theta) + 2q(12.8)$$

$$+ q_B(121.2)$$

$$= 2{,}136{,}790 + 1{,}164{,}800 +$$

$$+ 3{,}356{,}445$$

or

$$V = 6.66 \times 10^6\ \text{N} \approx |I_x|.$$

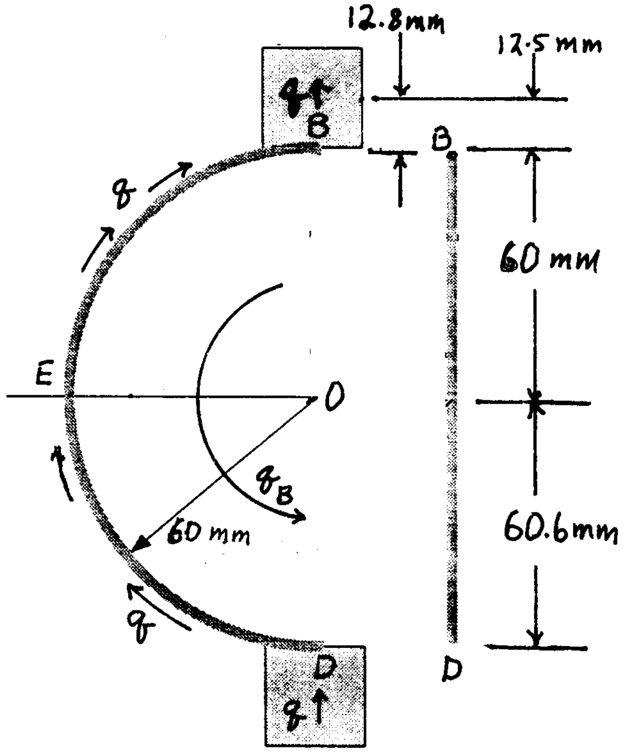


Figure b

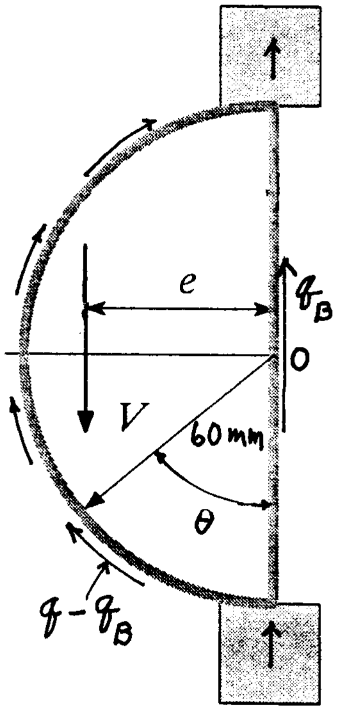


Figure c

8.41 The area of the T-section is 144 mm^2 and its centroid is located 6.44 mm from its base (Fig. a). The area of the angle section is given as 146 mm^2, with centroid 6.40 mm from its sides. Therefore, the area moment of inertia of the T-sections is

$$I_{x_1} \approx 2A_1'(\bar{y}_1')^2 = 2(144)(61.74)^2 = 1,097,806 \text{ mm}^4$$

and for the angle sections

$$I_{x_2} \approx 2A_2'(\bar{y}_2')^2 = 2(146)(48.30)^2 = 681,204 \text{ mm}^4$$

So, for the cross section

$$I_x = I_{x_1} + I_{x_2} = 1,779,000 \text{ mm}^4 \qquad (a)$$

Take saw cuts at A and B (Fig. b). Then,

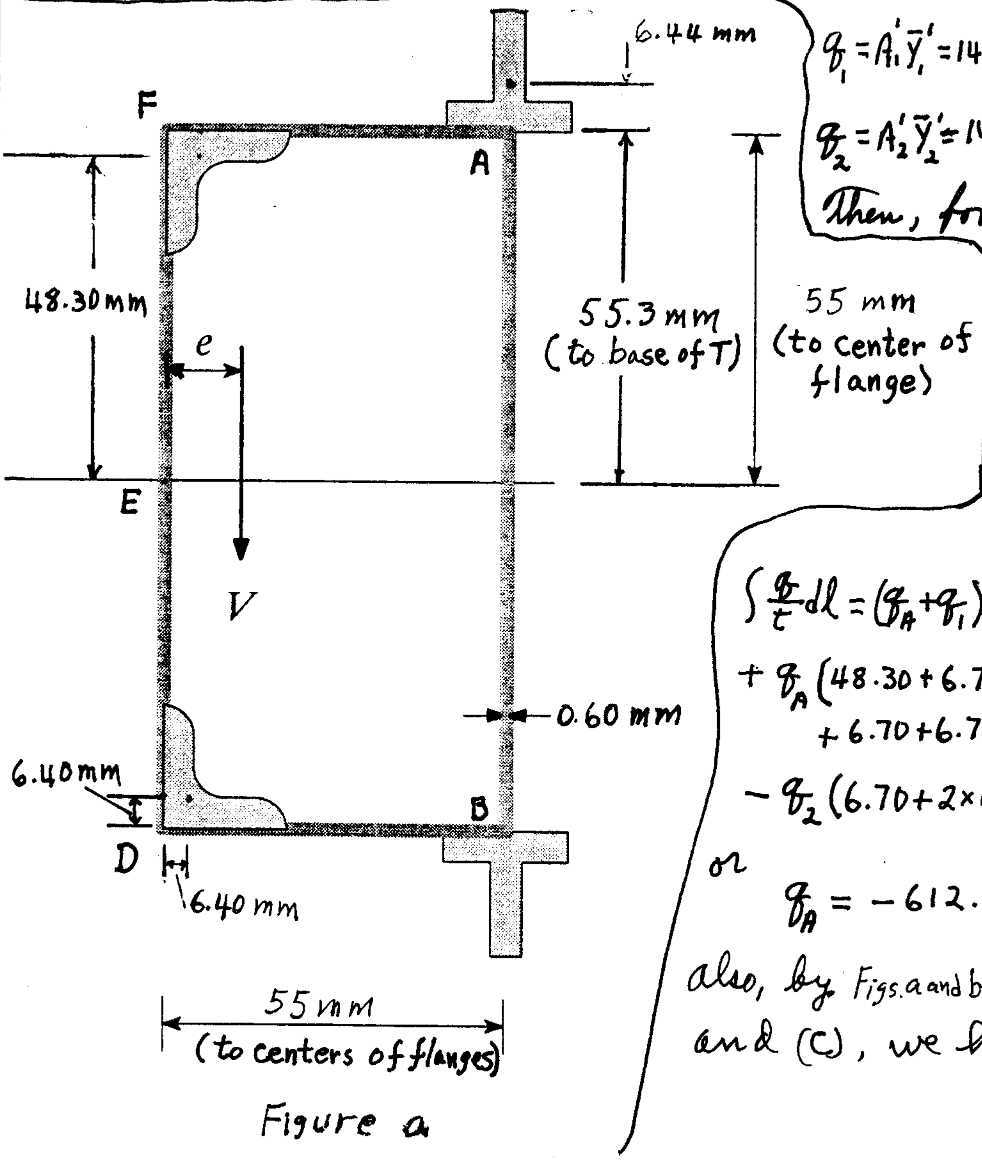


Figure a

$$q_1 = A_1'\bar{y}_1' = 144(61.74) = 8890.56 \text{ N/mm}$$

$$q_2 = A_2'\bar{y}_2' = 146(48.30) = 7051.80 \text{ N/mm} \qquad (b)$$

Then, for zero rotation of the cross section, starting at B and traversing counterclockwise, we have, with Eqs. (b) and Figs. a and b,

$$\int \frac{q}{t}dl = (q_A + q_1)\left(\frac{2\times 55}{0.60}\right) + q_A(48.30 + 6.70 + 6.70 + 2\times 48.30 + 6.70 + 6.70 + 48.30)/0.60 - q_2(6.70 + 2\times 48.30 + 6.70)/0.60 = 0$$

or

$$q_A = -612.92 \text{ N/mm} \qquad (c)$$

also, by Figs. a and b, and Eqs. (b) and (c), we have

(cont.)

8.41 Cont.

$$\circlearrowleft + \sum M_E = Ve - q_A(110\times 55 + 48.3\times 55 - 6.70\times 6.70 + 6.70\times 48.30 + 6.70\times 48.30 - 6.70\times 6.70 + 48.5\times 55) - q_1(2\times 61.74\times 55) + q_2(6.70\times 48.3 + 6.70\times 48.30) = 0$$

or

$$e = \frac{48{,}509{,}000}{1{,}779{,}000} = 27.27 \text{ mm}$$

Then, as a check, by Fig. b, we have

$$V = q_A(-6.7 - 2\times 48.3 - 6.70 + 110) + q_2(2\times 48.30) + q_1(2\times 61.74)$$

or

$$V = 1{,}779{,}000 \text{ N} = |I_x|$$

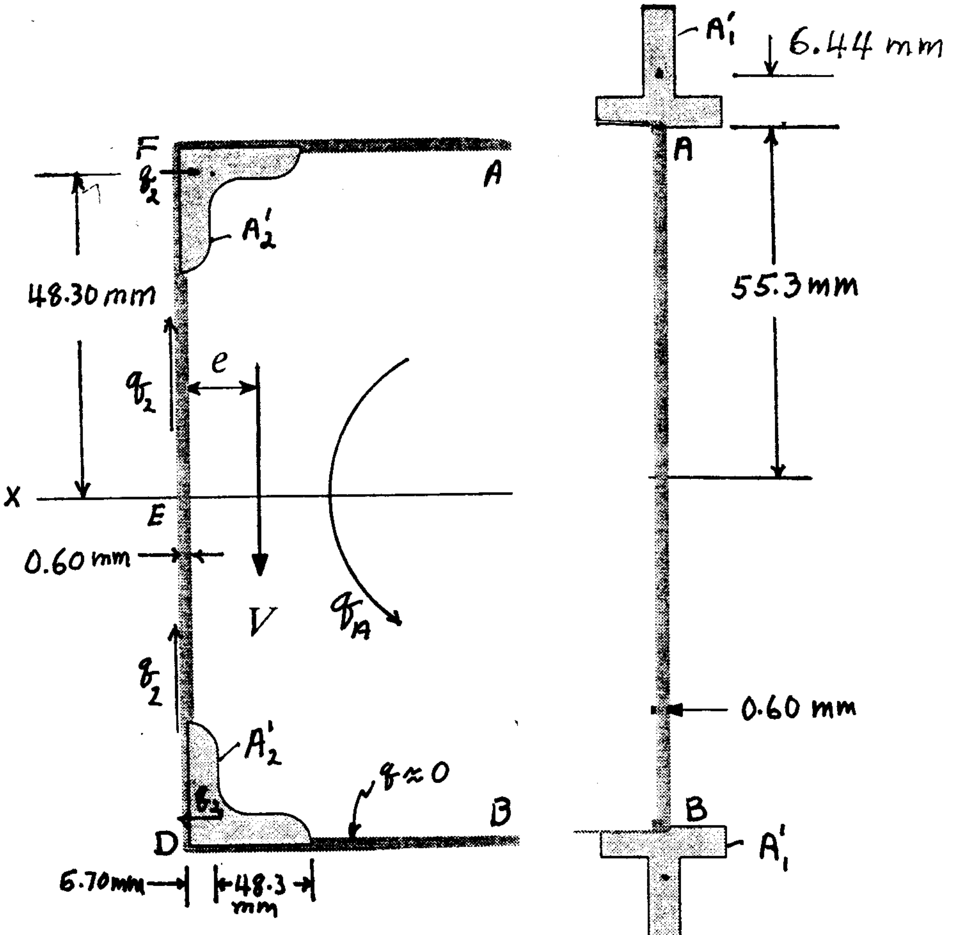


Figure b

8.42 The centroid of each T-section is located 9.67 mm from its base. The distance from the x axis to the centroid of is T-section is

$$\bar{y}_1 = 100 + 10 + 1 + 9.67 = 120.67\text{ mm}$$

So, the approximate area moment of inertia of a T-section is

$$I_{x1} \approx 2A_1\bar{Y}_1^{\,2} = 2(324)(120.67)^2 = 9,435,690\ \text{mm}^4$$

and for the square stringer

$$I_{x2} \approx 2A_2\bar{Y}_2^{\,2} = 2(400)(100)^2 = 8,000,000\ \text{mm}^4$$

So, the moment of inertia of the cross section is

$$I_x = I_{x1} + I_{x2} = 1.7436\times10^7\ \text{mm}^4 \qquad (a)$$

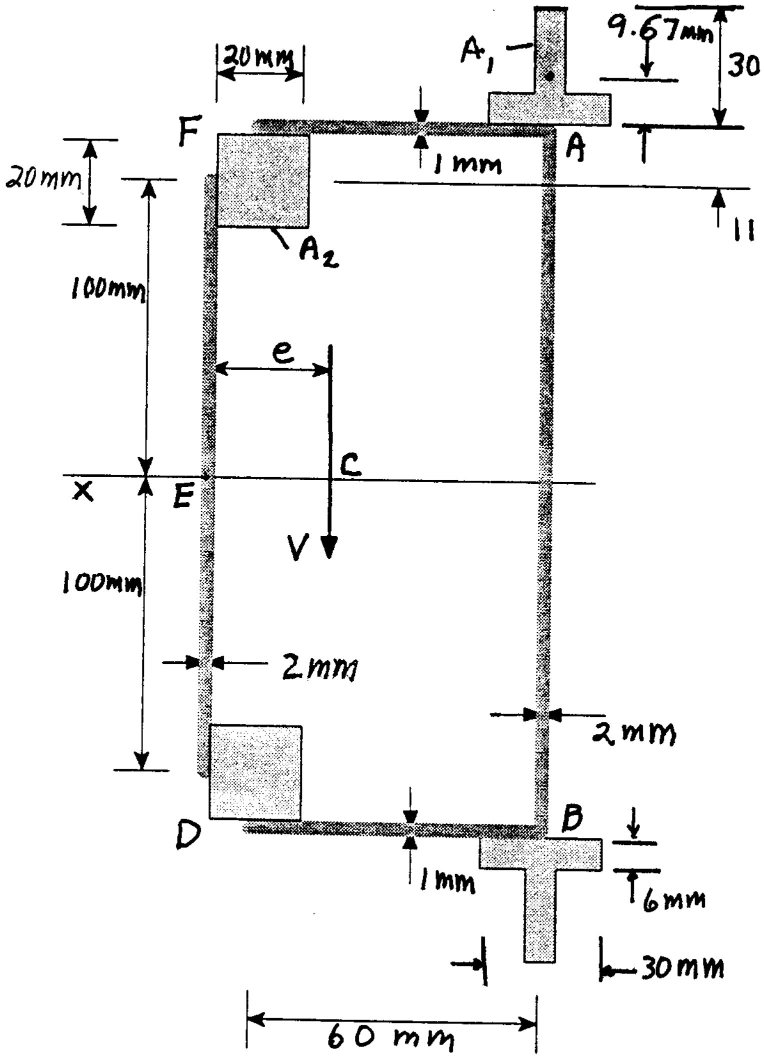


Figure a

Take saw cuts at A and B (Fig. b). Hence, with $V = I_x$,

$$q_1 = A_1'\bar{Y}_1' = (324)(120.67)$$

$$q_2 = A_2'\bar{Y}_2' = (400)(100)$$

or

$$q_1 = 39,097\ \text{N/mm} \qquad (b)$$

$$q_2 = 40,000\ \text{N/mm}$$

Then for zero rotation of the cross section, starting at B, we have with Eqs. (b)

(Cont.)

8.42 cont.

$$\oint \frac{q}{t}\,dl = (q_A+q_1)\left(\frac{221}{2}\right) + q_A\left(\frac{60+10.5}{1}\right) + (q_A-q_2)\left(\frac{11+200+11}{2}\right)$$
$$+\, q_A\left(\frac{10.5+60}{1}\right) = 0$$

or

$$q_A = 330.43 \text{ N/mm} \qquad \text{(c)}$$

Then, by Fig. b and Eqs. (b) and (c), we have

$$\circlearrowright + \Sigma M_E = Ve - q_A(221\times 71 + 60\times 110.5 - 10.5\times 11 + 11\times 100 + 11\times 100$$
$$- 10.5\times 11 + 60\times 110.5) + q_2(11\times 100 + 11\times 100)$$
$$- q_1(241.34\times 71) = 0$$

or

$$e = \frac{5.9215\times 10^8}{V} = \frac{5.9215\times 10^8}{1.7436\times 10^7} = 33.96 \text{ mm}$$

As a check, by Fig. b, we have

$$V = q_1(241.34) + q_A(221.0) - q_A(10.5\times 2) + (q_2 - q_A)(200) = 1.7436\times 10^7 \text{ N}$$

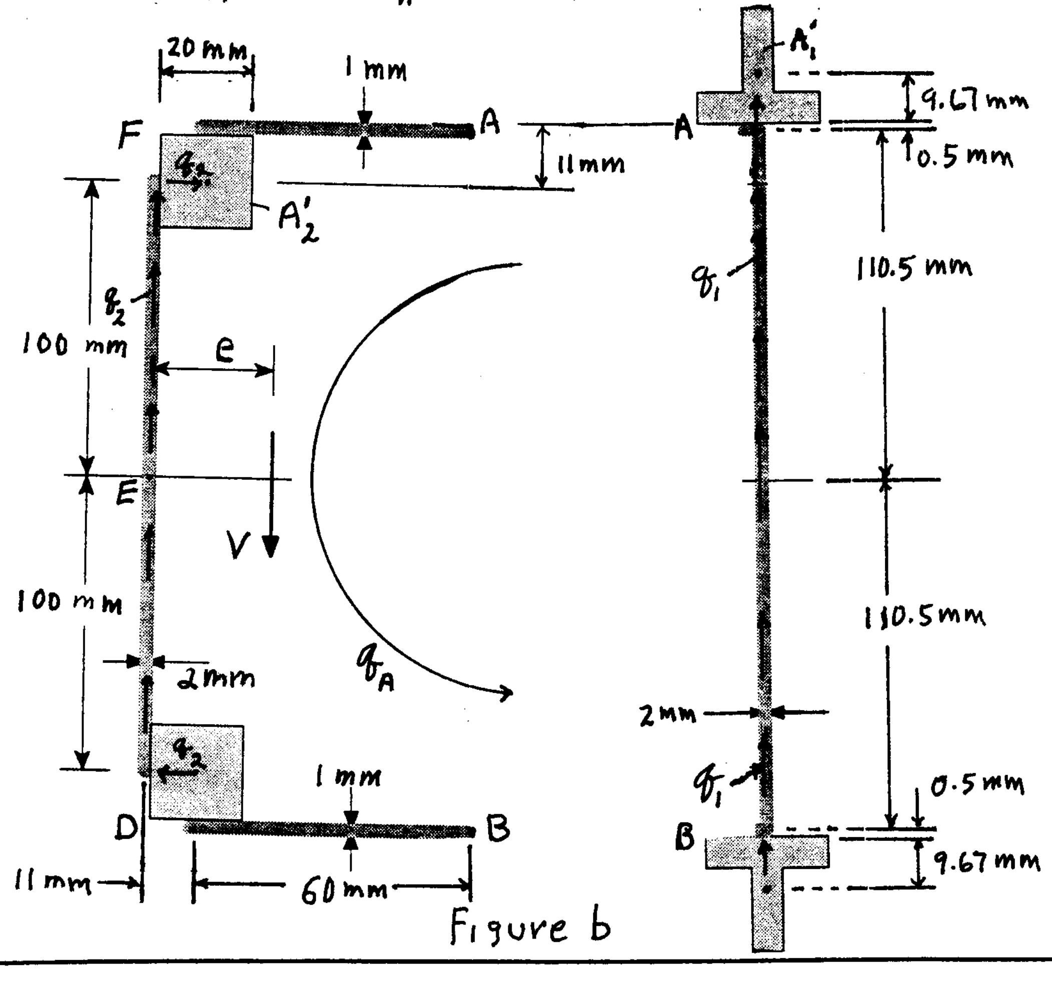


Figure b

8.43 By Fig. a, for the section ABQDE,

$$I_{x1} = 2[\tfrac{1}{12}(300)(4)^3 + 300(4)(250)^2] + \tfrac{1}{12}(4)(500)^3 = 1.9167\times10^8\ \mathrm{mm}^4$$

For section AHE,

$$I_{x2} = \tfrac{1}{12}(4)(500)^3 = 4.1667\times10^8\ \mathrm{mm}^4$$

For section EFA,

$$I_{x3} = \int y^2 dA = 2\int_0^{\pi/2}(250\cos\theta)^2(5)(250\,d\theta) = 1.2272\times10^8\ \mathrm{mm}^4$$

Hence, for the cross section

$$I_x = I_{x1} + I_{x2} + I_{x3} = 3.561\times10^8\ \mathrm{mm}^4 \qquad (a)$$

Take saw cuts at A and E, to the left and the right of the section AHE (Fig. b). Let V' for the beam be equal to the magnitude of I_x, so that $V_1' = I_{x1}$, $V_2' = I_{x2}$ and $V_3' = I_{x3}$. Then, the shear flows for the separated sections are determined as follows (with $V_1' = V_1$, $V_2' = V_2$, and $V_3' = V_3$):

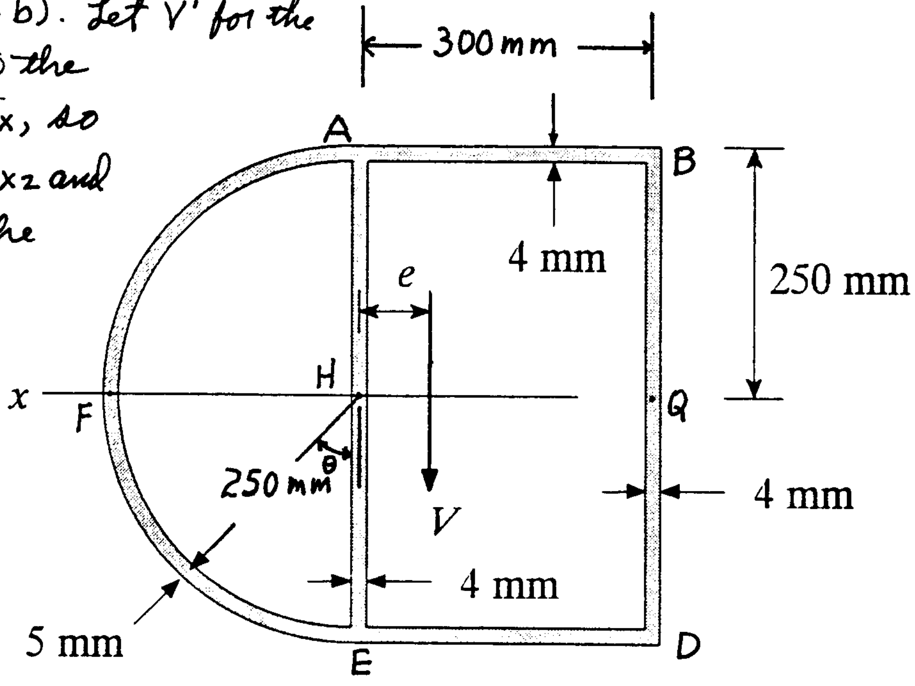


Figure a

For section ABQDE,

$$q_D = q_B = \frac{V_1}{I_{x1}} A_1' Y_1' = (300)(4)(250) = 300\ \frac{\mathrm{kN}}{\mathrm{mm}}$$

$$q_Q = q_D + (250)(4)(125) = 425\ \frac{\mathrm{kN}}{\mathrm{mm}} \qquad (b)$$

For section AHE

$$q_H = A_2'\bar{Y}_2' = 250(4)(125) = 125\ \frac{\mathrm{kN}}{\mathrm{mm}} \qquad (c)$$

(cont.)

8.43 cont. For section AFE, consider the arc θ and area $A_3' = (250\,\theta)(5) = 1250\theta$. The centroid G of the arc is located at $HG = (250 \sin\frac{\theta}{2})/(\theta/2)$. Hence, by Fig. b, $\bar{y}_3' = HG\cos\frac{\theta}{2} = \frac{250\sin\theta}{\theta}$. Therefore, the shear flow in section AFE is

$$q(\theta) = \frac{V_3}{I_{x3}} A_3' \bar{y}_3' = 312.5 \sin\theta \left[\frac{kN}{mm}\right] \qquad (d)$$

Then, for zero rotation of the cross section (Figs. a and b)

$$\oint \frac{q}{t}dl = \int \frac{q(\theta)}{5}dl - q_2\left(\frac{250\pi}{5}\right) - \left(q_2 + \frac{2}{3}q_H\right)\frac{500}{4} + q_1\left(\frac{500}{4}\right) = 0$$

$$(e)$$

$$\oint \frac{q}{t}dl = \left(q_1 + \frac{1}{2}q_D\right)\frac{300}{4} + [q_1 + q_D + \frac{2}{3}(q_Q - q_D) + \left(q_1 + \frac{1}{2}q_B\right)\frac{300}{4} + \left(q_1 - q_2 - \frac{2}{3}q_H\right)\frac{500}{4} = 0$$

where $\int \frac{q(\theta)}{5}dl = \int_0^{\pi} \frac{(312.5\sin\theta)(250\,d\theta)}{5} = 31250$ kN/mm (f)

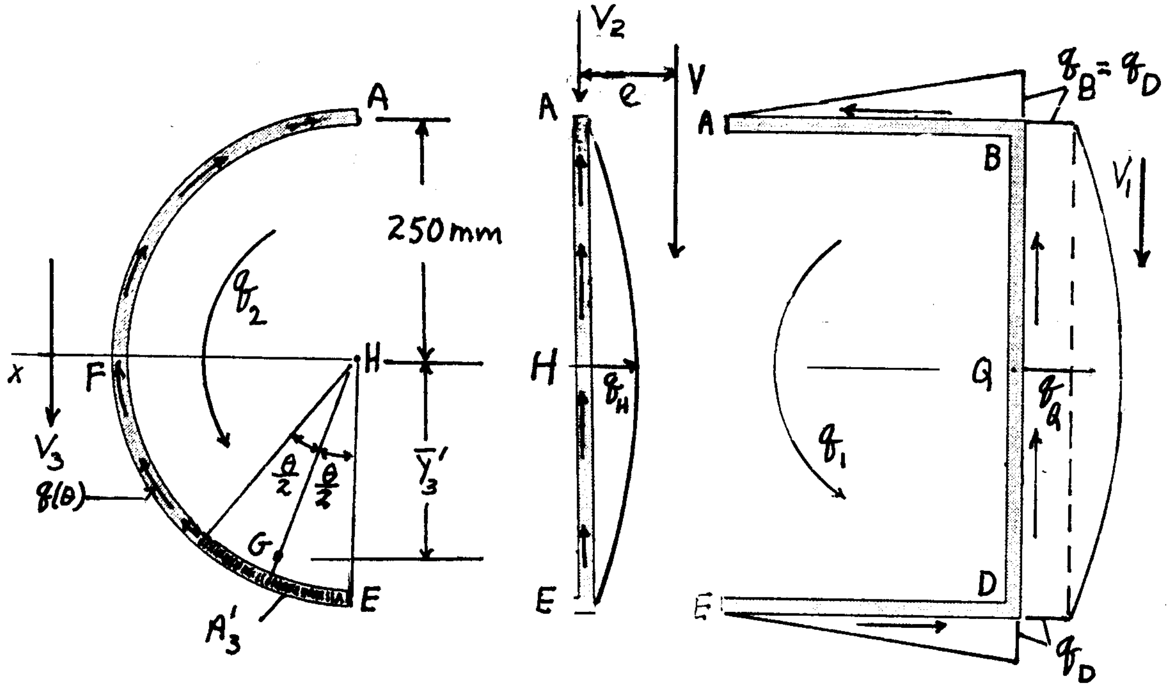


Figure b

The solution of Eqs. (e) is, with Eqs. (b), (c), and (f)

$$q_1 = -147.32 \text{ kN/mm}$$

$$q_2 = 8.572 \text{ kN/mm} \qquad (g)$$

(Cont.)

8.43 cont. Hence, by Figs. a and b, and Eqs. (b), (c), (d), and (g), we find

$$\curvearrowleft + \sum M_H = Ve + \int_0^{\pi} [(q(\theta) - q_2)(250\,d\theta)250 - (q_1 + \tfrac{1}{2} q_D)(300)(250)$$
$$- [q_1 + q_D + \tfrac{2}{3}(q_Q - q_D)](500)(300) - (q_1 + \tfrac{1}{2} q_B)(300)(250) = 0$$

Therefore,

$$Ve = 3.2585 \times 10^9 \text{ N}\cdot\text{mm}$$

or with $V = 3.561 \times 10^8$ N $(= |I_x|)$,

$$e = 9.15 \text{ mm}$$

As a check, by Figs. a and b, and Eqs. (b), (c), (d), and (g),

$$V = \int_0^{\pi} [(q(\theta) - q_2)\sin\theta](250\,d\theta) + (q_2 + \tfrac{2}{3} q_H)500 - q_1(500)$$
$$+ [q_1 + q_D + \tfrac{2}{3}(q_Q - q_D)]500 = 3.561 \times 10^8 \text{N} (= |I_x|).$$

8.44 For section ABQDE (Fig. a)

$$I_{x1} = 2[\tfrac{1}{12}(80)(3)^3 + 80(3)(80)^2] + \tfrac{1}{12}(3)(160)^3 = 4.096 \times 10^6 \text{ mm}^4$$

For section AHE, $I_{x2} = \frac{1}{12}(4)(160)^3 = 1.365 \times 10^6$ mm⁴.

For section AFE, $I_{x3} = \int y^2 dA$, where $y = \frac{\sqrt{2}}{2} l$, $dl = \sqrt{2}\,dy$, $dA = 3\,dl = 3\sqrt{2}\,dy$.

Hence,

$$I_{x3} = 2\int_0^{80} y^2 (3\sqrt{2}\,dy)$$
$$= 1.448 \times 10^6 \text{mm}^4$$

So, for the cross section, $I_x = I_{x1} + I_{x2} + I_{x3}$ or

$$I_x = 6.909 \times 10^6 \text{mm}^4 \quad (a)$$

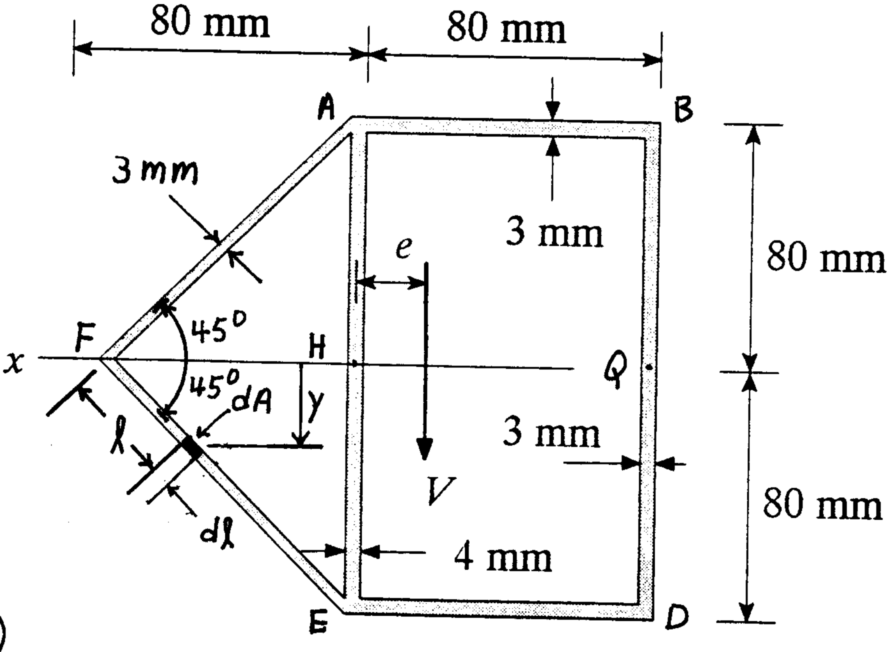


Figure a

(Cont.)

8.44 cont. Make saw cuts to the left and right of section AHE. Let $V' = |I_x|$, so that $V_1' = I_{x1}$, $V_2' = I_{x2}$, and $V_3' = I_{x3}$. Also, let $V_1' = V_1$, $V_2' = V_2$, and $V_3' = V_3$ (Fig. b).

For section ABQDE,

$$q_D = q_B = \frac{V_1}{I_{x1}} A_1' \bar{y}_1' = 80(3)(80) = 19\,200 \frac{\text{N}}{\text{mm}}$$

$$q_Q = q_D + 80(3)(40) = 28{,}800 \frac{\text{N}}{\text{mm}} \qquad \text{(b)}$$

For section AHE,

$$q_H = (80)(4)(40) = 12{,}800 \frac{\text{N}}{\text{mm}} \qquad \text{(c)}$$

For section AFE, by Figs. a and b,

$$q(l) = \frac{V_3}{I_{x3}} A_3' \bar{y}_3' = (80\sqrt{2} - l)(3)\left(\frac{80\sqrt{2} + l}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = 9600\sqrt{2} - \frac{3\sqrt{2}}{4} l^2 \qquad \text{(d)}$$

For zero rotation of the cross section by Figs. a and b, we have for the right-hand bay, starting at E

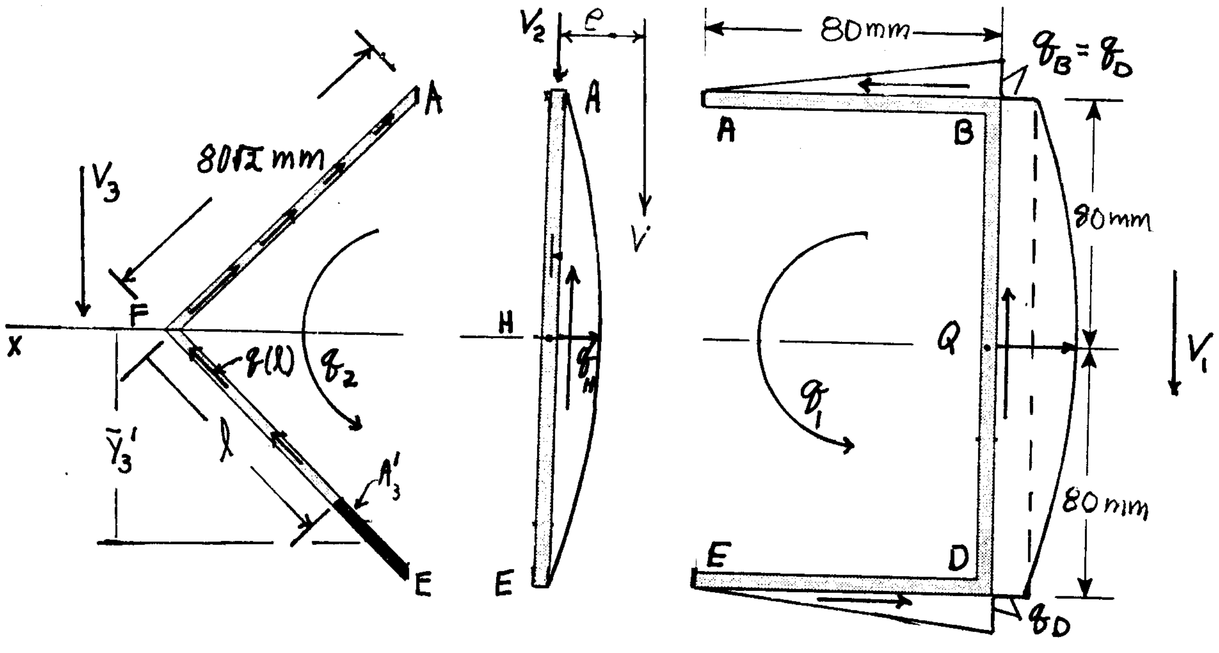


Figure b

(cont)

8.44 cont.

$$\oint \frac{q}{t}dl = \left(q_2 + \tfrac{2}{3}q_H - q_1\right)\frac{160}{4} - \left(q_1 + \tfrac{1}{2}q_B\right)\frac{80}{3} - \left[q_1 + q_B + \tfrac{2}{3}(q_Q - q_B)\right]\frac{160}{3}$$
$$- \left(q_1 + \tfrac{1}{2}q_D\right)\frac{80}{3} = 0 \qquad (e)$$

Similarly, for the left-hand bay, starting at E,

$$\oint \frac{q}{t}dl = 2\int_0^{80\sqrt{2}} [q(l) - q_2]\frac{dl}{3} + \left(q_1 - q_2 - \tfrac{2}{3}q_H\right)\frac{160}{4} = 0 \qquad (f)$$

where, with Eq. (d),

$$2\int_0^{80\sqrt{2}} [q(l) - q_2]\frac{dl}{3} = 682{,}666.7 - 75.4247 q_2 \qquad (g)$$

By Eqs. (b), (c), (e), (f), and (g),

$$q_1 = -2964.6, \quad q_2 = 1929.8 \ \frac{N}{mm} \qquad (h)$$

Then, by Figs. a and b, with Eqs. (b), (c), and (h),

$$\circlearrowright + \sum M_F = V(80+e) - \left(q_2 + \tfrac{2}{3}q_H - q_1\right)(160)(80) - \left(q_1 + \tfrac{1}{2}q_D\right)(80)(80)$$
$$- \left[q_1 + q_D + \tfrac{2}{3}(q_Q - q_D)\right](160)(160) - \left(q_1 + \tfrac{1}{2}q_B\right)(80)(80) = 0$$

Hence,

$$V(80+e) = 8.3627 \times 10^8 \ N\cdot mm$$

or

$$e = \frac{8.3627\times 10^8}{6.609\times 10^6} - 80 = 46.5 \ mm$$

As a check, by Figs. a and b, with Eqs. (b), (c), (d), and (h), we have,

$$V = 2\int_0^{80\sqrt{2}} \left[(q(l) - q_2)\frac{\sqrt{2}}{2}\right]dl + \left(q_2 + \tfrac{2}{3}q_H - q_1\right)160$$
$$+ \left[q_1 + q_D + \tfrac{2}{3}(q_Q - q_D)\right]160$$
$$= 1{,}139{,}390 + 2{,}148{,}440 + 3{,}621{,}660$$

or

$$V = 6.909 \times 10^6 \ N \ (= |I_x|).$$

8.45 By Fig. a, for section BDQEF,

$$I_{x_1} = 2[\tfrac{1}{12}(50)(6)^3 + 50(6)(50)^2 + \tfrac{1}{12}(6)(100)^3 = 2.0018\times10^6 \text{ mm}^4$$

For section ABHFG,

$$I_{x2} = 2[\tfrac{1}{12}(25)(5)^3 + 25(5)(50)^2 + \tfrac{1}{12}(5)(100)^3 = 1.0422\times10^6 \text{ mm}^4$$

Therefore, for the cross section,

$$I_x = I_{x1} + I_{x2} = 3.0440\times10^6 \text{ mm}^4 \qquad (a)$$

Take saw cuts to the right of section BHF (Fig. b) and let $V' = |I_x|$, so that $V_1' = I_{x1}$ and $V_2' = I_{x2}$ (and let $V_1' = V_1$, $V_2' = V_2$).

For section BDQEF, by Fig. b,

$$q_D = q_E = \frac{V_1}{I_{x1}} A_1'\bar{y}_1' = 50(6)(50) = 15000 \frac{\text{N}}{\text{mm}}$$

$$q_Q = q_E + 50(6)(25) = 22{,}500 \frac{\text{N}}{\text{mm}} \qquad (b)$$

Similarly for section ABHFG,

$$q_B = q_F = \frac{V_2}{I_{x2}} A_2'\bar{y}_2' = 25(5)(50) = 6250 \frac{\text{N}}{\text{mm}}$$

$$q_H = q_F + 50(5)(25) = 12500 \frac{\text{N}}{\text{mm}} \qquad (c)$$

For zero rotation of the cross section, starting at point F,

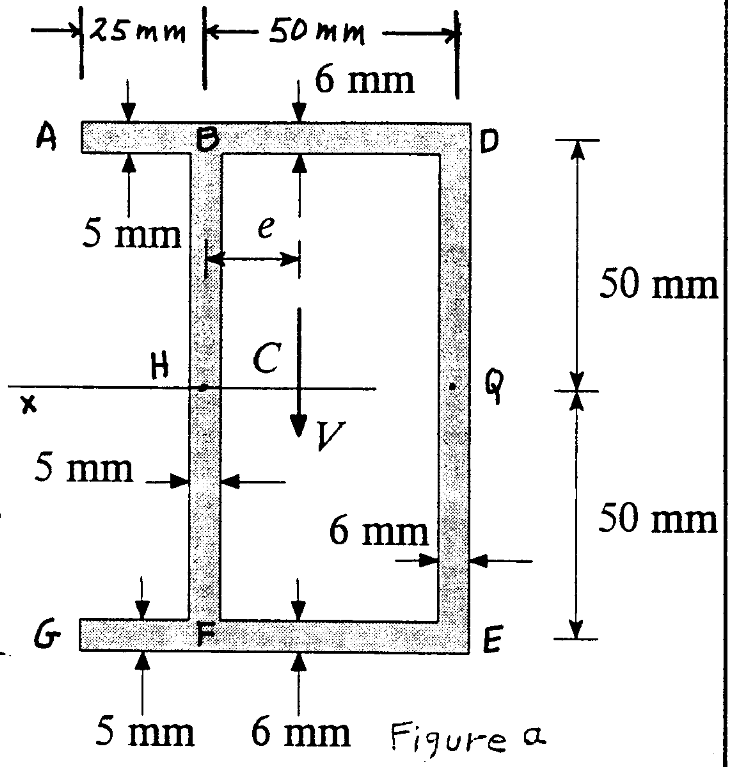


Figure a

$$\oint \frac{q}{t} dl = (q_1 + \tfrac{1}{2}q_E)(\tfrac{50}{6}) + [q_1 + q_E + \tfrac{2}{3}(q_Q - q_E)](\tfrac{100}{6})$$
$$+ (q_1 + \tfrac{1}{2}q_D)(\tfrac{50}{6}) + [q_1 - q_F - \tfrac{2}{3}(q_H - q_F)](\tfrac{100}{5}) = 0$$

or

$$q_1 = -4687.5 \text{ N/mm} \qquad (d)$$

Then, by Figs. a and b, with Eqs. (b), (c), and (d), we have

(cont.)

8.45 cont.

$$\circlearrowright \sum M_B = Ve - \tfrac{1}{2} q_F(25)(100) - (q_1 + \tfrac{1}{2} q_E)(50)(100)$$
$$- [q_1 + q_E + \tfrac{2}{3}(q_Q - q_E)](100)(50) = 0$$

or

$$Ve = 9.8438 \times 10^7 \text{ N}\cdot\text{mm}$$

Therefore,

$$e = \frac{9.8438\times 10^7}{3.044\times 10^6} = 32.34 \text{ mm}$$

as a check, by Fig. b, with Eqs. (b), (c), and (d),

$$V = [q_1 + q_E + \tfrac{2}{3}(q_Q - q_E)]100 + [q_F + \tfrac{2}{3}(q_H - q_F) - q_1]100$$

or $$V = 3.042 \times 10^6 (\approx |I_x|) \text{ N}$$

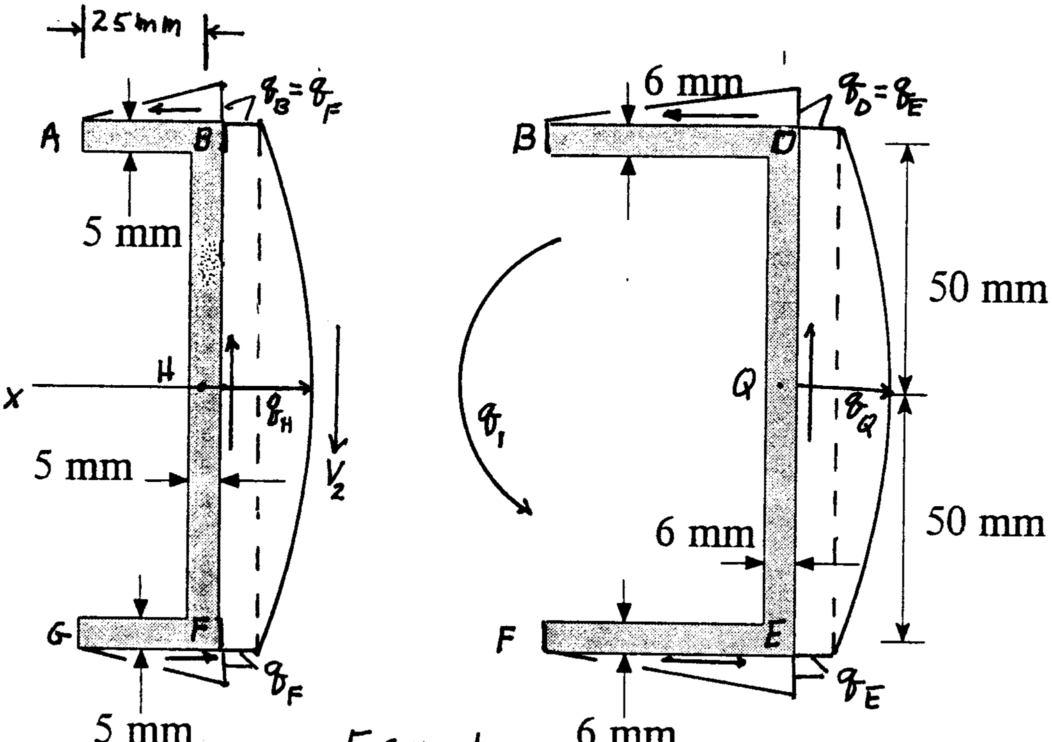


Figure b

9.1 The cross section of the curved beam is shown in Fig. a. For a maximum allowable circumferential stress $\sigma_{\theta\theta} = 250\,\text{MPa}$, we wish to determine the allowable moment. For a given moment, $\sigma_{\theta\theta}$ is, by Eq. (9.11),

$$\sigma_{\theta\theta} = \frac{M(A - rA_m)}{Ar(RA_m - A)} \qquad (a)$$

To calculate values of A, R, and A_m for the cross section divide the cross section into two areas A_1 and A_2 (Fig. a).

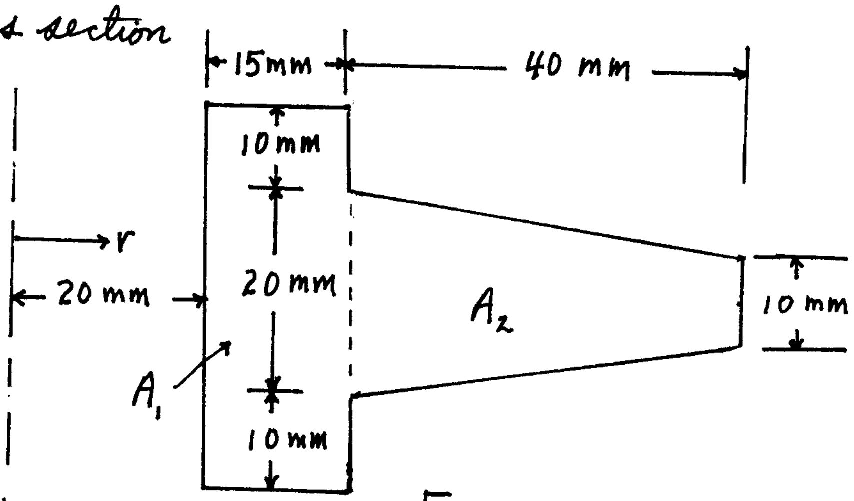


Figure a

For area A_1 (see Table 9.2),

$$\begin{aligned} A_1 &= 600\ \text{mm}^2 \\ a_1 &= 20\ \text{mm} \\ b_1 &= 40\ \text{mm} \qquad (b) \\ c_1 &= 35\ \text{mm} \\ R_1 &= \frac{a_1 + c_1}{2} = 27.5\ \text{mm} \\ A_{m_1} &= b_1 \ln \frac{c_1}{a_1} = 22.38\ \text{mm} \end{aligned}$$

Similarly, for area A_2 (Fig. c of Table 9.2)

$$\begin{aligned} A_2 &= 600\ \text{mm}^2 \\ R_2 &= 52.78\ \text{mm} \qquad (c) \\ A_{m_2} &= 11.91\ \text{mm} \end{aligned}$$

Then, by Eqs. (9.12) - (9.14) and Eqs. (b) and (c),

(cont.)

9.1 cont.

$$A = A_1 + A_2 = 1200 \text{ mm}^2$$

$$R = \frac{R_1 A_1 + R_2 A_2}{A_1 + A_2} = 40.14 \text{ mm} \qquad \text{(d)}$$

$$A_m = A_{m_1} + A_{m_2} = 34.29 \text{ mm}$$

By Eqs. (a) and (d), we find

$$\sigma_{\theta\theta} = M\left(\frac{0.005669}{r} - 0.0001620\right) \qquad \text{(e)}$$

M in N·mm

For positive moment M, the maximum tensile stress $\sigma_{\theta\theta}$ occurs at $r = 20$ mm (Fig. a). Hence, Eq. (e) yields

$$M = 2.058 \times 10^6 \text{ N·mm} = 2.058 \text{ kN·m} \qquad \text{(f)}$$

For negative moment M, the maximum tensile stress $\sigma_{\theta\theta}$ occurs at $r = 75$ mm. Then, Eq. (e) yields

$$|M| = 2.893 \times 10^6 \text{ N·mm} = 2.893 \text{ kN·m} \qquad \text{(g)}$$

Hence, the maximum allowable moment is M = 2.058 kN·m.

9.2 (a) The free-body diagram of the section A-A of the hook is shown in Fig. a. By Fig. a,

$$\Sigma F_y = N - P = 0$$

$$\circlearrowleft\!+ \Sigma M_{\bar{O}} = M - PR = 0$$

or

$$N = P$$
$$M = PR \qquad \text{(a)}$$
$$R = r_0 + \frac{d}{2}$$

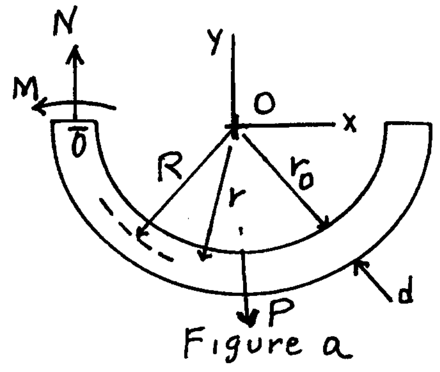


Figure a

(Cont.)

9.2 cont. By Eq. (9.11),

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M(A - rA_m)}{Ar(RA_m - A)} \qquad \text{(b)}$$

where $A = \pi d^2/4$ and by Table 9.2, $A_m = \left[2r_0 + d - \sqrt{(2r_0+d)^2 - d^2}\,\right]$

Hence, by Eqs. (a) and (b)

$$\sigma_{\theta\theta} = \frac{4P}{\pi d^2}\left\{1 + \frac{(2r_0+d)\left[d^2 - 4r\left(2r_0+d - \sqrt{(2r_0+d)^2 - d^2}\right)\right]}{2r\left[2(2r_0+d)\left(2r_0+d - \sqrt{(2r_0+d)^2 - d^2}\right) - d^2\right]}\right\} \qquad \text{(c)}$$

The maximum tensile stress occurs at $r = r_0$ (Fig. a).
Therefore, by Eq. (c),

$$(\sigma_{\theta\theta})_{\text{max. tensile}} = \frac{4P}{\pi d^2}\left\{1 + \frac{(2r_0+d)\left[d^2 - 4r_0\left(2r_0+d - \sqrt{(2r_0+d)^2 - d^2}\right)\right]}{2r_0\left[2(2r_0+d)\left(2r_0+d - \sqrt{(2r_0+d)^2 - d^2}\right) - d^2\right]}\right\} \qquad \text{(d)}$$

The maximum compressive stress occurs at $r = r_0 + d$.
Therefore, by Eq. (c).

$$(\sigma_{\theta\theta})_{\text{max. compressive}} = \frac{4P}{\pi d^2}\left\{1 + \frac{(2r_0+d)\left[d^2 - 4(r_0+d)\left(2r_0+d - \sqrt{(2r_0+d)^2 - d^2}\right)\right]}{2(r_0+d)\left[2(2r_0+d)\left(2r_0+d - \sqrt{(2r_0+d)^2 - d^2}\right) - d^2\right]}\right\} \qquad \text{(e)}$$

(b) For a maximum allowable design tensile stress of 375 MPa, with $r_0 = 75$ mm and $d = 50$ mm, Eq. (d) yields

$$375\ \frac{\text{N}}{\text{mm}^2} = 0.010523\, P_{\text{allowable}}$$

or

$$P_{\text{allowable}} = 35.64\ \text{kN}$$

$\boxed{9.3}$ (a) Divide the cross section into two areas A_1 and A_2 (Fig. a). By Fig. a and Table 9.2,

$$A_1 = 50 \times 25 = 1250 \text{ mm}^2; \quad A_2 = \tfrac{1}{2}(50)(50) = 1250 \text{ mm}^2$$

$$R_1 = 1470 + 12.5 = 1482.5 \text{ mm}, \quad R_2 = 1470 + 25 + \tfrac{1}{3}(50) = 1511.7 \text{ mm} \qquad (a)$$

$$A_{m_1} = 50 \ln \frac{1495}{1470} = 0.84319 \text{ mm}, \quad A_{m_2} = \frac{50(1545)}{50} \ln \frac{1545}{1495} = 0.82695 \text{ mm}$$

Hence, by Eqs. (9.12)—(9.14) and (a),

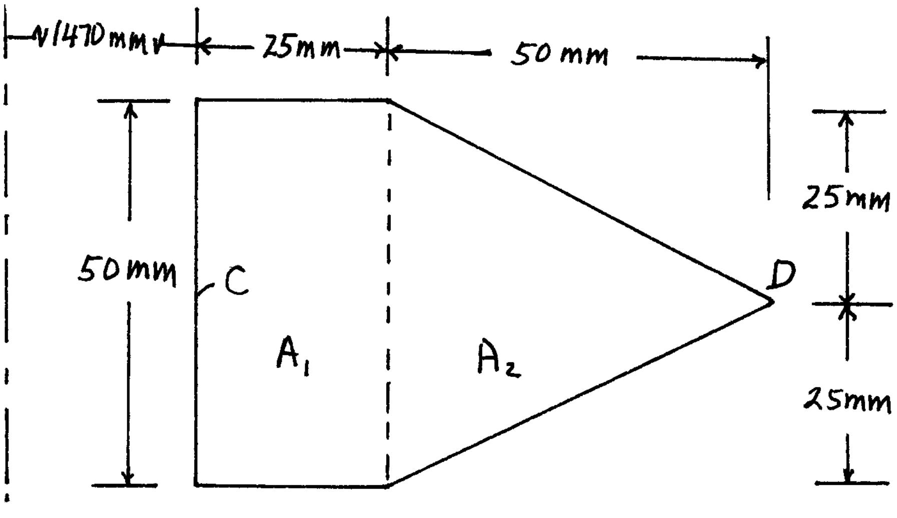


Figure a

$$A = A_1 + A_2 = 2500 \text{ mm}^2 = 0.0025 \text{ m}^2$$

$$A_m = A_{m_1} + A_{m_2} = 1.67014 \text{ mm} = 0.00167014 \text{ m} \qquad (b)$$

$$R = \frac{R_1 A_1 + R_2 A_2}{A} = 1497.1 \text{ mm} = 1.4971 \text{ m}$$

By Fig. b,

$$\Sigma F_y = V + P \sin\theta = 0$$

$$\Sigma F_\theta = N + P \cos\theta = 0$$

$$\circlearrowleft + \Sigma M_0 = PR(1 - \cos\theta) - M = 0$$

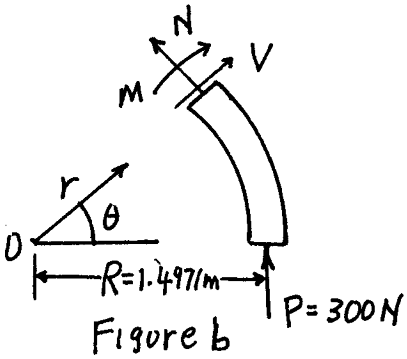


Figure b

or

(Cont.)

9.3 cont.

$$V = -P\sin\theta = -300\sin\theta \quad [N]$$

$$N = -P\cos\theta = -300\cos\theta \quad [N] \qquad (c)$$

$$M = PR(1-\cos\theta) = 449{,}130(1-\cos\theta) \quad [N\cdot mm]$$

By Eqs. (b), (c), and (9.11), we have

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M(A - rA_m)}{Ar(RA_m - A)}$$

or

$$\sigma_{\theta\theta} = -120\cos\theta + \left(\frac{12.2514}{r} - 8.1846\right)(1-\cos\theta)\times 10^5 \quad [kPa] \quad (d)$$

For maximum (or minimum) $\sigma_{\theta\theta}$, $d\sigma_{\theta\theta}/d\theta = 0$, or with Eq. (d),

$$\theta = 0 \quad \text{or} \quad \theta = \pi$$

For $\theta = 0$, $\sigma_{\theta\theta}$ is a minimum and for $\theta = \pi$, $\sigma_{\theta\theta}$ is a maximum [see Eq (d)]. The maximum tensile and compressive stresses at $\theta = \pi$ are, by Eq. (d) and Fig. a;

For $r = 1.470$ m : $\sigma_{\theta\theta} = 120 + 29{,}937 = 30.06$ MPa: Tension at C (e)

For $r = 1.545$ m : $\sigma_{\theta\theta} = 120 - 50{,}978 = -50.86$ MPa: Compression at D (f)

By Eq. (d), with $\theta = \pi/2$ rad, and with Fig. a,

For $r = 1.470$ m, $\sigma_{\theta\theta} = 0 + 14{,}969 = 14.97$ MPa ; Tension at C (g)

For $r = 1.545$m, $\sigma_{\theta\theta} = 0 - 25{,}489 = -25.49$ MPa; compression at D (h)

Using straight beam theory, we have

$$\sigma_{\theta\theta} = -\frac{My}{I} \qquad (i)$$

(cont.)

9.3 cont. By Fig. c and Appendix B,

$$y_0 = \frac{A_1 y_1 + A_2 y_2}{A} = \frac{1250(12.5) + 1250(41.67)}{2500} = 27.08\text{ mm}$$

$$I_{x1} = \frac{1}{12}(50)(25)^3 + 1250(27.08-12.5)^2 = 330{,}825\text{ mm}^4$$

$$I_{x2} = \frac{1}{36}(50)(50)^3 + 1250(41.67-27.08)^2 = 439{,}696\text{ mm}^4$$

or

$$I_x = I_{x1} + I_{x2} = 7.7052\times10^5\text{ mm}^4 = 7.7052\times10^{-7}\text{ m}^4$$

For $\theta = \pi$ (Fig. b), $M = 2PR = 2(300)(1.4971) = 898.26$ N·m

Hence, by Eq. (i),

$$\sigma_{\theta\theta} = -1.166\times10^9 y \quad (j)$$

For $y = -0.02708$ m (point C in Fig. c), Eq. (j) yields $\sigma_{\theta\theta} = 31.57$ MPa, compared to 30.06 MPa by curved beam theory.

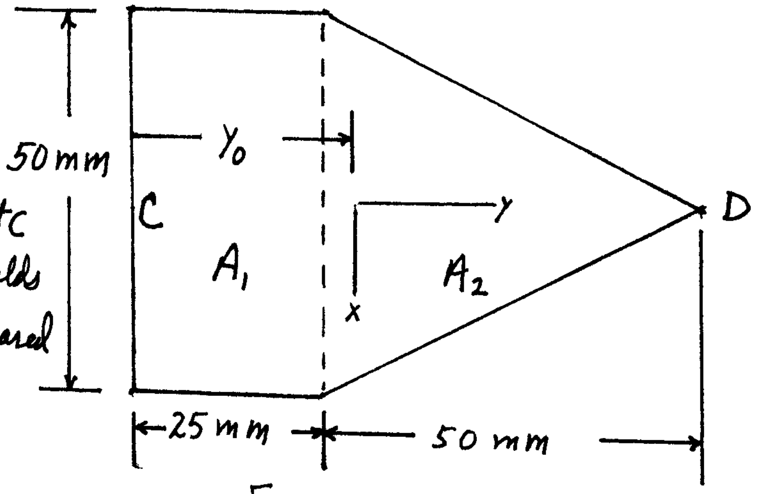


Figure c

For $y = 0.04792$ m (point D in Fig. c), Eq. (j) yields $\sigma_{\theta\theta} = -55.87$ MPa, compared to -50.86 MPa by curved beam theory.

For $\theta = \frac{\pi}{2}$ (Fig. b), $M = PR = 449.13$ N·m. Hence, by Eq. (i),

$$\sigma_{\theta\theta} = -5.829\times10^8 y \qquad (k)$$

For $y = -0.02708$ m (point C in Fig. c), Eq. (k) yields $\sigma_{\theta\theta} = 15.78$ MPa, compared to 14.97 MPa by curved beam theory. For $y = 0.04792$ m (point D in Fig. c), Eq. (k) yields $\sigma_{\theta\theta} = -27.93$ MPa, compared to -25.49 MPa by curved beam theory.

(cont.)

9.3 cont. (b) Compared to the results of Example 9.2, the maximum tensile stress at C (Fig. c) is reduced by approximately $(38.76-30.06)\times100/38.76 \approx 22\%$. However, the maximum compressive stress at D (Fig. c) is increased by $(-50.86+37.46)\times100/(-37.46) \approx 36\%$. This increase in compressive stress at D may result in crushing at D.

(c) In addition to possible crushing at D, the redesigned section has a larger cross-sectional area, 2500 mm² versus 2400 mm², resulting in a heavier member. It does not appear that the redesign has merit.

9.4 (a) Consider the free-body diagram of the hook part ABC (Fig. a). Summing forces in the y direction, we have, where t is the thickness of the hook,

$$\Sigma F_y = P - \int_0^{\pi} [p(r_i\,d\phi)t] \sin\phi = 0$$

or

$$p = \frac{P}{2r_i t} \qquad \text{(a)}$$

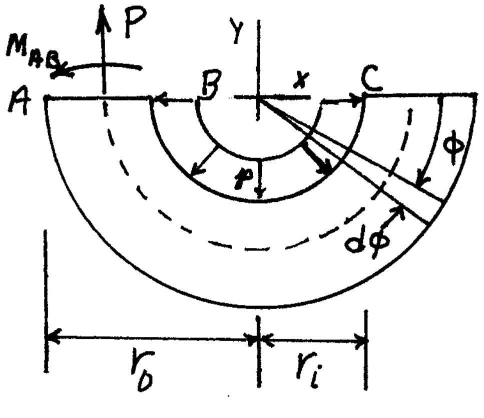


Figure a

Then, by the free-body diagram of an element of the hook (Fig. b), we have

$$\Sigma F_x = -N\sin\theta - V\cos\theta + pr_i t\int_0^{\theta}\cos\phi\,d\phi = 0$$

or

$$N\sin\theta + V\cos\theta = pr_i t\sin\theta \qquad \text{(b)}$$

$$\Sigma F_y = -N\cos\theta + V\sin\theta - pr_i t\int_0^{\theta}\sin\phi\,d\phi$$

or

$$N\cos\theta - V\sin\theta = -pr_i t(1-\cos\theta) \qquad \text{(c)}$$

$$\circlearrowleft + \Sigma M_O = M - NR = 0$$

or

$$M = NR \qquad \text{(d)}$$

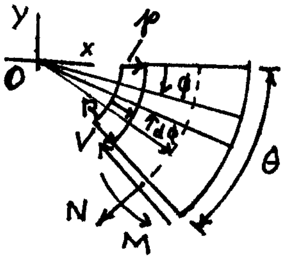


Figure b

(Cont.)

9.4 cont. The solution of Eqs. (b), (c), and (d) is

$$N = p r_i t (1 - \cos\theta) \qquad (e)$$

$$V = p r_i t \sin\theta \qquad (f)$$

$$M = p r_i R t (1 - \cos\theta) \qquad (g)$$

Hence, by Eqs. (e), (g) and (9.11),

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M(A - rA_m)}{Ar(RA_m - A)} \qquad (h)$$

where

$$A = (r_o - r_i)t$$
$$A_m = t \ln\frac{r_o}{r_i} \qquad (i)$$
$$R = \frac{1}{2}(r_o + r_i)$$

and by Eqs. (a), (e), and (g)

$$N = \frac{1}{2}P(1 - \cos\theta) \qquad (j)$$
$$M = \frac{1}{2}PR(1 - \cos\theta)$$

So, by Eqs. (h), (i), and (j), in terms of P, r_o, r_i, r, and θ,

$$\sigma_{\theta\theta} = \frac{P(1-\cos\theta)}{2(r_o - r_i)t}\left\{1 + \frac{(r_o + r_i)(r_o - r_i - r\ln\frac{r_o}{r_i})}{r[(r_o + r_i)\ln\frac{r_o}{r_i} - 2(r_o - r_i)]}\right\} \qquad (k)$$

(b) For $r_i = 60$ mm, $r_o = 180$ mm, and $t = 50$ mm, Eq. (k) yields

$$\sigma_{\theta\theta} = P(1 - \cos\theta)\left(\frac{0.1014}{r} - 0.000845\right) \qquad (l)$$

For $\theta = \pi/2$, Eq. (l) yields

$$\sigma_{\theta\theta} = P\left(\frac{0.1014}{r} - 0.000845\right) \qquad (m)$$

(cont.)

9.4 Cont. For maximum tensile stress, $r = r_i = 60\,\text{mm}$. Then, Eq. (m) yields

$$(\sigma_{\theta\theta})_{\text{max tension}} = 0.000845\,P \left[\frac{N}{mm^2}\right] = 845\,P \left[\frac{N}{m^2}\right] \quad (n)$$

For maximum compressive stress, $r = r_0 = 180\,\text{mm}$. Then Eq. (m) yields

$$(\sigma_{\theta\theta})_{\text{max. compression}} = -0.000282\,P\left[\frac{N}{mm^2}\right] = -282\,P\left[\frac{N}{m^2}\right] \quad (o)$$

For $\theta = \pi$, Eq. (l) yields

$$\sigma_{\theta\theta} = P\left(\frac{0.2028}{r} - 0.00169\right) \quad (p)$$

For maximum tensile stress, $r = r_i = 60\,\text{mm}$. So, Eq (p) yields

$$(\sigma_{\theta\theta})_{\text{max. tension}} = 0.00169\,P\left[\frac{N}{mm^2}\right] = 1690\,P\left(\frac{N}{m^2}\right] \quad (q)$$

For maximum compressive stress, $r = r_0 = 180\,\text{mm}$, and

$$(\sigma_{\theta\theta})_{\text{max. compression}} = -0.000563\,P\left[\frac{N}{mm^2}\right] = -563\,P\left[\frac{N}{m^2}\right] \quad (r)$$

(c) For a maximum allowable tensile stress of 340MPa and a safety factor of 2.2, Eq. (q) yields the maximum allowable load $P = 91.45\,\text{kN}$.

Comparing the above results to those of Example 9.4, we see that the pressure distribution exerted by the pin affects the stress $\sigma_{\theta\theta}$ in the range $0 \le \theta < \pi$. However, it does not affect the maximum tensile and compressive stresses at $\theta = \pi$.

9.5

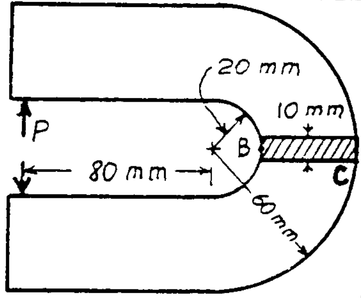
$A = 10(40) = 400 \text{ mm}^2$

$A_m = 10 \ln \frac{60}{20} = 10.986 \text{ mm}$

$$\sigma_{\theta\theta B} = Y = \frac{SF(P)}{A} + \frac{SF(P)(R+80)[A - r_B A_m]}{A r_B [R A_m - A]}$$

$$430 = \frac{1.75P}{400} + \frac{1.75P(120)[400 - 20(10.986)]}{400(20)[40(10.986) - 400]} = 0.1244P$$

$P = 3458 \text{ N} = \underline{3.458 \text{ kN}}$



9.6

$A_m = 10 \ln \frac{55}{15} = 12.993 \text{ mm}$

$$430 = \frac{1.75P}{400} + \frac{1.75P(120)[400 - 15(12.993)]}{400(15)[35(12.993) - 400]} = 0.1355P$$

$P = 3174 \text{ N} = \underline{3.174 \text{ kN}}$

9.7

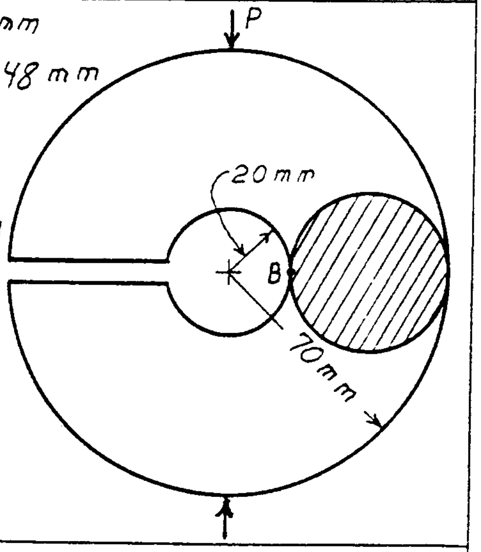
$A = \frac{\pi(50^2)}{4} = 1963 \text{ mm}^2$; $R = 20 + 25 = 45 \text{ mm}$

$A_m = 2\pi[R - \sqrt{R^2 - b^2}] = 2\pi[45 - \sqrt{45^2 - 25^2}] = 47.648 \text{ mm}$

$$\sigma_{\theta\theta B} = -\frac{P}{A} - \frac{PR[A - r_B A_m]}{A r_B [R A_m - A]}$$

$$= -\frac{20{,}000}{1963} - \frac{20{,}000(45)[1963 - 20(47.648)]}{1963(20)[45(47.648) - 1963]}$$

$= \underline{-138.0 \text{ MPa}}$



9.8

$A = \frac{25 + 10}{2}(80 - 45) = 612.5 \text{ mm}^2$

$$R = \frac{45[2(25) + 10] + 80[25 + 2(10)]}{3(25 + 10)} = 60.00 \text{ mm}$$

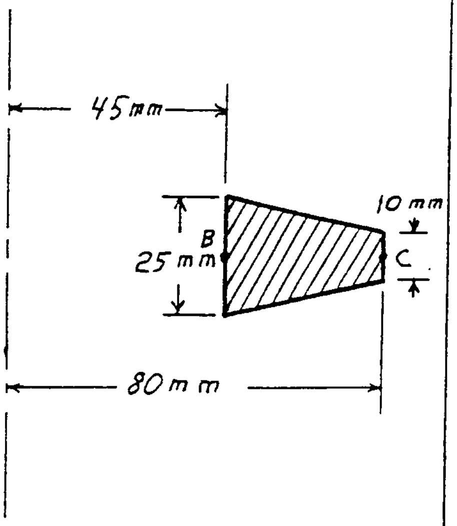
$$A_m = \frac{25(80) - 10(45)}{80 - 45} \ln \frac{80}{45} - 25 + 10 = 10.480 \text{ m}$$

$$\sigma_{\theta\theta B} = \frac{P}{A} + \frac{PR[A - r_B A_m]}{A r_B [R A_m - A]}$$

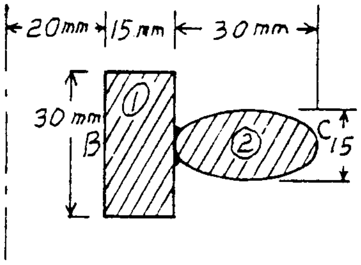
$$150 = \frac{P}{612.5} + \frac{P(60)[612.5 - 45(10.48)]}{612.5(45)[60(10.48) - 612.5]}$$

$= 0.02045 P$

$P = 7{,}335 \text{ N} = \underline{7.335 \text{ kN}}$



9.9



$$A_1 = 15(30) = 450 \text{ mm}^2; \quad R_1 = 20 + \frac{15}{2} = 27.5 \text{ mm}$$

$$A_{m1} = 30 \ln \frac{35}{20} = 16.788 \text{ mm}$$

$$A_2 = \pi\left(\frac{15}{2}\right)\left(\frac{30}{2}\right) = 353.43 \text{ mm}^2; \quad R_2 = 20 + 15 + \frac{30}{2} = 50 \text{ mm}$$

$$A_{m2} = \frac{2\pi(7.5)}{15}\left(50 - \sqrt{50^2 - 15^2}\right) = 7.235 \text{ mm}$$

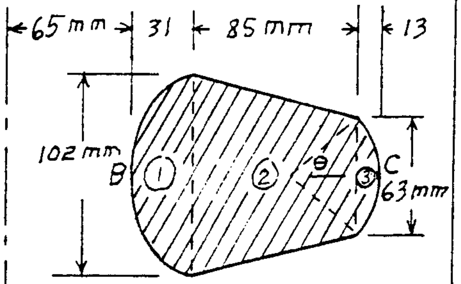
$$A = A_1 + A_2 = 450 + 353.43 = 803.43 \text{ mm}^2; \quad A_m = A_{m1} + A_{m2} = 16.788 + 7.235 = 24.023 \text{ mm}$$

$$R = \frac{R_1 A_1 + R_2 A_2}{A} = \frac{27.5(450) + 50(353.43)}{803.43} = 37.40 \text{ mm}$$

$$\sigma_{\theta\theta B} = \frac{1000 M_x (A - r_B A_m)}{A r_B (R A_m - A)} = \frac{1000 M_x [803.43 - 20(24.023)]}{803.43(20)[37.40(24.023) - 803.43]} = \underline{0.2115 M_x (\text{MPa})}$$

$$\sigma_{\theta\theta C} = \frac{1000 M_x [803.43 - 65(24.023)]}{803.43(65)[37.40(24.023) - 803.43]} = \underline{-0.1528 M_x (\text{MPa})}$$

9.10 $a_2 = 65 + 31 = 96 \text{ mm}; \quad C_2 = a + 85 = 181 \text{ mm}$



$$A_2 = \frac{102 + 63}{2}(85) = 7012.50 \text{ mm}^2$$

$$R_2 = \frac{96[2(102) + 63] + 181[102 + 2(63)]}{3(102 + 63)} = 135.15 \text{ mm}$$

$$A_{m2} = \frac{102(181) - 63(96)}{85} \ln \frac{181}{96} - 102 + 63 = 53.6156 \text{ mm}$$

$$b_3 = 44.663 \text{ mm}; \quad a_3 = 149.337 \text{ mm}; \quad \theta = 0.7828 \text{ rad}$$

$$A_3 = (44.663)^2(0.7828) - \frac{44.663}{2} \sin 2(0.7828) = 564.14 \text{ mm}^2$$

$$R_3 = 149.337 + \frac{4(44.663) \sin^3 0.7828}{3[2(0.7828) - \sin 2(0.7828)]} = 186.27 \text{ mm}$$

$$A_{m3} = 2(149.337)(0.7828) - 2(44.663) \sin 0.7828 - \pi\sqrt{149.337^2 - 44.663^2}$$
$$+ 2\sqrt{149.337^2 - 44.663^2} \sin^{-1} \frac{44.663 + 149.337 \cos 0.7828}{149.337 + 44.663 \cos 0.7828} = 3.0296$$

$$A_1 = \frac{\pi(51)(31)}{2} = 2483.43 \text{ mm}^2; \quad R_1 = 96 - \frac{4(31)}{3\pi} = 82.84 \text{ mm}$$

$$A_{m1} = 102 + \frac{51\pi}{31}\left(96 - \sqrt{96^2 - 31^2}\right) - \frac{102}{31}\sqrt{96^2 - 31^2} \sin^{-1} \frac{31}{96} = 30.2838 \text{ mm}$$

$$A = 7012.50 + 564.14 + 2483.43 = 10{,}060.1 \text{ mm}^2; \quad A_m = 53.6156 + 3.0296 + 30.2838 = 86.929 \text{ mm}$$

$$R = \frac{7012.50(135.15) + 564.14(186.27) + 2483.43(82.84)}{10{,}060.1} = 125.11 \text{ mm}$$

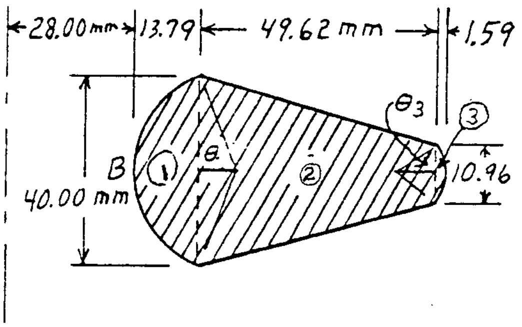
$$\sigma_{\theta\theta B} = \frac{100{,}000}{10{,}060.1} + \frac{100{,}000(125.11)[10{,}060.1 - 65(86.929)]}{10{,}060.1(65)[125.11(86.929) - 10{,}060.1]} = \underline{113.3 \text{ MPa}}$$

$$\sigma_{\theta\theta C} = \frac{100{,}000}{10{,}060.1} + \frac{100{,}000(125.11)[10{,}060.1 - 194(86.929)]}{10{,}060.1(194)[125.11(86.929) - 10{,}060.1]} = \underline{-43.5 \text{ MPa}}$$

9.11

$a_2 = 28.00 + 13.79 = 41.79 \text{ mm}$

$c_2 = a + 49.62 = 91.41 \text{ mm}$



$$A_2 = \frac{40.00+10.96}{2}(49.62) = 1264.3 \text{ mm}^2$$

$$R_2 = \frac{41.79[2(40.00)+10.96] + 91.41[40.00+2(10.96)]}{3(40.00+10.96)}$$

$$= 61.89 \text{ mm}$$

$$A_{m_2} = \frac{40.00(91.41) - 10.96(41.79)}{49.62} \ln\frac{91.41}{41.79} - 40.00 + 10.96 = 21.411 \text{ mm}$$

$\theta_3 = 0.5647 \text{ rad}; \; a_2 = 82.76 \text{ mm}; \; b_2 = 10.24 \text{ mm}$

$$A_3 = 10.24^2(0.5647) - \frac{10.24^2}{2}\sin 2(0.5647) = 11.8 \text{ mm}^2$$

$$R_3 = 82.76 + \frac{4(10.24)\sin^3 0.5647}{3[2(0.5647) - \sin 2(0.5647)]} = 92.05 \text{ mm}$$

$$A_{m_3} = 2(82.76)(0.5647) - 2(10.24)\sin 0.5647 - \pi\sqrt{82.76^2 - 10.24^2}$$
$$+ 2\sqrt{82.76^2 - 10.24^2}\,\sin^{-1}\frac{10.24 + 82.76\cos 0.5647}{82.76 + 10.24\cos 0.5647} = 0.128 \text{ mm}$$

$\theta_1 = 1.2072 \text{ rad}; \; a_1 = 49.40 \text{ mm}; \; b_1 = 21.40 \text{ mm}$

$$A_1 = 21.40^2(1.2072) - \frac{21.40^2}{2}\sin 2(1.2072) = 400.6 \text{ mm}^2$$

$$R_1 = 49.40 - \frac{4(21.40)\sin^3 1.2072}{3[2(1.2072) - \sin 2(1.2072)]} = 36.09 \text{ mm}$$

$$A_{m1} = 2(49.40)(1.2072) + 2(21.40)\sin 1.2072 - \pi\sqrt{49.40^2 - 21.40^2}$$
$$- 2\sqrt{49.40^2 - 21.40^2}\,\sin^{-1}\frac{21.40 - 49.40\cos 1.2072}{49.40 - 21.40\cos 1.2072} = 11.221 \text{ mm}$$

$$A = 1264.3 + 11.8 + 400.6 = 1676.7 \text{ mm}^2$$

$$A_m = 21.411 + 0.128 + 11.221 = 32.760 \text{ mm}$$

$$R = \frac{1264.3(61.89) + 11.8(92.05) + 400.6(36.09)}{1676.7} = 55.94$$

$$\sigma_{\theta\theta B} = \frac{30{,}000}{1676.7} + \frac{30{,}000(55.94)[1676.7 - 28(32.760)]}{1676.7(28)[55.94(32.760) - 1676.7]} = \underline{192.0 \text{ MPa}}$$

$$\sigma_{\theta\theta C} = \frac{30{,}000}{1676.7} + \frac{30{,}000(55.94)[1676.7 - 93(32.760)]}{1676.7(93)[55.94(32.760 - 1676.7]} = \underline{-76.7 \text{ MPa}}$$

9.12

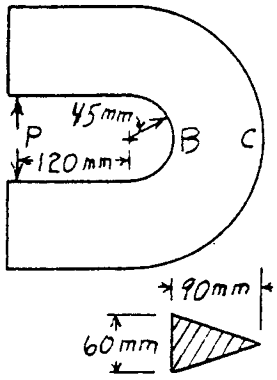
$$A = \frac{60}{2}(90) = 2700\ mm^2$$

$$R = \frac{2(45)+135}{3} = 75.0\ mm$$

$$A_m = \frac{60(135)}{90} \ln\frac{135}{45} - 60 = 38.875\ mm$$

$$\sigma_{\theta\theta B} = \frac{40,000}{2700} + \frac{40,000(195)[2700-45(38.875)]}{2700(45)[75(38.875)-2700]} = \underline{297.8\ MPa}$$

$$\sigma_{\theta\theta C} = \frac{40,000}{2700} + \frac{40,000(195)[2700-135(38.875)]}{2700(135)[75(38.875)-2700]} = \underline{-238.1\ MPa}$$



9.13 $N = -200\sin 45^\circ = -141.4\,kN$, $M = -200(750/\cos 45^\circ) = -212,132\ kN\cdot mm$

$A = 50(500) = 25,000\ mm^2$, $R = 750\ mm$, $A_m = 50\ln(1000/50) = 34.65\ mm$

FRom Eq (9.11):

@ $r = 500\ mm$

$$\sigma_{\theta\theta} = \frac{-141.4}{25,000} - \frac{212,132[25,000-500(34.65)]}{25,000(500)[750(34.65)-25,000]}$$

$$\sigma_{\theta\theta} = -0.00566 - 0.13190$$

$$\underline{\underline{\sigma_{\theta\theta} = -0.1376\ kN/mm^2 = 137.6 MPa}}$$

@ $r = 1000\ mm$

$$\sigma_{\theta\theta} = \frac{-141.4}{25,000} - \frac{212,132[25,000-1000(34.65)]}{25,000(1000)[750(34.65)-25,000]}$$

$$\sigma_{\theta\theta} = -0.00566 + 0.08292$$

$$\underline{\underline{\sigma_{\theta\theta} = 0.07726\ kN/mm^2 = 77.26 MPa}}$$

9.14 From Example 9.1, the dimensions of the beam are shown in Figs. a and b.

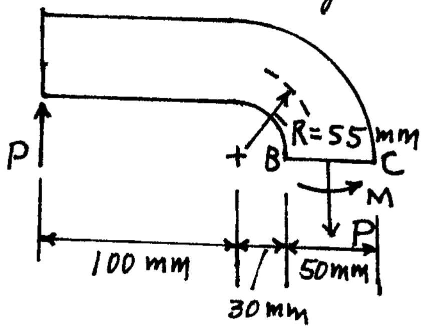


Figure a

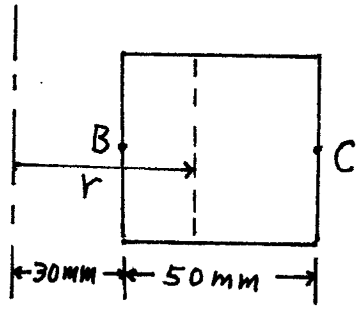


Figure b

By Eqs. (9.18) and (9.19) and Fig. a, we have

$$\sigma_{rr} = \frac{A'}{A}\frac{P}{tr} + \frac{AA'_m - A'A_m}{trA(RA_m - A)}M \qquad (a)$$

where by Fig. b, Eqs. (9.20) and Example 9.1,

$$A' = \int_{30}^{r} 50\,d\rho = 50(r-30)\ [\mathrm{mm}^2],\ A'_m = \int_{30}^{r} 50\frac{d\rho}{\rho} = 50\ln\frac{r}{30}\ [\mathrm{mm}]$$

$$A = 50\times 50 = 2500\ \mathrm{mm}^2,\ A_m = 50\ln\frac{80}{30} = 49.0415\ \mathrm{mm} \qquad (b)$$

$$R = 30 + 25 = 55\ \mathrm{mm},\quad t = 50\ \mathrm{mm}$$

$$P = 9500\ \mathrm{N},\quad M = P(155) = 1472.5\times 10^3\ \mathrm{N\cdot mm}$$

By Eqs. (a) and (b), we obtain

$$\sigma_{rr} = \frac{50(r-30)}{2500}\cdot\frac{9500}{50r} + \frac{[2500(50\ln\frac{r}{30}) - 50(r-30)(49.0415)]\times 1472.5\times 10^3}{50(2500)r[55(49.0415) - 2500]}$$

or

$$\sigma_{rr} = -142.6166 - \frac{21107.75}{r} + 7463.916\frac{\ln r}{r} \qquad (c)$$

For $r = 30$ mm, $\sigma_{rr} \approx 0$, and for $r = 80$ mm, $\sigma_{rr} \approx 2.4 \neq 0$. This results because of the sensitivity of the calculations involving A_m.

(Cont.)

9.14 For σ_{rr} to be exactly zero at $r = 30$ mm and $r = 80$ mm, A_m must be calculated with more significant figures.

By Eq. (c), the maximum value of σ_{rr} is determined by the condition

$$\frac{d\sigma_{rr}}{dr} = \frac{21107.75}{r^2} - \frac{7463.916}{r^2}\ln r + \frac{7463.916}{r^2} = 0$$

or

$$\ln r \simeq 3.82797, \quad r \simeq 45.969 \text{ mm} \qquad \text{(d)}$$

By Eqs. (c) and (d),

$$(\sigma_{rr})_{max} = 19.75 \text{ MPa} \quad \text{at } r \simeq 45.969 \text{ mm}$$

The distribution of σ_{rr} as a function of r is shown in Fig. C and in Table A.

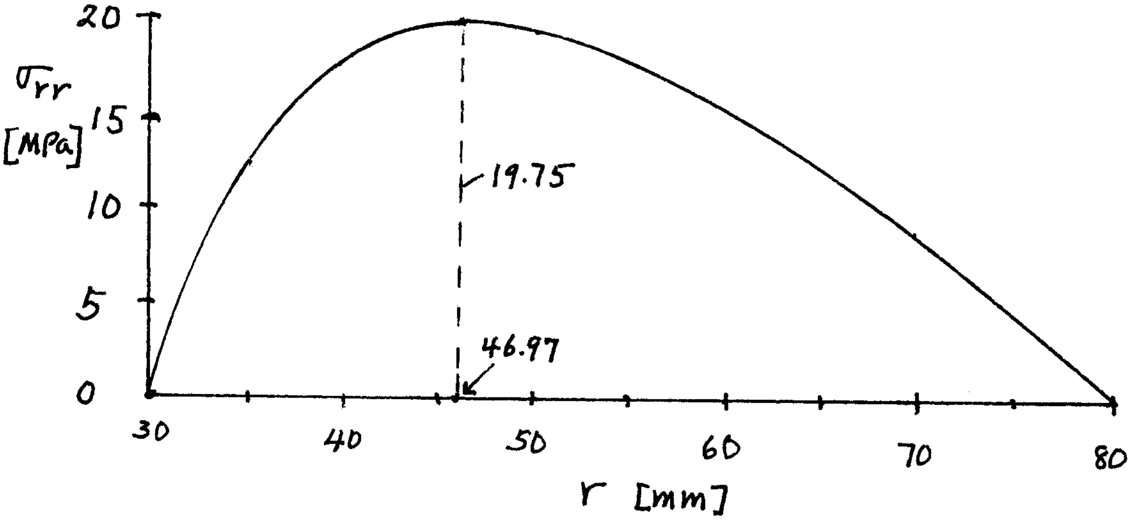


Table A

r	30	35	40	45.97	50	60	70	80
σ_{rr}	0	12.5	18.0	19.75	19.2	14.9	8.85	0

$\boxed{9.15}$ By Fig. a and Table 9.2, we find

$$A = \tfrac{1}{2}(60)(90) = 2700 \text{ mm}^2$$

$$R = 45 + \tfrac{1}{3}(90) = 75 \text{ mm} \quad \text{(a)}$$

$$A_m = \frac{60(45+90)}{135-45} \ln\frac{135}{45} - 60 = 38.8751 \text{mm}$$

By Fig. a and Eqs. (9.20), we obtain for $r = 75$ mm (then $t = 40$mm)

$$A' = \int dA = \int_{45}^{75} \left(-\tfrac{2}{3}\rho + 90\right) d\rho = 1500 \text{ mm}^2$$

$$\text{(b)}$$

$$A'_m = \int \frac{dA}{\rho} = \int_{45}^{75} \left(-\tfrac{2}{3} + \frac{90}{\rho}\right) d\rho = 25.97431 \text{ mm}$$

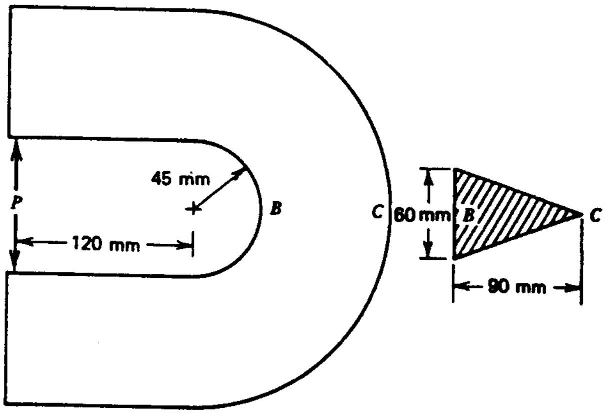


Figure a

Hence, by Eqs. (a), (b), (9.18), and (9.19), we have

$$\sigma_{rr} = \frac{A'}{A}\frac{P}{tr} + \frac{AA'_m - A'A_m}{trA(RA_m - A)} M$$

or

$$\sigma_{rr} = \left(\frac{1500}{2700}\right)\left(\frac{40{,}000}{40(75)}\right) + \frac{2700(25.97431) - 1500(38.8751)}{40(75)(2700)[75(38.8751) - 2700]}[40{,}000(195)]$$

or

$$\sigma_{rr} = 60.18 \text{ MPa}$$

$\boxed{9.16}$ By Prob. 9.9,

$A = 803.43\ mm^2$, $A_m = 24.023\ mm$, $R = 37.40\ mm$

$A' = A_1 = 450\ mm^2$, $t = 10\ mm$, $r = 35\ mm$

$A'_m = A_{m1} = 16.788\ mm$

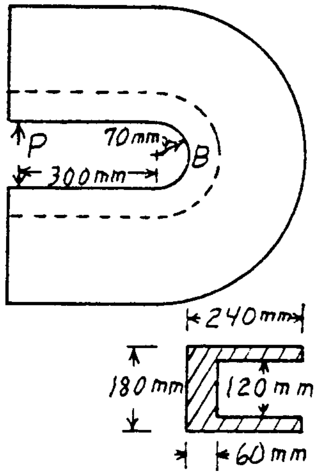
So, $\sigma_{rr} = \dfrac{AA'_m - A'A_m}{trA(RA_m - A)} 1000 M_x$, $[M_x] = [N \cdot mm]$

$$= \frac{1000[(803.43)(16.788) - 450(24.023)]}{10(35)(803.43[37.40(24.023) - 803.43]} M_x$$

$$= 0.1002 M_x \quad [MPa]$$

9.17

$A_1 = 180(60) = 10,800\ mm^2;\ R_1 = 70+30 = 100\ mm$



$A_{m1} = 180\ \ln\frac{130}{70} = 111.427\ mm$

$A_2 = 60(180) = 10,800\ mm^2;\ R_2 = 130+90 = 220\ mm$

$A_{m2} = 60\ \ln\frac{310}{130} = 52.142\ mm$

$A = 10,800 + 10,800 = 21,600\ mm^2$

$A_m = 111.427 + 52.142 = 163.569\ mm$

$R = \frac{10,800(100)+10,800(220)}{21,600} = 160.0\ mm$

(a) $\sigma_{\theta\theta B} = Y = SF\frac{P}{A} + SF\frac{460P[A - r_B A_m]}{A r_B [R A_m - A]}$

$320 = \frac{4P}{21,600} + \frac{4(460P)[21,600-70(163.569)]}{21,600(70)[160(163.569-21,600]} = 0.002887P$

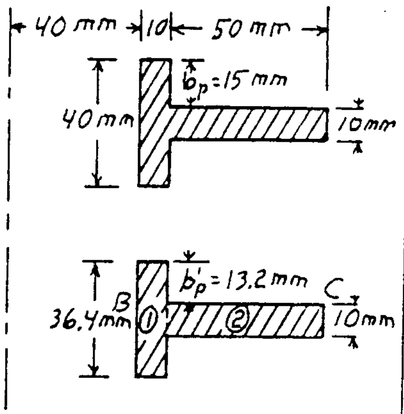
$P = \underline{110.8\ kN}$

(b) $A' = A_1 = 10,800\ mm^2;\ A_m' = A_{m1} = 111.427\ mm$

$\sigma_{rr} = \frac{110,800(460)[21,600(111.427)-10,800(163.569)]}{60(130)(21,600)[160(163.569)-21,600]} = \underline{42.4\ MPa}$

(c) Yes

9.18



$$\frac{b_p^2}{rt} = \frac{15^2}{45(10)} = 0.5$$

From Table 9.3, $\alpha = 0.878$ and $\beta = 1.238$

$$b'_p = \alpha b_p = 0.878(15) = 13.2 \text{ mm}$$

$$A_1 = [10 + 2(13.2)](10) = 364 \text{ mm}^2$$

$$R_1 = 40 + 5 = 45 \text{ mm}$$

$$A_{m1} = 36.4 \ln \frac{50}{40} = 8.122 \text{ mm}$$

$$A_2 = 50(10) = 500 \text{ mm}^2$$

$$R_2 = 40 + 10 + 25 = 75 \text{ mm}$$

$$A_{m2} = 10 \ln \frac{100}{50} = 6.931 \text{ mm}$$

$$A = A_1 + A_2 = 364 + 500 = 864 \text{ mm}^2$$

$$A_m = A_{m1} + A_{m2} = 8.122 + 6.931 = 15.053 \text{ mm}$$

$$R = \frac{R_1 A_1 + R_2 A_2}{A} = \frac{45(364) + 75(500)}{864} = 62.4 \text{ mm}$$

$$\sigma_B = \frac{M_x (A - r_B A_m)}{A r_B (R A_m - A)} = \frac{2{,}500{,}000[864 - 40(15.053)]}{864(40)[62.4(15.053) - 864]} = 251.6 \text{ MPa}$$

$$\sigma_C = \frac{2{,}500{,}000[864 - 100(15.053)]}{864(100)[62.4(15.053) - 864]} = -246.4 \text{ MPa}$$

$$\bar{\sigma}_{\theta\theta} = \frac{2{,}500{,}000[864 - 45(15.053)]}{864(45)[62.4(15.053) - 864]} = 159.3 \text{ MPa}$$

$$\sigma_{xx(max)} = -\beta \bar{\sigma}_{\theta\theta} = -197.3 \text{ MPa}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{251.6 - (-197.3)}{2} = \underline{224.5 \text{ MPa}}$$

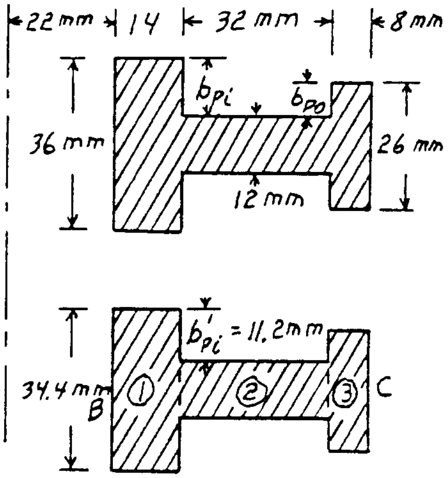
9.19 Use corrected cross section.

$$A' = A_1 = 364 \text{ mm}^2$$

$$A'_m = A_{m1} = 8.122 \text{ mm}$$

$$\sigma_{rr} = \frac{M_x (A A'_m - A' A_m)}{t r A (R A_m - A)} = \frac{2{,}500{,}000[864(8.122) - 364(15.053)]}{10(50)(864[62.4(15.053) - 864]} = \underline{118.2 \text{ MPa}}$$

9.20 For area A_1



$$\frac{b_{pi}^2}{rt} = \frac{12^2}{29(14)} = 0.355;\ \alpha = 0.932;\ \beta = 0.957$$

$$b'_{pi} = \alpha b_{pi} = 0.932(12) = 11.2 \text{ mm}$$

For area A_3, $\frac{b_{po}^2}{rt} = \frac{7^2}{72(8)} = 0.085$ (No correction)

$$A_1 = [12 + 2(11.2)](14) = 481.6 \text{ mm}^2$$

$$R_1 = 22 + 7 = 29 \text{ mm}$$

$$A_{m1} = [12 + 2(11.2)] \ln \frac{36}{22} = 16.941 \text{ mm}$$

$$A_2 = 32(12) = 384 \text{ mm}^2$$

$$R_2 = 36 + 16 = 52 \text{ mm}$$

$$A_{m2} = 12 \ln \frac{68}{36} = 7.632 \text{ mm}$$

$$A_3 = 26(8) = 208 \text{ mm}^2$$

$$R_3 = 68 + 4 = 72 \text{ mm}$$

$$A_{m3} = 26 \ln \frac{76}{68} = 2.892 \text{ mm}$$

$$A = A_1 + A_2 + A_3 = 481.6 + 384 + 208 = 1073.6 \text{ mm}^2$$

$$A_m = A_{m1} + A_{m2} + A_{m3} = 16.941 + 7.632 + 2.892 = 27.465 \text{ mm}$$

$$R = \frac{R_1 A_1 + R_2 A_2 + R_3 A_3}{A} = \frac{29(481.6) + 52(384) + 72(208)}{1073.6} = 45.56 \text{ mm}$$

$$\sigma_B = \frac{P}{A} + \frac{(51+R)P[A - r_B A_m]}{A r_B [R A_m - A]} = \frac{12{,}000}{1073.6} + \frac{96.56(12{,}000)[1073.6 - 22(27.465)]}{1073.6(22)[45.56(27.465) - 1073.6]} = \underline{140.8 \text{ MPa}}$$

$$\sigma_C = \frac{12{,}000}{1073.6} + \frac{96.56(12{,}000)[1073.6 - 76(27.465)]}{1073.6(76)[45.56(27.465) - 1073.6]} = \underline{-69.8 \text{ MPa}}$$

9.21 Use corrected cross section.

$$A' = A_1 = 481.6 \text{ mm}^2$$

$$A'_m = A_{m1} = 16.941 \text{ mm}$$

$$\sigma_{rr} = \frac{M_x[A A'_m - A' A_m]}{t r A [R A_m - A]} = \frac{96.56(12{,}000)[1073.6(16.941) - 481.6(27.465)]}{12(36)(1073.6)[45.56(27.465) - 1073.6]} = \underline{69.7 \text{ MPa}}$$

9.22 Since M_x and N are applied simultaneously, the increment of strain energy due to M_x and N is

$$dU = \frac{1}{2} M_x \omega d\theta + \frac{1}{2} N \bar{\epsilon}_{\theta\theta} R d\theta \qquad (a)$$

where ω is a result due to both M_x and N. Thus, by Eq. (9.10),

$$\omega = \frac{1}{E}\left[\frac{A_m M_x}{A(RA_m - A)} - \frac{N}{A}\right] \qquad (b)$$

Also, by Eq. (9.3), with $r = R$,

$$R\bar{\epsilon}_{\theta\theta} = R_n \omega - R\omega \qquad (c)$$

Then, with Eqs. (9.9) and (b), we write Eq. (c) as

$$R\bar{\epsilon}_{\theta\theta} = \frac{1}{E} \cdot \frac{M_x}{RA_m - A} - \frac{R}{E} \cdot \frac{A_m M_x}{A(RA_m - A)} + \frac{R}{E}\frac{N}{A} \qquad (d)$$

So, by Eqs. (a), (b), and (d),

$$dU = \left\{\frac{1}{2}\frac{M_x}{E}\left[\frac{A_m M_x}{A(RA_m - A)} - \frac{N}{A}\right] + \frac{1}{2}N\left[\frac{1}{E}\frac{M_x}{RA_m - A} - \frac{R}{E}\frac{A_m M_x}{A(RA_m - A)} + \frac{NR}{EA}\right]\right\} d\theta$$

$$= \left\{\frac{1}{2}\frac{A_m M_x^2}{EA(RA_m - A)} - \frac{1}{2}\frac{M_x N}{EA} + \frac{1}{2}\frac{N}{EA}(-M_x + NR)\right\} d\theta$$

$$= \left\{\frac{1}{2}\frac{N^2 R}{EA} + \frac{1}{2}\frac{A_m M_x^2}{AE(RA_m - A)} - \frac{M_x N}{EA}\right\} d\theta$$

Integrating, we obtain

$$U = \int \frac{N^2 R}{2AE} d\theta + \int \frac{A_m M_x^2}{2AE(RA_m - A)} d\theta - \int \frac{M_x N}{EA} d\theta \qquad (e)$$

With the addition of the strain energy due to shear, Eq. (e) is the same as Eq. (9.31).

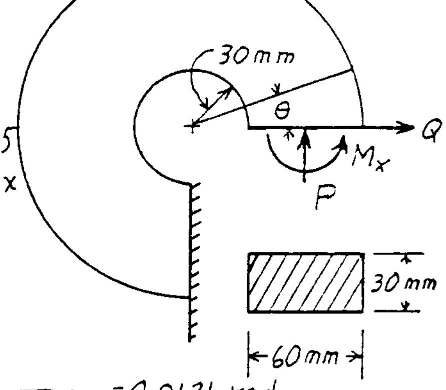
9.23

$A = 60(30) = 1800\ \text{mm}^2;\quad R = 30 + 30 = 60\ \text{mm}$

$A_m = 30 \ln \frac{90}{30} = 32.958\ \text{mm}$

$\sigma_{max} = Y = 420 = \dfrac{M_x[1800 - 30(32.958)]}{1800(30)[60(32.958) - 1800]} = 0.00008465\, M_x$

$M_x = 4.96\ \text{kN.m}$



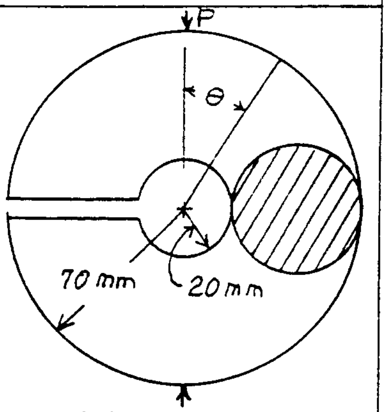
At angle θ, $M = M_x + PR(1 - \cos\theta) + QR\sin\theta$

$$\theta = \frac{\partial U}{\partial M_x} = \int_0^{\frac{3\pi}{2}} \frac{A_m M_x}{AE(RA_m - A)} \frac{\partial M}{\partial M_x} d\theta = \frac{3\pi(32.958)(4{,}960{,}000)}{2(1800)(200{,}000)[60(32.958) - 1800]} = 0.0121\ \text{rad}$$

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^{3\pi/2} \frac{A_m M_x}{AE(RA_m - A)} \frac{\partial M}{\partial Q} d\theta = \frac{32.958(4{,}960{,}000)(60)}{1800(200{,}000)[60(32.958) - 1800]} = \underline{0.154\ \text{mm}}$$

$$\delta_P = \frac{\partial U}{\partial P} = \int_0^{3\pi/2} \frac{A_m M_x}{AE(RA_m - A)} \frac{\partial M}{\partial P} d\theta = \frac{32.958(4{,}960{,}000)(60)}{1800(200{,}000)[60(32.958) - 1800]}\left[\frac{3\pi}{2} + 1\right] = \underline{0.877\ \text{mm}}$$

9.24 $A = 1963\ \text{mm}^2;\ A_m = 47.648\ \text{mm};\ R = 45\ \text{mm}$



$M = -PR\sin\theta \qquad \dfrac{\partial M}{\partial P} = -R\sin\theta$

$N = -P\sin\theta \qquad \dfrac{\partial N}{\partial P} = -\sin\theta$

$V = -P\cos\theta \qquad \dfrac{\partial V}{\partial P} = -\cos\theta$

$$\delta_P = \int_0^{\pi} \frac{A_m M}{AE(RA_m - A)} \frac{\partial M}{\partial P} d\theta + \int_0^{\pi} \frac{N}{AE} \frac{\partial N}{\partial P} R\, d\theta + \int_0^{\pi} \frac{kV}{GA} \frac{\partial V}{\partial P} R\, d\theta$$

$$= \frac{47.648(20{,}000)(45^2)\pi}{1963(72{,}000)[45(47.648) - 1963](2)} + \frac{20{,}000(45)\pi}{72{,}000(1963)(2)} + \frac{1.3(20{,}000)(45)\pi}{27{,}100(1963)(2)}$$

$$= 0.1184 + 0.0100 + 0.0345 = \underline{0.1629\ \text{mm}}$$

9.25 $A = 2700\ \text{mm}^2;\ A_m = 38.875;\ R = 75\ \text{mm};\ I = \frac{bh^3}{36} = \frac{60(90)^3}{36} = 1{,}215{,}000\ \text{mm}^4$

0 – 120 mm	$M = Pz$	$N = 0$	$V = P$
$0 - \pi$	$M = P(120 + R\sin\theta)$	$N = P\sin\theta$	$V = P\cos\theta$

$$\delta_P = 2\int_0^{120} \frac{M}{EI} \frac{\partial M}{\partial P} dz + \int_0^{\pi} \frac{A_m M}{AE(RA_m - A)} \frac{\partial M}{\partial P} d\theta + \int_0^{\pi} \frac{N}{AE} \frac{\partial N}{\partial P} R\, d\theta + 2\int_0^{120} \frac{kV}{GA} \frac{\partial V}{\partial P} dz + \int_0^{\pi} \frac{kV}{GA} \frac{\partial V}{\partial P} R\, d\theta$$

$$= \frac{2(40{,}000)(120)^3}{200{,}000(1{,}215{,}000)(3)} + \frac{38.875(40{,}000)[120^2\pi + 2(120)(75)(2) + \frac{75^2\pi}{2}]}{2700(200{,}000)[75(38.875) - 2700]} + \frac{40{,}000(75)\pi}{2700(200{,}000)(2)}$$

$$+ \frac{2(1.5)(40{,}000)(120)}{2700(77{,}500)} + \frac{1.5(40{,}000)(75)\pi}{2700(77{,}500)(2)}$$

$$\delta_P = 0.1896 + 1.2029 + 0.0087 + 0.0688 + 0.0338 = \underline{1.504\ \text{mm}}$$

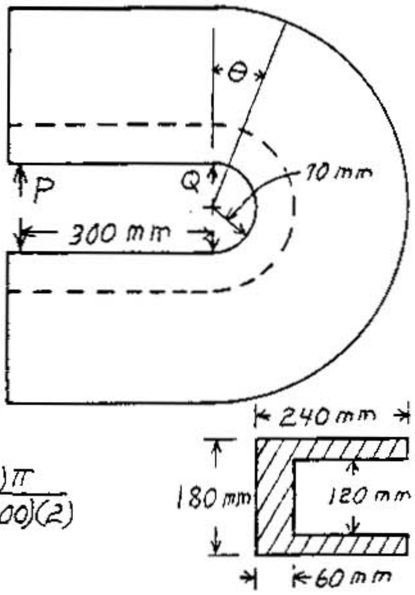
9.26 $A = 21{,}600\ mm^2$; $A_m = 163.569\ mm$; $R = 160\ mm$

$A_w = 240(180-120) = 14{,}400\ mm$

$M = P(300 + R\sin\theta) + QR\sin\theta \qquad \frac{\partial M}{\partial Q} = R\sin\theta$

$N = (P+Q)\sin\theta \qquad \frac{\partial N}{\partial Q} = \sin\theta$

$V = (P+Q)\cos\theta \qquad \frac{\partial V}{\partial Q} = \cos\theta$



$$\delta_Q = \int_0^{\pi} \frac{A_m M}{EA(RA_m - A)} \frac{\partial M}{\partial Q} d\theta + \int_0^{\pi} \frac{N}{EA} \frac{\partial N}{\partial Q} R\,d\theta + \int_0^{\pi} \frac{V}{GA_w} \frac{\partial V}{\partial Q} R\,d\theta$$

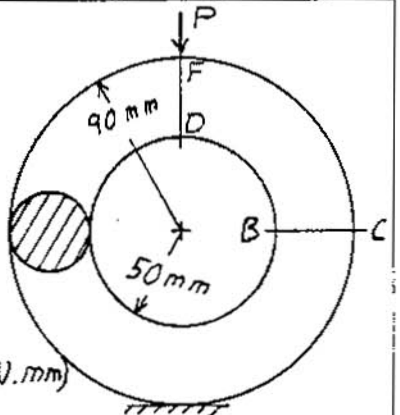
$$= \frac{163.569(126{,}000)(160)\left[300(2) + \frac{160\pi}{2}\right]}{102{,}000(21{,}600)[160(163.569) - 21{,}600]} + \frac{126{,}000(160)\pi}{102{,}000(21{,}600)(2)}$$

$$+ \frac{126{,}000(160)\pi}{42{,}500(14{,}400)(2)}$$

$\delta_Q = 0.2786 + 0.0144 + 0.0517 = \underline{0.3447\ mm}$

9.27 $A = \frac{\pi D^2}{4} = \frac{\pi(40)^2}{4} = 1257\ mm^2$

$R = 50 + 20 = 70\ mm$



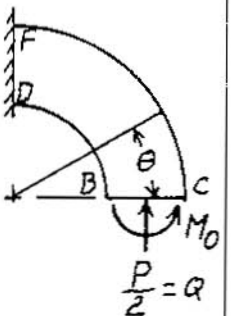
$A_m = 2\pi(70 - \sqrt{70^2 - 20^2}) = 18.334\ mm$

Since $\frac{R}{h} = \frac{R}{D} = \frac{70}{40} = 1.75 < 2$, Eq.(9.36) is valid.

$M_O = -\frac{PR}{2}\left(1 - \frac{2A}{RA_m\pi}\right) = -\frac{70P}{2}\left[1 - \frac{2(1257)}{70(18.334)\pi}\right] = -13.18P\ (N.mm)$

The moment at section DF is

$M_{DF} = \frac{PR}{2} + M_O = 21.82\ R\ (N.mm)$



This moment multiplied by SF will initiate yielding at D.

$$\sigma_D = Y = 520 = \frac{SF(M_{DF})[A - r_D A_m]}{A r_O[RA_m - A]} = \frac{1.75(21.82P)[1257 - 50(18.334)]}{1257(50)[70(18.334) - 1257]} = 0.00784P$$

$P = \underline{66.33\ kN}$

9.28 Let $Q = \frac{P}{2}$

$$M = M_0 + QR(1-\cos\theta) \qquad \frac{\partial M}{\partial Q} = R(1-\cos\theta)$$

$$N = -Q\cos\theta \qquad \frac{\partial N}{\partial Q} = -\cos\theta$$

$$V = -Q\sin\theta \qquad \frac{\partial V}{\partial Q} = -\sin\theta$$

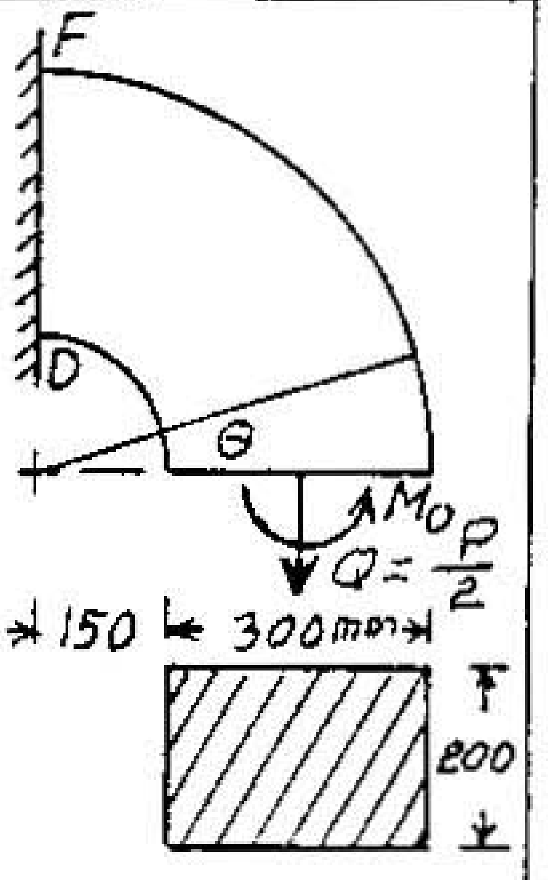
$$\frac{\delta_P}{2} = \delta_Q = \int_0^{\pi/2} \frac{A_m M}{EA(RA_m - A)}\frac{\partial M}{\partial Q}d\theta - \int_0^{\pi/2}\frac{M}{EA}\frac{\partial N}{\partial Q}d\theta - \int_0^{\pi/2}\frac{N}{EA}\frac{\partial M}{\partial Q}d\theta + \int_0^{\pi/2}\frac{N}{EA}\frac{\partial N}{\partial Q}R\,d\theta + \int_0^{\pi/2}\frac{kV}{GA}\frac{\partial V}{\partial Q}R\,d\theta$$

$$\delta_P = \frac{2(18.334)(70)\left[-13.18(60{,}000)(\frac{\pi}{2}-1) + \frac{60{,}000(70)}{2}(\frac{3\pi}{2}-2)\right]}{200{,}000(1257)[70(18.334)-1257]} + \frac{2\left[-13.18(60{,}000) + \frac{60{,}000(70)}{2}(1-\frac{\pi}{4})\right]}{200{,}000(1257)}$$

$$+ \frac{2(60{,}000)(70)(1-\frac{\pi}{4})}{2(200{,}000)(1257)} + \frac{2(60{,}000)(70)\pi}{2(200{,}000)(1257)(4)} + \frac{2(1.3)(60{,}000)(70)\pi}{2(77{,}500)(1257)(4)}$$

$$\delta_P = 2.0298 - 0.0027 + 0.0036 + 0.0131 + 0.0440 = \underline{2.0878 \text{ mm}}$$

9.29 $A = 300(200) = 60{,}000 \text{ mm}^2; \quad R = 150 + 150 = 300 \text{ mm}$



$$A_m = 200 \ln\frac{450}{150} = 219.72 \text{ mm}$$

$$M_0 = \frac{PR}{2}\left(1 - \frac{2A}{RA_m\pi}\right) = \frac{300P}{2}\left[1 - \frac{2(60{,}000)}{300(219.72)\pi}\right] = 63.08P \text{ (N.mm)}$$

$$M_{DF} = -\frac{PR}{2} + M_0 = -86.92P \text{ (N.mm)}$$

$$\sigma_B = \frac{P}{2A} + \frac{M_0(A - r_B A_m)}{Ar_B(RA_m - A)}$$

$$= \frac{4{,}000{,}000}{2(60{,}000)} + \frac{63.08(4{,}000{,}000)[60{,}000 - 150(219.72)]}{60{,}000(150)[300(219.72) - 60{,}000]} = \underline{161.5 \text{ MPa}}$$

$$\sigma_D = \frac{-86.92(4{,}000{,}000)[60{,}000 - 150(219.72)]}{60{,}000(150)[300(219.72) - 60{,}000]} = \underline{-176.6 \text{ MPa}}$$

9.30

$$M = M_0 - QR(1-\cos\theta) \qquad \frac{\partial M}{\partial Q} = -R(1-\cos\theta)$$

$$N = Q\cos\theta \qquad \frac{\partial N}{\partial Q} = \cos\theta$$

$$V = Q\sin\theta \qquad \frac{\partial V}{\partial Q} = \sin\theta$$

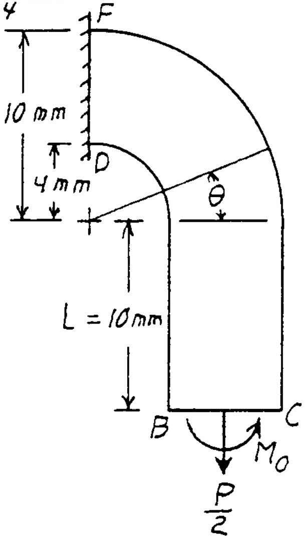
$$\frac{\delta_P}{2} = \delta_Q = \int_0^{\pi/2} \frac{A_m M}{EA(RA_m - A)}\frac{\partial M}{\partial Q}d\theta - \int_0^{\pi/2}\frac{M}{EA}\frac{\partial N}{\partial Q}d\theta - \int_0^{\pi/2}\frac{N}{EA}\frac{\partial M}{\partial Q}d\theta + \int_0^{\pi/2}\frac{N}{EA}\frac{\partial N}{\partial Q}R\,d\theta + \int_0^{\pi/2}\frac{kV}{GA}\frac{\partial V}{\partial Q}R\,d\theta$$

$$\delta_P = \frac{2(219.72)(300)\left[-63.08(4{,}000{,}000)(\frac{\pi}{2}-1) + \frac{4{,}000{,}000(300)}{2}(\frac{3\pi}{2}-2)\right]}{72{,}000(60{,}000)[300(219.72)-60{,}000]} + \frac{2(4{,}000{,}000\left[-63.08 + \frac{300}{2}(1-\frac{\pi}{4})\right]}{72{,}000(60{,}000)}$$

$$+ \frac{4{,}000{,}000(300)(1-\frac{\pi}{4})}{72{,}000(60{,}000)} + \frac{4{,}000{,}000(300)\pi}{72{,}000(60{,}000)(4)} + \frac{1.5(4{,}000{,}000)(300)\pi}{27{,}100(60{,}000)(4)}$$

$$\delta_P = 7.6519 - 0.0572 + 0.0596 + 0.2182 + 0.8694 = \underline{8.7419 \text{ mm}}$$

9.31 $A = \frac{\pi 6^2}{4} = 28.27\ mm^2;\ R = 4+3 = 7\,mm;\ I = \frac{\pi 6^4}{64} = 63.62\ mm^4$



$A_m = 2\pi(7-\sqrt{7^2-3^2}) = 4.244\ mm;\ \frac{R}{2b} = \frac{70}{60} = 1.167 < 2$

0 to $L \qquad M = M_0 \qquad \frac{\partial M}{\partial M_0} = 1 \qquad N = \frac{P}{2}$

0 to $\frac{\pi}{2} \qquad M = M_0 - \frac{PR}{2}(1-\cos\theta) \qquad \frac{\partial M}{\partial M_0} = 1 \qquad N = \frac{P}{2}\cos\theta$

$$\theta_{BC} = 0 = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_0}dz + \int_0^{\pi/2}\frac{A_m M}{EA(RA_m - A)}\frac{\partial M}{\partial M_0}d\theta - \int_0^{\pi/2}\frac{N}{EA}\frac{\partial M}{\partial M_0}d\theta$$

$$= \frac{M_0 L}{EI} + \frac{A_m}{EA(RA_m - A)}\left(M_0\frac{\pi}{2} - \frac{PR\pi}{4} + \frac{PR}{2}\right) - \frac{P}{2EA}$$

$$0 = \frac{M_0(10)}{63.62} + \frac{4.244}{28.27[7(4.244)-28.27]}\left(\frac{M_0\pi}{2} - \frac{7\pi P}{4} + \frac{7P}{2}\right) - \frac{P}{2(28.27)}$$

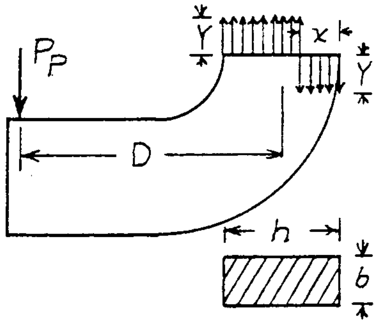
$M_0 = 0.7045\,P$

$M_{DF} = M_0 - \frac{PR}{2} = 2.7955\,P$

$$\sigma_D = Y = \frac{M_{DF}(A - r_D A_m)}{A r_D (RA_m - A)}$$

$$P = \frac{250(28.27)(4)[7(4.244)-28.27]}{2.7955[28.27-4(4.244)]} = \underline{1.288\ kN}$$

9.32 $P_P = Yb(h-x) - Ybx = Yb(h-2x)$



$$M = Yb(h-x)\left(\frac{h}{2} - \frac{h-x}{2}\right) + Ybx\left(\frac{h}{2} - \frac{x}{2}\right)$$

$$= Ybx(h-x) = P_P D = YbD(h-2x)$$

$$x = \frac{h}{2} + D - \sqrt{\left(\frac{h}{2}\right)^2 + D^2}$$

$$\frac{M}{M_P} = \frac{Ybx(h-x)}{\frac{Ybh^2}{4}} = \underline{\frac{4D}{h}\sqrt{1+\frac{4D^2}{h^2}} - \frac{8D^2}{h^2}}$$

9.33

$$M \approx M_P = \frac{Ybh^2}{4} = \frac{430(10)(40)^2}{4} = 120\,P_P$$

$P_P = 14.33\ kN$

$$\frac{P_P}{P_Y} = \frac{14.33}{6.05} = \underline{2.37}$$

9.34

$A_T = A_C = 100(20) = 2000\ mm^2$

The distance between centroids of A_T and A_C is 60 mm.

$M_P = 60 A_T Y = 60(2000)(280) = \underline{33.6\ kN.m}$

10.1 (a) By Eqs. (10.13) and (10.15),

$$\theta = -\frac{P\beta^2}{k} B_{\beta z}, \quad V_y = -\frac{P}{2} D_{\beta z} \qquad (a)$$

Hence, by superposition and Eq. (a), with Fig. a,

$$\theta = +\frac{P_1\beta^2}{k} B_{\beta(z+a)} - \frac{P_2\beta^2}{k} B_{\beta z} \qquad (b)$$

$$V_y = +\frac{P_1}{2} D_{\beta(z+a)} - \frac{P_2}{2} D_{\beta z} \qquad (c)$$

(b) Let $|P_1| = |P_2| = P$ and multiply and divide the right-hand side of Eq. (b) by δ. This yields

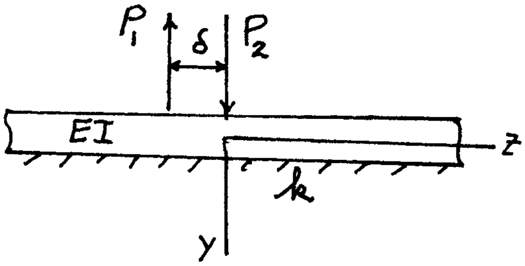


Figure a

$$\theta = \frac{(P\delta)\beta^2}{k}\left[\frac{B_{\beta(z+a)} - B_{\beta z}}{\delta}\right] \qquad (d)$$

Then, let $\delta \to 0$ and $P\delta \to M_0$. Then, by Eq. (d), we obtain

$$\theta = \lim_{\delta \to 0}\left\{\frac{(P\delta)\beta^2}{k}\left[\frac{B_{\beta(z+a)} - B_{\beta z}}{\delta}\right]\right. = \frac{M_0\beta^2}{k}\frac{dB_{\beta z}}{dz} \qquad (e)$$

or by Eqs. (e) and (10.17)

$$\theta = \frac{M_0\beta^3}{k} C_{\beta z}; \quad z \geq 0$$

Similarly, by Eq. (c), we obtain

$$V_y = -\frac{M_0\beta}{2} A_{\beta z}; \quad z \geq 0$$

10.2 By Eq. (f) of Example 10.3,

$$y = \frac{M_0 \beta^2}{k} B_{\beta z}; \quad z \geq 0 \qquad (a)$$

By Eqs. (a) and (10.17),

$$\theta = \frac{dy}{dz} = \frac{M_0 \beta^2}{k} \frac{dB_{\beta z}}{dz} = \frac{M_0 \beta^3}{k} C_{\beta z}$$

The shear V_y is given similarly by Eq. (g) of Example 10.3, with Eqs. (10.1) and (10.17),

$$V_y = \frac{dM_x}{dz} = \frac{d}{dz}\left(\frac{M_0}{2} D_{\beta z}\right) = \frac{M_0}{2} \frac{dD_{\beta z}}{dz}$$

or

$$V_y = -\frac{M_0 \beta}{2} A_{\beta z}$$

10.3 By Eqs. (10.16),

$$\frac{dA_{\beta z}}{dz} = \frac{d}{dz}\left[e^{-\beta z}(\sin\beta z + \cos\beta z)\right] = -2\beta e^{-\beta z} \sin\beta z$$

or

$$\frac{dA_{\beta z}}{dz} = -2\beta B_{\beta z}$$

Similarly,

$$\frac{dB_{\beta z}}{dz} = \frac{d}{dz}\left(e^{-\beta z} \sin\beta z\right) = \beta e^{-\beta z} \cos\beta z - \beta e^{-\beta z} \sin\beta z$$

or

$$\frac{dB_{\beta z}}{dz} = \beta e^{-\beta z}(\cos\beta z - \sin\beta z) = \beta C_{\beta z}$$

$$\frac{dC_{\beta z}}{dz} = \frac{d}{dz}\left[e^{-\beta z}(\cos\beta z - \sin\beta z)\right] = -2\beta e^{-\beta z} \cos\beta z = -2\beta D_{\beta z}$$

$$\frac{dD_{\beta z}}{dz} = \frac{d}{dz}\left(e^{-\beta z} \cos\beta z\right) = -\beta e^{-\beta z}(\sin\beta z + \cos\beta z) = -\beta A_{\beta z}$$

10.4 By Eqs. (10.12) and (10.17), with Eq. (10.5),

$$\theta = \frac{dy}{dz} = \frac{P\beta}{2k}\frac{dA_{\beta z}}{dz} = \frac{P\beta}{2k}(-2\beta B_{\beta z}) = -\frac{P\beta^2}{k}B_{\beta z} \qquad (a)$$

Similarly, by Eqs. (a) and (10.14), with Eq. (10.17)

$$M_x = -EI_x\frac{d\theta}{dz} = EI_x\frac{P\beta^2}{k}\frac{dB_{\beta z}}{dz} = \frac{k}{4\beta^4}\left(\frac{P\beta^2}{k}\right)(\beta C_{\beta z}) \qquad (b)$$

or $$M_x = \frac{P}{4\beta}C_{\beta z}$$

and by Eqs. (10.15) and (b), with Eq. (10.17),

$$V_y = \frac{dM_x}{dz} = \frac{P}{4\beta}\frac{dC_{\beta z}}{dz} = \frac{P}{4\beta}(-2\beta D_{\beta z}) = -\frac{P}{2}D_{\beta z}$$

10.5 By Eqs. (10.17),

$$\frac{d^2A_{\beta z}}{dz^2} = \frac{d}{dz}(-2\beta B_{\beta z}) = -2\beta\frac{dB_{\beta z}}{dz} = -2\beta^2C_{\beta z}$$

$$\frac{d^2B_{\beta z}}{dz^2} = \frac{d}{dz}(\beta C_{\beta z}) = \beta\frac{dC_{\beta z}}{dz} = -2\beta^2D_{\beta z}$$

$$\frac{d^2C_{\beta z}}{dz^2} = \frac{d}{dz}(-2\beta D_{\beta z}) = -2\beta\frac{dD_{\beta z}}{dz} = 2\beta^2A_{\beta z}$$

$$\frac{d^2D_{\beta z}}{dz^2} = \frac{d}{dz}(-\beta A_{\beta z}) = -\beta\frac{dA_{\beta z}}{dz} = 2\beta^2B_{\beta z}$$

These results may also be obtained by the length process of direct differentiation of Eqs. (10.16).

10.6

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{7}{4(200{,}000)(36{,}900{,}000)}} = 0.000698 \text{ mm}^{-1}$$

$$y_{max} = \frac{P\beta}{2k} = \frac{170{,}000(0.000698)}{2(7)} = 8.48 \text{ mm}; \quad M_{max} = \frac{P}{4\beta} = \frac{170{,}000}{4(0.000698)} = 60.9 \text{ kN.m}$$

Percentage increase in $y_{max} = 100\,\frac{8.48-5.04}{5.04} = \underline{68.3}$

Percentage increase in $M_{max} = 100\,\frac{60.9-51.2}{51.2} = \underline{18.9}$

10.7 $k = 76\,k_0 = 76(0.270) = 20.52 \text{ N/mm}^2$

$$\beta = \sqrt[4]{\frac{20.52}{4(200{,}000)(5{,}120{,}000)}} = 0.001496 \text{ mm}^{-1}; \quad L = 4000 \text{ mm} > \frac{3\pi}{2\beta} = 3150 \text{ mm}$$

$$y_{max} = \frac{P\beta}{2k} = \frac{60{,}000(0.001496)}{2(20.52)} = \underline{2.187 \text{ mm}}$$

$$M_{max} = \frac{P}{4\beta} = \frac{60{,}000}{4(0.001496)} = 10.03 \text{ kN.m}$$

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{10{,}030{,}000(63.5)}{5{,}120{,}000} = \underline{124.4 \text{ MPa}}$$

10.8

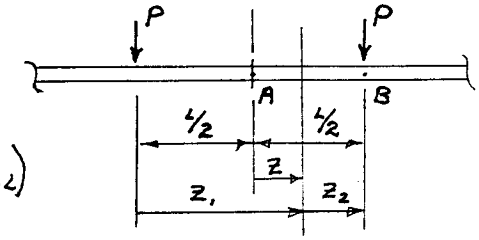
$$\beta = \sqrt[4]{\frac{20.52}{4(72{,}000)(5{,}120{,}000)}} = 0.001931 \text{ mm}^{-1}$$

$$y_{max} = \frac{60{,}000(0.001931)}{2(20.52)} = \underline{2.823 \text{ mm}}$$

$$M_{max} = \frac{60{,}000}{4(0.001931)} = 7.77 \text{ kN.m}$$

$$\sigma_{max} = \frac{7{,}770{,}000(63.5)}{5{,}120{,}000} = \underline{96.3 \text{ MPa}}$$

10.9



a) $$Y_A = 2Y\Big|_{z=L/2} = 2\left(\frac{P\beta}{2k}\right)A_{\beta z}\Big|_{z=L/2} = 2\left(\frac{P\beta}{2k}\right)A_{\beta L/2}$$

$$Y_B = Y\Big|_{z=0} + Y\Big|_{z=L} = \frac{P\beta}{2k}\left(A_{\beta 0} + A_{\beta L}\right) = \frac{P\beta}{2k}\left(1 + A_{\beta L}\right)$$

IF $Y_A = Y_B$ THEN $2A_{\beta L/2} = 1 + A_{\beta L}$

$$2e^{-\beta L/2}\left(\sin\beta L/2 + \cos\beta L/2\right) = 1 + e^{-\beta L}\left(\sin\beta L + \cos\beta L\right)$$

BY NUMERICAL STUDY, THE ROOTS ARE $\beta L = 0$ AND $\beta L = 1.8585$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ └ TRIVIAL SOLUTION

$\underline{\underline{\therefore L = 1.8585/\beta}}$

(cont.)

10.9 (CONTINUED)

b) IF $P=1$, $EI=1$, AND $k=1$, THEN $\beta = \frac{1}{\sqrt{2}}$

LET z_1 = DISTANCE FROM LEFT LOAD TO POINT @ z,
z_2 = DISTANCE FROM RIGHT LOAD TO POINT @ z.

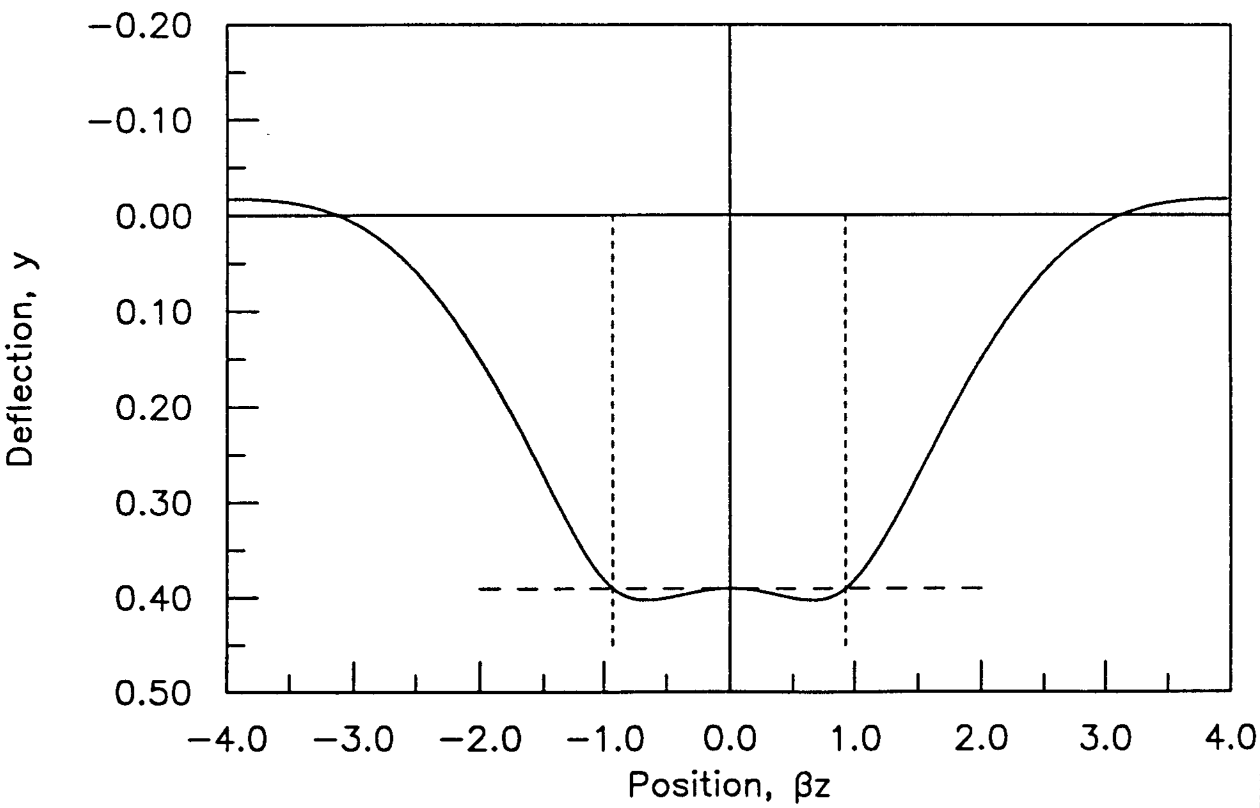
$0 \le z \le L/2 :\quad z_1 = L/2 + z \quad , \quad z_2 = L/2 - z$

$z > L/2 :\quad z_1 = L/2 + z \quad , \quad z_2 = z - L/2$

$y(z) = y(z_1) + y(z_2)$

$$\underline{\underline{y(z) = \frac{\beta}{2}\left[e^{-\beta z_1}(\sin\beta z_1 + \cos\beta z_1) + e^{-\beta z_2}(\sin\beta z_2 + \cos\beta z_2)\right]}}$$

c) Deflection Plot, Problem 10.9



10.10 $\beta = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{17}{4(200,000)\,36.9\times10^{6}}} = 0.0008299\ mm^{-1}$

FOR A SINGLE LOAD, $y = 0$ @ $\beta z = 3\pi/4$ $\Rightarrow$ $z = \frac{3\pi}{4\beta} = 2839\ mm$

IF WHEELS ARE SPACED @ $> 2z$, THEN THE TRACK LIFTS UP.

$\therefore$ LIMIT WHEEL SPACING TO $\underline{\underline{S \approx 5.5\ m < 2z}}$.

ERRATA: IN THE TEXT, THE MOMENT OF INERTIA OF THE TRAIN RAIL SHOULD BE $\underline{\underline{I = 36.9\times10^{6}\ mm^{4}}}$.

10.11 $I_x = \frac{200(300)^3}{12} = 450,000,000\ mm^4;\quad P = \frac{60,000(9.81)}{4} = 147.2\ kN$

$k = 200\,k_0 = 200(0.029) = 5.80\ N/mm^2$

$\beta = \sqrt[4]{\frac{5.80}{4(12,400)(450,000,000)}} = 0.000714\ mm^{-1}$

$\beta z_1 = 0.000714(1500) = 1.071;\quad A_{\beta z_1} = 0.4651;\quad C_{\beta z_1} = -0.1363$

$y_{max} = \frac{P\beta}{2k}\left[1 + A_{\beta z_1}\right] = \frac{147,200(0.000714)(1+0.4651)}{2(5.80)} = \underline{13.27\ mm}$

$M_{max} = \frac{P}{4\beta}\left[1 + C_{\beta z_1}\right] = \frac{147,200(1-0.1363)}{4(0.000714)} = 44.52\ kN.m$

$\sigma_{max} = \frac{44,520,000(150)}{450,000,000} = \underline{14.84\ MPa}$

$L \geq = 1500 + \frac{3\pi}{2\beta} = 1500 + \frac{3\pi}{2(0.000714)} = 8100\ mm$

$= \underline{8.10\ m}$

10.12 Let Q be the load in one of the rods of length L'.

$$e = \frac{QL'}{AE} = \frac{Q}{K}; \quad K = \frac{AE}{L'} = \frac{\pi(18)^2(200{,}000)}{4(2500)} = 20{,}360 \text{ N/mm}$$

$$k = \frac{K}{L} = \frac{20{,}360}{500} = 40.7 \text{ N/mm}^2$$

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{40.7}{4(200{,}000)(11{,}000{,}000)}} = 0.001467 \text{ mm}^{-1}; \quad \ell = \underline{500} < \frac{\pi}{4\beta} = \underline{536 \text{ mm}}$$

$$y_{max} = \frac{P\beta}{2k} = \frac{60{,}000(0.001467)}{2(40.7)} = 1.081 \text{ mm}; \quad Q_{max} = Ky_{max} = 20{,}360(1.081) = 22.02 \text{ kN}$$

$$\sigma_{max}(\text{rod}) = \frac{Q_{max}}{A} = \frac{22{,}020(4)}{\pi(18)^2} = 86.5 \text{ MPa}$$

$$M_{max} = \frac{P}{4\beta} = \frac{60{,}000}{4(0.001467)} = 10.22 \text{ kN.m}$$

$$\sigma_{max}(\text{beam}) = \frac{M_{max}c}{I_x} = \frac{10{,}220{,}000(76)}{11{,}000{,}000} = \underline{70.6 \text{ MPa}}$$

10.13
$$e = Q\left[\frac{L_1'}{A_1E_1} + \frac{L_2'}{A_2E_2}\right] = Q\left[\frac{2500(4)}{\pi(18)^2(200{,}000)} + \frac{800(4)}{\pi(18)^2(72{,}000)}\right] = 0.0000928Q \text{ (mm)}$$

$$K = \frac{Q}{e} = \frac{1}{0.0000928} = 10{,}780 \text{ N/mm}; \quad k = \frac{K}{\ell} = \frac{10{,}780}{500} = 21.56 \text{ N/mm}^2$$

$$\beta = \sqrt[4]{\frac{21.56}{4(200{,}000)(11{,}000{,}000)}} = 0.001251 \text{ mm}^{-1}; \quad \ell = 500 < \frac{\pi}{4\beta} = 628 \text{ mm}$$

$$y_{max} = \frac{P\beta}{2k} = \frac{60{,}000(0.001251)}{2(21.56)} = 1.74 \text{ mm}; \quad Q_{max} = Ky_{max} = 10{,}780(1.74) = 18.77 \text{ kN}$$

$$\sigma_{max}(\text{rod}) = \frac{Q_{max}}{A} = \frac{18{,}770(4)}{\pi(18)^2} = \underline{73.7 \text{ MPa}}$$

$$M_{max} = \frac{P}{4\beta} = \frac{60{,}000}{4(0.001251)} = 11.99 \text{ kN.m}$$

$$\sigma_{max}(\text{beam}) = \frac{M_{max}c}{I_x} = \frac{11{,}990{,}000(76)}{11{,}000{,}000} = \underline{82.8 \text{ MPa}}$$

10.14 $I_x = \frac{60(200)^3}{12} = 40{,}000{,}000 \text{ mm}^4$

$$K = k_0(\text{contact area}) = 0.330(100)(60) = 1980 \text{ N/mm}$$

$$k = \frac{K}{\ell} = \frac{1980}{600} = 3.30 \text{ N/mm}^2$$

$$\beta = \sqrt[4]{\frac{3.30}{4(12{,}400)(40{,}000{,}000)}} = 0.001136 \text{ mm}^{-1}; \ell = \underline{600} < \frac{\pi}{4\beta} = \underline{692 \text{ mm}}; \text{ yes}$$

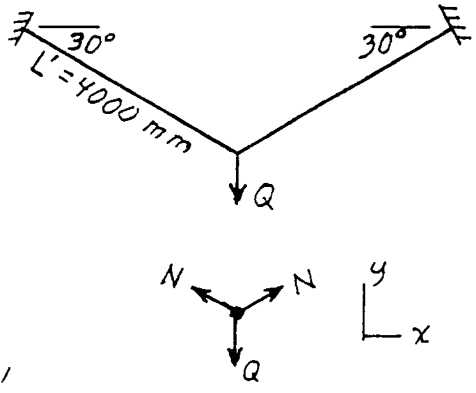
$$M_{max} = \frac{\sigma_{max}I_x}{c} = \frac{40.0(40{,}000{,}000)}{100} = 16{,}000{,}000 = SF\frac{P}{4\beta}$$

$$P = \frac{16{,}000{,}000(4)(0.001136)}{2.50} = \underline{29.08 \text{ kN}}$$

$$y_{max} = \frac{P\beta}{2k} = \frac{29{,}080(0.001136)}{2(3.30)} = 5.01 \text{ mm}$$

Maximum pressure $= y_{max}k_0 = 5.01(0.330) = \underline{1.652 \text{ MPa}}$

10.15 $\Sigma F_y = 0 = 2N \sin 30° - Q; \quad Q = N$



$$I_x = \frac{\pi(50)^4}{64} = 306{,}800 \text{ mm}^4$$

$$q_Q = \frac{\partial U}{\partial Q} = 2\frac{NL'}{AE}\frac{\partial N}{\partial Q} = \frac{2QL'}{AE} = \frac{Q}{K}$$

$$K = \frac{AE}{2L'} = \frac{\pi(2)^2(200{,}000)}{4(2)(4000)} = 78.54 \text{ N/mm}$$

$$k = \frac{K}{\ell} = \frac{78.54}{900} = 0.0873 \text{ N/mm}^2$$

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{0.0873}{4(200{,}000)(306{,}800)}} = 0.000772 \text{ mm}^{-1}$$

(a) Assume beam fails first.

$$\sigma_{max} = Y = SF\frac{M_{max}c}{I_x}; \quad M_{max} = \frac{300(306{,}800)}{2.00(25)} = 1{,}841{,}000 \text{ N.mm} = \frac{P}{4\beta}$$

$P = 4(0.000772)(1{,}841{,}000) = 5.68$ kN; Assume wire fails first.

$$\sigma_{max} = Y = SF\frac{N}{A}; \quad N = Q = \frac{1200\pi(2)^2}{2.00(4)} = 1885 \text{ N}$$

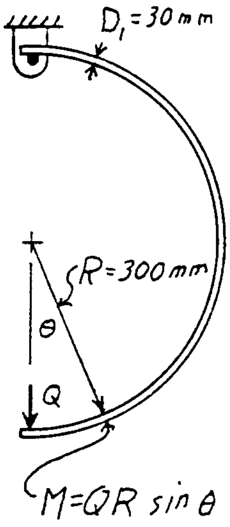
$$y_{max} = \frac{P\beta}{2k} = \frac{P(0.000772)}{2(0.0873)} = 0.004422P; \quad Q = Ky_{max} = 78.54(0.004422P) = 0.3473P$$

$P = \frac{1885}{0.3473} = \underline{5.43 \text{ kN}}$. This is allowable load.

(b) $\ell = \underline{800} < \frac{\pi}{4\beta} = \underline{1017 \text{ mm}}$ $\quad$ Yes

---

10.16 $A_1 = \frac{\pi D_1^2}{4} = \frac{\pi(30)^2}{4} = 706.9 \text{ mm}^2; \quad I_{x1} = \frac{\pi D_1^4}{64} = \frac{\pi(30)^4}{64} = 39{,}760 \text{ mm}^4$



$$q_Q = \frac{\partial U}{\partial Q} = \int_0^{\pi}\frac{M}{E_1 I_{x1}}\frac{\partial M}{\partial Q}R\,d\theta = \frac{QR^3}{E_1 I_{x1}}\int_0^{\pi}\sin^2\theta\,d\theta = \frac{QR^3\pi}{2E_1 I_{x1}} = \frac{Q}{K}$$

$$K = \frac{2E_1 I_{x1}}{\pi R^3} = \frac{2(200{,}000)(39{,}760)}{\pi(300)^3} = 187.5 \text{ N/mm}$$

$$k = \frac{K}{\ell} = \frac{187.5}{550} = 0.3409 \text{ N/mm}^2; \quad I_x = \frac{\pi D^4}{64} = \frac{\pi(40)^4}{64} = 125{,}700 \text{ mm}^4$$

$$\beta = \sqrt[4]{\frac{0.3409}{4(200{,}000)(125{,}700)}} = 0.001357 \text{ mm}^{-1}$$

Maximum stress in beam.

$$M_{max} = \frac{P}{4\beta} = \frac{3000}{4(0.001357)} = 552{,}700 \text{ N.mm}$$

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{552{,}700(20)}{125{,}700} = \underline{87.94 \text{ MPa}}$$

Maximum stress in curved beams.

$$y_{max} = \frac{P\beta}{2k} = \frac{3000(0.001357)}{2(0.3409)} = 5.971 \text{ mm}; \quad Q = Ky_{max} = 187.5(5.971) = 1120 \text{ N}$$

$$\sigma_{max} = \frac{Q}{A} + \frac{QRc}{I_{x1}} = \frac{1120}{706.9} + \frac{1120(300)(15)}{39{,}760} = \underline{128.3 \text{ MPa}}$$

10.17

$$q_Q = \frac{QL'^3}{48E_1I_{x1}} = \frac{Q}{K}; \quad K = \frac{48E_1I_{x1}}{L'^3} = \frac{48(200{,}000)(24{,}000{,}000)}{3000^3} = 8533 \text{ N/mm}$$

$$k = \frac{K}{\ell} = \frac{8533}{600} = 14.22 \text{ N/mm}^2$$

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{14.22}{4(200{,}000)(24{,}000{,}000)}} = 0.000928 \text{ mm}^{-1}; \quad \ell = 600 < \frac{\pi}{4\beta} = 846 \text{ mm}$$

In long beam, $M_{max} = \frac{P}{4\beta} = \frac{90{,}000}{4(0.000928)} = 24.25$ kN.m

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{24{,}250{,}000(101.5)}{24{,}000{,}000} = \underline{102.6 \text{ MPa}}$$

$$y_{max} = \frac{P\beta}{2k} = \frac{90{,}000(0.000928)}{2(14.22)} = 2.937 \text{ mm}; \quad Q = Ky_{max} = 8533(2.937) = 25.06 \text{ kN}$$

In cross beams, $M'_{max} = \frac{QL'}{4} = \frac{25{,}060(3000)}{4} = 18.80$ kN.m

$$\sigma_{max} = \frac{M'_{max}c'}{I_{x1}} = \frac{18{,}800{,}000(101.5)}{24{,}000{,}000} = \underline{79.5 \text{ MPa}}$$

10.18

$$K = \frac{2E_1I_{x1}}{\pi R^3} = \frac{2(72{,}000)(39{,}760)}{\pi(300)^3} = 67.50 \text{ N/mm}$$

$$k = \frac{K}{\ell} = \frac{67.50}{550} = 0.1227 \text{ N/mm}^2$$

$$\beta = \sqrt[4]{\frac{0.1227}{4(200{,}000)(125{,}700)}} = 0.001051 \text{ mm}^{-1}; \quad \ell = 550 < \frac{\pi}{4\beta} = 747 \text{ mm}$$

Maximum stress in beam

$$M_{max} = \frac{P}{4\beta} = \frac{3000}{4(0.001051)} = 713{,}600 \text{ N.mm}$$

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{713{,}600(20)}{125{,}700} = \underline{113.5 \text{ MPa}}$$

Maximum stress in curved beams.

$$y_{max} = \frac{P\beta}{2k} = \frac{3000(0.001051)}{2(0.1227)} = 12.85 \text{ mm}; \quad Q = Ky_{max} = 67.50(12.85) = 867.4 \text{ N}$$

$$\sigma_{max} = \frac{Q}{A} + \frac{QRc}{I_{x1}} = \frac{867.4}{706.9} + \frac{867.4(300)(15)}{39{,}760} = \underline{99.4 \text{ MPa}}$$

10.19

$$\beta = \sqrt[4]{\frac{14.22}{4(72{,}000)(24{,}000{,}000)}} = 0.001198 \text{ mm}^{-1}; \quad \ell = 600 < \frac{\pi}{4\beta} = 656 \text{ mm}$$

In long beam, $M_{max} = \frac{P}{4\beta} = \frac{90{,}000}{4(0.001198)} = 18.78$ kN.m

$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{18{,}780{,}000(101.5)}{24{,}000{,}000} = \underline{79.4 \text{ MPa}}$; In cross beams

$$y_{max} = \frac{P\beta}{2k} = \frac{90{,}000(0.001198)}{2(14.22)} = 3.791 \text{ mm}; \quad Q = Ky_{max} = 8533(3.791) = 32.35 \text{ kN}$$

$$M'_{max} = \frac{QL'}{4} = \frac{32{,}350(3000)}{4} = 24.26 \text{ kN.m}; \quad \sigma_{max} = \frac{M'_{max}c'}{I_{x1}} = \frac{24{,}260{,}000(101.5)}{24{,}000{,}000} = \underline{102.6 \text{ MPa}}$$

10.20

SOLUTION IS PRESENTED FOR TWO DIFFERENT METHODS:

A) SPECIALIZE THE GENERAL SOLUTION, EQ. (10.7), FOR THE APPROPRIATE BOUNDARY CONDITIONS; SEE SEC. 10.2.

B) APPLY THE MOMENT AS A COUPLE OF TWO FORCES AND USE SUPERPOSITION.

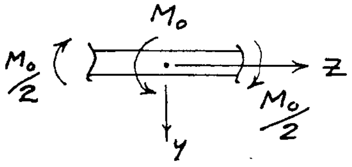
METHOD A:

ANTISYMMETRY CONDITIONS REQUIRE THAT:

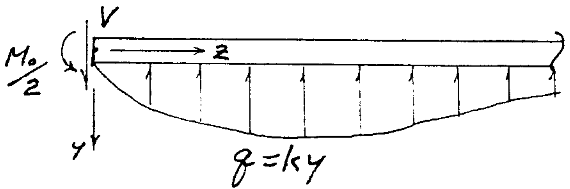
a) DEFLECTION AT THE LOAD POINT IS ZERO.

b) HALF OF THE APPLIED MOMENT IS RESISTED BY THE ELASTIC FOUNDATION UNDER HALF OF THE BEAM FOR WHICH $z \geq 0$.

F.B.D. OF LOADED SEGMENT



F.B.D. OF RIGHT HALF OF BEAM



GENERAL SOLUTION [EQ.(10.7)]: $y(z) = e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z)$

CONDITION a): $@ z=0,\ y=0 \Rightarrow 0 = e^0(C_3 \sin 0 + C_4 \cos 0)$

$\therefore\ C_4 = 0$ AND $y(z) = C_3 e^{-\beta z} \sin\beta z$

CONDITION b): FOR EQUILIBRIUM $\Sigma M = 0 \Rightarrow M_0/2 + \int_0^\infty q z\, dz = 0$

$$M_0/2 = -\int_0^\infty k z C_3 e^{-\beta z} \sin\beta z\, dz.$$

USE INTEGRATION BY PARTS, TABLES, OR SOFTWARE SUCH AS DERIVE AND MATHCAD TO EVALUATE THE INTEGRAL.

$$M_0/2 = -k C_3 \left(\frac{1}{2\beta^2}\right) \Rightarrow C_3 = \frac{-M_0\beta^2}{k}$$

$$\therefore\ \underline{\underline{y(z) = \frac{-M_0\beta^2}{k} e^{-\beta z} \sin\beta z = \frac{-M_0\beta^2}{k} B_{\beta z}}}\ ;$$

$$\underline{\underline{\theta(z) = dy/dz = \frac{-M_0\beta^3}{k} C_{\beta z}}}\ ; \qquad \underline{\underline{M_x(z) = -EI\, d^2y/dz^2 = \frac{-M_0}{2} D_{\beta z}}}\ ;$$

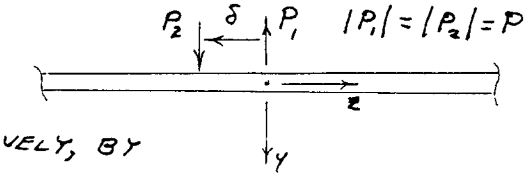
$$\underline{\underline{V_y(z) = \frac{dM_x}{dz} = \frac{M_0\beta}{2} A_{\beta z}}}.$$

(cont.)

10.20 (CONTINUED)

METHOD B:

CONSIDER TWO LOADS P_1, P_2 AS SHOWN:



DEFLECTIONS DUE TO P_1 & P_2 ARE, RESPECTIVELY, BY Eq(10.12):

$$y_1 = \frac{-P_1\beta}{2k} A_{\beta z} \ , \quad y_2 = \frac{-P\beta}{2k} A_{\beta(z+\delta)}$$

TOTAL DEFLECTION, BY SUPERPOSITION IS $y = \frac{P\beta}{2k}\left[A_{\beta(z+\delta)} - A_{\beta z}\right]$

LIKEWISE, FROM Eq. (10.13), (10.14), & (10.15):

$$\theta = -P\beta^2/k\left[B_{\beta(z+\delta)} - B_{\beta z}\right]$$

$$M_x = P/4\beta\left[C_{\beta(z+\delta)} - C_{\beta z}\right]$$

$$V_y = -P/2\left[D_{\beta(z+\delta)} - D_{\beta z}\right]$$

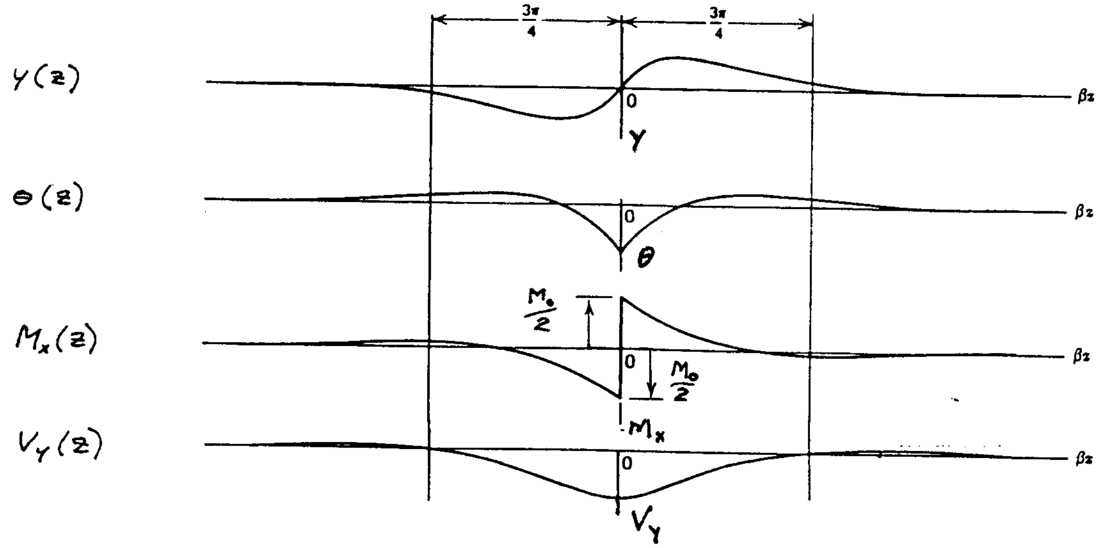
LET $\delta \to 0$ AND $P\delta = M_0$. THEN FOR $z \geq 0$ AND FROM THE DEFINITION OF A DERIVATIVE AS A LIMIT:

$$y(z) = \lim_{\delta\to 0} \frac{(P\delta)\beta}{2k}\left[\frac{A_{\beta(z+\delta)} - A_{\beta z}}{\delta}\right] = \frac{M_0\beta}{2k}\frac{dA_{\beta z}}{dz} = \underline{\underline{\frac{-M_0\beta^2}{k} B_{\beta z}}} \ .$$

$$\theta(z) = \lim_{\delta\to 0} \frac{-(P\delta)\beta^2}{k}\left[\frac{B_{\beta(z+\delta)} - B_{\beta z}}{\delta}\right] = \frac{-M_0\beta^2}{k}\frac{dB_{\beta z}}{dz} = \underline{\underline{\frac{-M_0\beta^3}{k} C_{\beta z}}}$$

$$M_x(z) = \lim_{\delta\to 0} \frac{(P\delta)}{4\beta}\left[\frac{C_{\beta(z+\delta)} - C_{\beta z}}{\delta}\right] = \frac{M_0}{4\beta}\frac{dC_{\beta z}}{dz} = \underline{\underline{-\frac{M_0}{2} D_{\beta z}}}$$

$$V_y(z) = \lim_{\delta\to 0} \frac{-(P\delta)}{2}\left[\frac{D_{\beta(z+\delta)} - D_{\beta z}}{\delta}\right] = \frac{-M_0}{2}\frac{dD_{\beta z}}{dz} = \underline{\underline{\frac{M_0\beta}{2} A_{\beta z}}}$$



10.21 By Eq. (10.23), the beam displacement under a uniform load is

$$y_H = \frac{w}{2k}\left(2 - e^{-\beta a}\cos\beta a - e^{-\beta b}\cos\beta b\right) \quad (a)$$

Since to the left of point H, $a \to \infty$ and to the right of point H, $b \to \infty$ (see Fig. 10.7), $e^{-\beta a} \to 0$ and $e^{-\beta b} \to 0$. Therefore, Eq. (a) yields

$$y_H = \frac{w}{k} \quad (b)$$

for all points in the beam; that is, the beam is pushed down as a rigid body. Similarly by Eqs. (10.25), (10.26), and (10.27), as $d \to \infty$ and $b \to \infty$

$$\theta_H \to 0, \quad M_H \to 0, \quad \text{and} \quad V_H \to 0.$$

10.22 Consider the equilibrium of an element dz of the beam in the loaded region (Fig. a). By Fig. a,

$$\Sigma F_y = 0: \ \frac{dV_y}{dz} = ky - w(z) \quad (a)$$

$$\Sigma M_0 = 0: \ V_y = \frac{dM_x}{dz} \quad (b)$$

But by Eq. (10.1),

$$M_x = -EI_x \frac{d^2y}{dz^2} \quad (c)$$

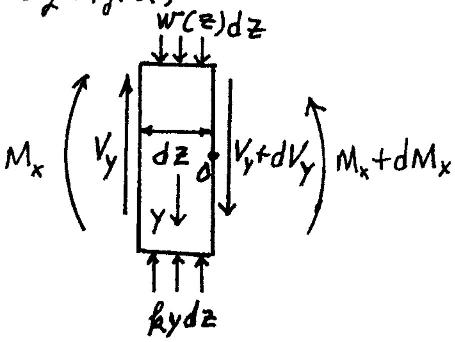


Figure a

Hence, by Eqs. (a), (b), and (c),

$$\frac{dV_y}{dz} = \frac{d^2M_x}{dz^2} = -EI_x\frac{d^4y}{dz^4} = ky - w(z)$$

or

$$EI_x \frac{d^4y}{dz^4} + ky = w(z)$$

10.23 The beam and load is shown in Fig. a. The maximum design pressure at the center of the loaded segment is

$$p_{max} = k y_H(l/2) = 15 \text{ N/mm} \qquad (a)$$

By Eq. (10-23),

$$y_H\left(\frac{l}{2}\right) = \frac{w}{2k}\left(2 - e^{-\beta a}\cos\beta a - e^{-\beta b}\cos\beta b\right)$$

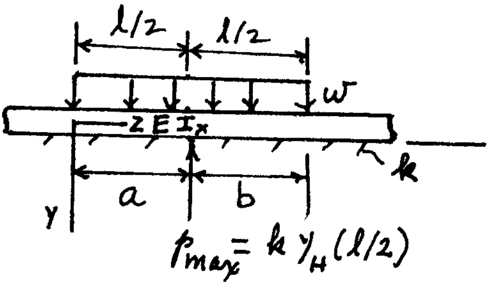


Figure a

or since $w = 60$ N/mm and $a = b = \frac{l}{2}$ (Fig. a),

$$y_H\left(\frac{l}{2}\right) = \frac{60}{k}\left(1 - e^{-\beta\frac{l}{2}}\cos\beta\frac{l}{2}\right) \qquad (b)$$

Hence, by Eqs. (a) and (b), $p_{max} = k y_H = 60\left(1 - e^{-\beta\frac{l}{2}}\cos\beta\frac{l}{2}\right) = 15$ or

$$e^{-\beta\frac{l}{2}}\cos\beta\frac{l}{2} = 0.75 \qquad (c)$$

The solution of Eq. (c) is

$$\beta\frac{l}{2} = 0.28767 \qquad (d)$$

Then, by Eq. (10.5),

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} \qquad (e)$$

Substituting Eq. (e) into Eq. (d) and solving for I_x, we obtain

$$I_x = 2.282\,\frac{kl^4}{E}$$

10.24 Refer to Fig. a of Problem 10.23 with $p_{max} = w/8$. Then,

$$p_{max} = \frac{w}{8} = k y_H(l/2) = w\left(1 - e^{-\beta\frac{l}{2}}\cos\beta\frac{l}{2}\right), \text{ or}$$

$$e^{-\beta l/2}\cos\beta\frac{l}{2} = 0.875 \qquad (a)$$

(cont.)

10.24 cont. The solution of Eq. (a) is

$$\beta \frac{l}{2} = \sqrt[4]{\frac{k}{4EI_x}} \cdot \left(\frac{l}{2}\right) = 0.13353 \quad (b)$$

Solving Eq. (b) for I_x, we obtain

$$I_x = 49.148 \frac{kl^4}{E}$$

10.25 The beam and load is shown in Fig. a. By Figs. a and 10.8, for point A, with $L' = 4m$,

$a = 0, b = L' = 4\ m$ (a)

Also, with $E = 200$ GPa, $I_x = 80 \times 10^6\ mm^4$, and $k = 8.0\ N/mm^2$, we find

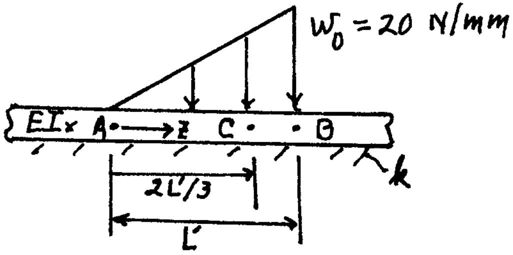


Figure a

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = 0.5946\ m^{-1} \quad (b)$$

Therefore, by Eqs. (a) and (b),

$$\beta a = 0, \quad \beta b = 2.3784 \quad (c)$$

Then, by Table 10.1, with $H = A$, and Eqs. (c), (10.29), and (10.31),

$$y_A = \frac{w_0}{4\beta k} \frac{1}{L'}(C_{\beta a} - C_{\beta b} - 2\beta L' D_{\beta b} + 4\beta a)$$

or

$$y_A = 0.26278(1 + 0.1312 + 0.31776 + 0) = 0.3808\ mm$$

$$M_A = -\frac{w_0}{8\beta^3} \frac{1}{L'}(A_{\beta a} - A_{\beta b} - 2\beta L' B_{\beta b})$$

or

$$M_A = -2.9731 \times 10^6 (1 + 0.002838 - 0.3052) = 2.074\ kN \cdot m$$

10.26 Referring to Problem 10.25, we have

$$L' = 4\,\text{m}, \quad E = 200\,\text{GPa}, \quad I_x = 80 \times 10^6\,\text{mm}^4$$

$$w_0 = 20\,\text{N/mm}, \text{ and } k = 8.0\,\text{N/mm}^2 \qquad (a)$$

Hence,

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = 5.946 \times 10^{-4}\,\text{mm}^{-1} \qquad (b)$$

For point B, Fig. a,

$$a = L' = 4000\,\text{mm}, \quad b = 0 \qquad (c)$$

By Eqs. (b) and (c),

$$\beta a = 2.3784, \quad \beta b = 0 \qquad (d)$$

Then, by Table 10.1, Eqs. (d), (10.30), and (10.32), with B = H,

$$\theta_B = -\frac{w_0}{2k}\frac{1}{L'}\left(D_{\beta a} - D_{\beta b} - \beta L' A_{\beta b} + 2\beta a\right)$$

or

$$\theta_B = -0.0003125(-0.0668 - 1 + 0.00675 + 4.7568) = -0.001155 \text{ rad}$$

$$V_B = \frac{w_0}{4\beta^2}\cdot\frac{1}{L'}\left(B_{\beta a} - B_{\beta b} + \beta L' C_{\beta b} - 2\beta a\right)$$

or

$$V_B = 3535.58(0.06416 - 0 - 0.31205 - 4.7568) = -17.69\,\text{kN}$$

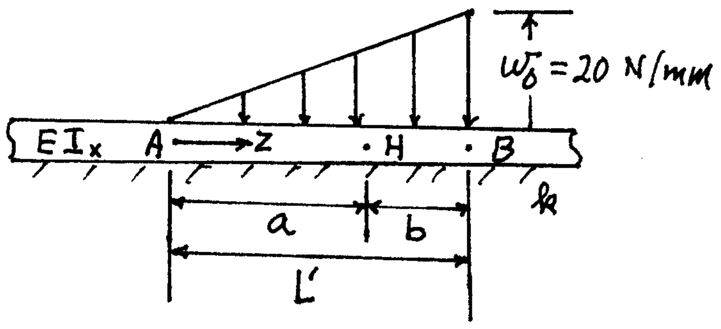


Figure a

$\boxed{10.27}$ Referring to Problem 10.25, we have

$$L' = 4\,m,\ E = 200\,GPa,\ I_x = 80\times10^6\ mm^4,$$
$$w_0 = 20\ N/mm,\ \text{and}\ k = 8.0\ N/mm^2$$

Hence,

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = 5.9460\times10^{-4}\ mm^{-1} \qquad (a)$$

For point C (Fig. a), under the centroid of the load,

$$a = \frac{2}{3}L' = 2666.7\ mm,\ b = 1333.3\ mm \qquad (b)$$

By Eqs. (a) and (b)

$$\beta a = 1.58562,\ \beta b = 0.79278 \qquad (c)$$

Then, by Table 10.1, Eqs. (c), (10.31), and (10.32), with H = C,

$$M_C = -\frac{w_0}{8\beta^3}\cdot\frac{1}{L'}\left(A_{\beta a} - A_{\beta b} - 2\beta L' B_{\beta b}\right)$$

or

$$M_C = -2.9731\times10^6(0.20186 - 0.63943 - 1.53312) = 5.859\ kN\cdot m$$

$$V_C = \frac{w_0}{4\beta^2}\cdot\frac{1}{L'}\left(B_{\beta a} - B_{\beta b} + \beta L' C_{\beta b} - 2\beta a\right)$$

or

$$V_C = 3535.58(0.2048 - 0.3223 - 0.01250 - 3.1712) = -11.67\ kN$$

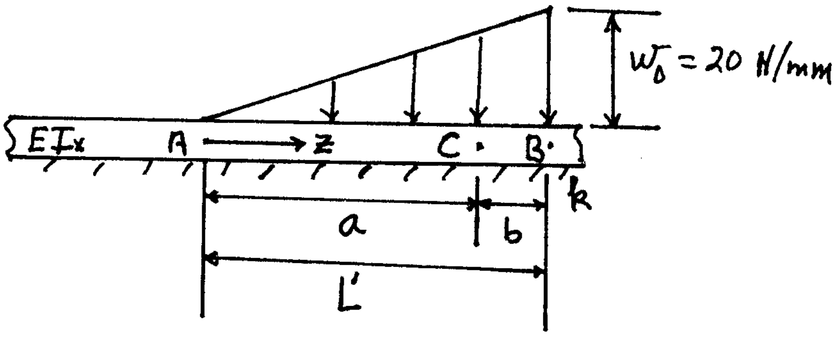


Figure a

10.28 $L' = 1000 \text{ mm}; \quad \beta = 0.001496 \text{ mm}^{-1}; \quad k = 20.52 \text{ N/mm}$

$$w = \frac{P}{L'} = \frac{60{,}000}{1000} = 60.0 \text{ N/mm}$$

$$\beta L' = 0.001496(1000) = 1.496 < \pi$$

The maximum deflection and maximum flexure stress occur at the center of L'.

$$\beta a = \beta b = 0.001496(500) = 0.748; \quad D_{\beta a} = 0.3470; \quad B_{\beta a} = 0.3219$$

$$y_{max} = \frac{w}{k}(1 - D_{\beta a}) = \frac{60.0}{20.52}(1 - 0.3470) = \underline{1.909 \text{ mm}}$$

$$M_{max} = \frac{w}{2\beta^2} B_{\beta a} = \frac{60.0(0.3219)}{2(0.001496)^2} = 4.315 \text{ kN.m}$$

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{4{,}315{,}000(63.5)}{5{,}120{,}000} = \underline{53.5 \text{ MPa}}$$

10.29 $\beta = 0.001136 \text{ mm}^{-1}; \quad \beta L' = 0.001136(3000) = 3.408 > \pi$

The maximum bending moment occurs outside the distributed load at a location for which βa is approximately equal to $\frac{\pi}{4}$

$$\beta a = \frac{\pi}{4}; \quad \beta b = \beta a + \beta L' = 4.193; \quad B_{\beta a} = 0.3224; \quad B_{\beta b} = -0.0127$$

$$\sigma_{max} = SF \frac{M_{max}c}{I_x} = Y; \quad M_{max} = \frac{40(40{,}000{,}000)}{2.00(100)} = 8.00 \text{ kN.m}$$

$$M_{max} = -\frac{w}{4\beta^2}(B_{\beta a} - B_{\beta b}) = -\frac{w[0.3224 - (-0.0127)]}{4(0.001136)^2} = -64{,}920\, w$$

$$w = \frac{-8{,}000{,}000}{-64{,}920} = \underline{123.2 \text{ N/mm}}$$

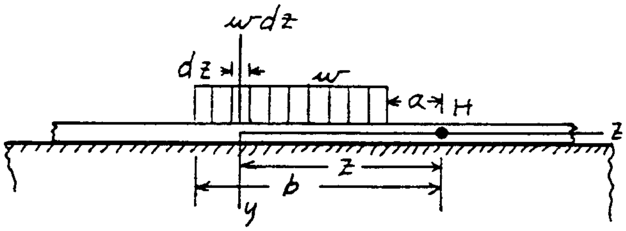
10.30

$$y_H = \int_a^b \frac{w\beta}{2k} e^{-\beta z}(\cos \beta z + \sin \beta z)\, dz$$

$$= \underline{\frac{w}{2k}(D_{\beta b} - D_{\beta a})}$$

$$M_H = \int_a^b \frac{w}{4\beta} e^{-\beta z}(\cos \beta z - \sin \beta z)\, dz$$

$$= \underline{-\frac{w}{4\beta^2}(B_{\beta a} - B_{\beta b})}$$



10.31 $k = 20.52\ N/mm^2$; $\beta = 0.001496\ mm^{-1}$

$$y_{max} = \frac{2P\beta}{k} = \frac{2(60{,}000)(0.001496)}{20.52} = \underline{8.75\ mm}$$

The maximum moment occurs where $V_y = 0$ or where $C_{\beta a} = 0$.

In Table 10.1, $C_{\beta a} = 0$ where $\beta a = \frac{\pi}{4}$. $B_{\beta a} = 0.3224$

$$a = \frac{\pi}{4\beta} = \frac{\pi}{4(0.001496)} = \underline{525\ mm}$$

$$M_{max} = -\frac{P B_{\beta a}}{\beta} = -\frac{60{,}000(0.3224)}{0.001496} = 12.93\ kN.m$$

$$\sigma_{max} = -\frac{M_{max}c}{I_x} = \frac{12{,}930{,}000(63.5)}{5{,}120{,}000} = \underline{160.4\ MPa}$$

10.32 $\beta = 0.001467\ mm^{-1}$; $K = 20{,}360\ N/mm$; $k = 40.7\ N/mm$

In order to determine the maximum stress in the rods, it is necessary to determine the deflection of the beam at $\ell/2$ from load P at the end of beam.

$$\frac{\beta\ell}{2} = \frac{0.001467(500)}{2} = 0.3668; \quad D_{\beta\ell/2} = 0.6470$$

$$y_{(end\ rod)} = \frac{2P\beta D_{\beta\ell/2}}{k} = \frac{2(40{,}000)(0.001467)(0.6470)}{37.7} = \underline{1.866\ mm}$$

$$Q = y_{(end\ rod)}K = 1.866(20{,}360) = 38{,}000\ N$$

$$\sigma_{max(rod)} = \frac{Q}{A} = \frac{38{,}000(4)}{\pi(18)^2} = \underline{149.3\ MPa}$$

$$M_{max} = -\frac{P B_{\beta z(max)}}{\beta} = -\frac{40{,}000(0.3224)}{0.001467} = -8.791\ kN.m$$

$$\sigma_{max\ (beam)} = -\frac{M_{max}c}{I_x} = \frac{8{,}791{,}000(76)}{11{,}000{,}000} = \underline{60.7\ MPa}$$

10.33 $I_x = \frac{15(20)^3}{12} = 10{,}000\ mm^4$; $k = 15\,k_0 = 15(0.200) = 3.00\ N/mm^2$

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{3.00}{4(82{,}700)(10{,}000)}} = 0.005488\ mm^{-1}; \quad M_{max} = M_0 \text{ (see Eq. 10.38)}$$

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{700(100)(10)}{10{,}000} = \underline{70.0\ MPa}$$

$$y = \frac{2P\beta}{k}D_{\beta z} - \frac{2\beta^2 M_0}{k}C_{\beta z} = \frac{2(700)(0.005488)}{3.00}D_{\beta z} - \frac{2(0.005488)^2(100)(700)}{3.00}C_{\beta z} = 2.561\,D_{\beta z} - 1.406\,C_{\beta z}$$

By trial and error y_{max} occurs at $\beta z = 0.10$ approximately.

$$z = \frac{0.10}{0.005488} = 18.2\ mm; \quad D_{\beta z} = 0.9003; \quad C_{\beta z} = 0.8100$$

$$y_{max} = 2.561(0.9003) - 1.406(0.8100) = \underline{1.167\ mm}$$

10.34 $\sigma_{max} = \frac{M_{max}\,c}{I_x} = \frac{(700)(200)(10)}{10,000} = 140.0$ MPa; $\beta = 0.005488$ mm^{-1}

$y = 2.561\,D_{\beta z} - 2.812\,C_{\beta z}$; By trial and error, y_{max} occurs at $\beta z \approx 0.875$. Therefore, $z = \frac{0.875}{0.005488} = 159$ mm; $D_{\beta z} = 0.2672$; $C_{\beta z} = -0.0528$.

Hence, $y_{max} = 2.561(0.2672) - 2.812(-0.0528) = 0.833$ mm.

10.35 In Problem 10.33, determine the value of b for which the slope at A is zero (Fig. a).

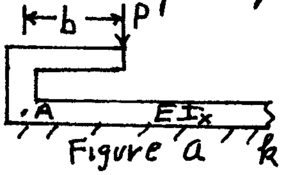


Figure a

By the given data in Problem 10.33, $P = 700$ N, $M_A = Pb = 700b$ N·mm, $k = 15k_0 = 15(0.200) = 3.0$ N/mm^2, and $\beta = \sqrt[4]{\frac{k}{4EI_x}} = 0.005488$ mm^{-1}

Then, by Eq. (10.37) and Table 10.1, with $\beta z = 0$,

$$\theta_A = -\frac{2P\beta^2}{k} A_{\beta z} + \frac{4\beta^3 M_A}{k} D_{\beta z}$$

or for zero slope at A

$$0 = -2.008 \times 10^{-5} P + 2.2035 \times 10^{-7} Pb$$

Therefore,

$$b = 91.1 \text{ mm}$$

Note that this value of b is valid for any load P.

10.36 $k = \frac{K}{\ell} = \frac{100}{500} = 0.200 \text{ N/mm}^2$

$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{0.200}{4(200,000)(2,530,000)}} = 0.0005607 \text{ mm}^{-1}; \ \ell = 500 < \frac{\pi}{4\beta} = 1401 \text{ mm}$

$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{3500(2000)(51)}{2,530,000} = \underline{141.1 \text{ MPa}}; \ M_0 = -1750(3500) = -6.125 \text{ kN.m}$

$y = \frac{2P\beta}{k} D_{\beta z} - \frac{2\beta^2 M_0}{k} C_{\beta z} = \frac{2(3500)(0.0005607)}{0.200} D_{\beta z} - \frac{2(0.0005607)^2(-6,125,000)}{0.2} C_{\beta z}$

$= 19.625\, D_{\beta z} + 19.256\, C_{\beta z}$

The origin for the z-axis is 1750 mm from load P. The maximum compression occurs in the first spring for which $z = 250$ mm and $\beta z = 0.0005607(250) = 0.1402$ ($D_{\beta z} = 0.8607$; $C_{\beta z} = 0.7396$). By trial and error the maximum tension occurs at $z = 3250$ mm.

$\beta z = 0.0005607(3250) = 1.8223$ ($D_{\beta z} = -0.0402$; $C_{\beta z} = -0.1967$)

$y_{comp} = 19.625(0.8607) + 19.256(0.7396) = 31.13 \text{ mm}$; Compression $= 100(31.13) = \underline{3.11 \text{ kN}}$

$y_{tension} = 19.625(-0.0402) + 19.256(-0.1967) = -4.577 \text{ mm}$; Tension $= 100(4.475) = \underline{457.7 \text{ N}}$

10.37 $\beta = \sqrt[4]{\frac{0.200}{4(72,000)(2,530,000)}} = 0.0007238; \ \ell = 500 < \frac{\pi}{4\beta} = 1085 \text{ mm}$

$y = \frac{2(3500)(0.0007238)}{0.200} D_{\beta z} - \frac{2(0.0007238)^2(-6,125,000)}{0.200} C_{\beta z}$

$= 25.333\, D_{\beta z} + 32.088\, C_{\beta z}$

$y_{comp} = 25.333(0.8607) + 32.088(0.7396) = \underline{42.33 \text{ mm}}$; Compression $= 100(42.33) = \underline{4.23 \text{ kN}}$

By trial and error the maximum tension occurs at $z = 2750$ mm. $\beta z = 0.0007238(2750) = 1.9905$ ($D_{\beta z} = -0.0556$; $C_{\beta z} = -0.1804$)

$y_{tension} = 25.333(-0.0556) + 32.088(-0.1804) = \underline{-7.196 \text{ mm}}$

Tension $= 100(7.196) = \underline{719.6 \text{ N}}$; σ_{max} = 141.1 MPa as in Problem 10.36

10.38 (a) By Eq.(10.19) and the data given in Problem 10.36, we find

$$k = \frac{K}{l} = \frac{100}{500} = 0.200 \text{ N/mm}^2 \qquad (a)$$

By Eqs.(10.5) and (a), $\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{0.200}{4(200{,}000)(2{,}530{,}000)}} = 0.0005607 \text{ mm}^{-1}$

Hence, by Eq.(10.20),

$$\frac{\pi}{4\beta} = \frac{\pi}{4(0.0005607)} = 1401 \text{ mm} \geq l = 500 \text{ mm}.$$

Therefore the spring spacing is sufficiently small.

(b) By Eq.(10.7),

$$y = e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z), \quad z \geq 0 \qquad (b)$$

Therefore,

$$\frac{d^2y}{dz^2} = 2\beta^2 e^{-\beta z}(C_4 \sin\beta z - C_3 \cos\beta z) \qquad (c)$$

$$\frac{d^3y}{dz^3} = 2\beta^3 e^{-\beta z}[C_3(\cos\beta z + \sin\beta z) + C_4(\cos\beta z - \sin\beta z)] \qquad (d)$$

The boundary conditions are (since $M_0 = 0$)

$$M_0 = -EI_x \frac{d^2y}{dz^2}\bigg|_{z=0} = 0 \; ; \quad P = EI_x \frac{d^3y}{dz^3}\bigg|_{z=0} \qquad (e)$$

Equations (c), (d), and (e) yield with Eq.(10.5),

$$C_3 = 0, \quad C_4 = \frac{P}{2\beta^3 EI_x} = \frac{2\beta P}{k} \qquad (f)$$

Then, by Eqs.(b),(f), and (10.16)

$$y = \frac{2\beta P}{k} e^{-\beta z} \cos\beta z = \frac{2\beta P}{k} D_{\beta z}$$

The function $C_{\beta z}$ does not enter. See also Eqs.(10.34) and (10.36).

10.39 $\beta = 0.001496\ \text{mm}^{-1}$; $\beta a = 0.001496(1000) = 1.496$; $k = 20.52\ \text{N/mm}^2$

$A_{\beta a} = 0.2402$; $B_{\beta a} = 0.2234$; $C_{\beta a} = -0.2066$; $D_{\beta a} = 0.0168$

$y = \frac{P\beta}{2k}\left[A_{\beta z} + 2D_{\beta a}D_{\beta(a+z)} + C_{\beta a}C_{\beta(a+z)}\right]$; y_{max} occurs at $\underline{z = 0}$.

$y_{max} = \frac{60{,}000(0.001496)}{2(20.52)}\left[1.000 + 2(0.0168)^2 + (-0.2066)^2\right] = \underline{2.282\ \text{mm}}$

$M_x = \frac{P}{4\beta}\left[C_{\beta z} - 2D_{\beta a}B_{\beta(a+z)} - C_{\beta a}A_{\beta(a+z)}\right]$; M_{max} occurs at $\underline{z = 0}$.

$M_{max} = \frac{60{,}000}{4(0.001496)}\left[1.0000 - 2(0.0168)(0.2234) - (-0.2066)(0.2406)\right] = 10.45\ \text{kN.m}$

$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{10{,}450{,}000(63.5)}{5{,}120{,}000} = \underline{129.6\ \text{MPa}}$

10.40 $\beta = 0.001496\ \text{mm}^{-1}$; $\beta a = 0.001496(500) = 0.748$; $k = 20.52\ \text{N/mm}^2$

$A_{\beta a} = 0.6689$; $B_{\beta a} = 0.3219$; $C_{\beta a} = 0.0251$; $D_{\beta a} = 0.3470$

$y = \frac{P\beta}{2k}\left[A_{\beta z} + 2D_{\beta a}D_{\beta(a+z)} + C_{\beta a}C_{\beta(a+z)}\right]$; y_{max} occurs at $\underline{z = -a}$.

$y_{max} = \frac{60{,}000(0.001496)}{2(20.52)}\left[0.6689 + 2(0.3470)(1.0000) + 0.0251(1.0000)\right] = \underline{3.036\ \text{mm}}$

$M_x = \frac{P}{4\beta}\left[C_{\beta z} - 2D_{\beta a}B_{\beta(a+z)} - C_{\beta a}A_{\beta(a+z)}\right]$; M_{max} occurs at $\underline{z = 0}$.

$M_{max} = \frac{60{,}000}{4(0.001496)}\left[1.000 - 2(0.3470)(0.3219) - 0.0251(0.6689)\right] = 7.62\ \text{kN.m}$

$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{7{,}620{,}000(63.5)}{5{,}120{,}000} = \underline{94.5\ \text{MPa}}$

10.41 $\beta = 0.001467\ \text{mm}^{-1}$; $K = 20{,}360\ \text{N/mm}$; $k = 40.7\ \text{N/mm}^2$

$\beta a = 0.001467(750) = 1.1003$; $A_{\beta a} = 0.4474$; $B_{\beta a} = 0.2967$; $C_{\beta a} = -0.1459$; $D_{\beta a} = 0.1508$

$y = \frac{P\beta}{2k}\left[A_{\beta z} + 2D_{\beta a}D_{\beta(a+z)} + C_{\beta a}C_{\beta(a+z)}\right]$; y_{max} occurs at $\underline{z = 0}$.

$y_{max} = \frac{60{,}000(0.001467)}{2(40.7)}\left[1.0000 + 2(0.1508)^2 + (-0.1459)^2\right] = \underline{1.153\ \text{mm}}$

$Q_{max} = Ky_{max} = 20{,}360(1.153) = 23.48\ \text{kN}$

$\sigma_{max(rod)} = \frac{Q}{A} = \frac{23{,}480(4)}{\pi(18)^2} = \underline{92.3\ \text{MPa}}$

$M_x = \frac{P}{4\beta}\left[C_{\beta z} - 2D_{\beta a}B_{\beta(a+z)} - C_{\beta a}A_{\beta(a+z)}\right]$; M_{max} occurs at $\underline{z = 0}$.

$M_{max} = \frac{60{,}000}{4(0.001467)}\left[1.0000 - 2(0.1508)(0.2967) - (-0.1459)(0.4474)\right] = 9.98\ \text{kN.m}$

$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{9{,}980{,}000(76)}{11{,}000{,}000} = \underline{68.95\ \text{MPa}}$

10.42 $\beta = 0.001467\ \text{mm}^{-1}$; $K = 20{,}360\ \text{N/mm}$; $k = 40.7\ \text{N/mm}^2$

$\beta a = 0.001467(250) = 0.3668$; $A_{\beta a} = 0.8951$; $B_{\beta a} = 0.2481$; $C_{\beta a} = 0.3989$; $D_{\beta a} = 0.6470$

$y = \frac{P\beta}{2k}\left[A_{\beta z} + 2D_{\beta a} D_{\beta(a+z)} + C_{\beta a} C_{\beta(a+z)}\right]$; y_{max} occurs at $\underline{z = -a.}$

$y_{max} = 1.081[0.8951 + 2(0.6470)(1.0000) + 0.3989(1.0000)] = \underline{2.798\ \text{mm}}$; 1st rod is at $z = 0$.

$y_{max\,(rod)} = 1.081[1.0000 + 2(0.6470)^2 + 0.3989^2] = 2.158\ \text{mm}$

$Q_{max} = K\, y_{max(rod)} = 20{,}360(2.158) = 43.94\ \text{kN}$

$\sigma_{max\,(rod)} = \frac{Q_{max}}{A} = \frac{43{,}940(4)}{\pi(18)^2} = \underline{172.7\ \text{MPa}}$

$M_x = \frac{P}{4\beta}\left[C_{\beta z} - 2D_{\beta a} B_{\beta(a+z)} + C_{\beta a} A_{\beta(a+z)}\right]$; By trial and error M_{max} occurs at $\underline{z = 650\ \text{mm}}$ approximately. $\beta z = 0.9536$; $\beta(a+z) = 1.3203$

$C_{\beta z} = -0.0911$; $B_{\beta(a+z)} = 0.2587$; $A_{\beta(a+z)} = 0.3251$

$M_{max} = \frac{60{,}000}{4(0.001467)}[-0.0911 - 2(0.6470)(0.2587) - 0.3989(0.3251)] = -5.678\ \text{kN.m}$

$\sigma_{max\,(beam)} = \frac{M_{max}c}{I_x} = \frac{5{,}678{,}000(76)}{11{,}000{,}000} = \underline{39.2\ \text{MPa}}$

10.43

$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{12.0}{4(200{,}000)(17{,}070{,}000)}} = 0.000968\ \text{mm}^{-1}$

$y = \frac{2P\beta}{k} D_{\beta z_1} + \frac{P\beta}{2k}\left[A_{\beta z_2} + 2D_{\beta a} D_{\beta(a+z_2)} + C_{\beta a} C_{\beta(a+z_2)}\right]$; y_{max} occurs at the free end for which $z_1 = 0$ and $z_2 = -a = -2500\ \text{mm}$. $\beta a = 2.4200$

$A_{\beta a} = -0.0079$; $D_{\beta a} = -0.0667$; $C_{\beta a} = -0.1255$

$y_{max} = \frac{2(80{,}000)(0.000968)}{12.0}(1.0000) + \frac{80{,}000(0.000968)}{2(12.0)}[-0.0079 + 2(-0.0667)(1.0000) - 0.1255(1.0000)]$

$= 12.91 - 0.86 = \underline{12.05\ \text{mm at free end}}$

$M_x = -\frac{P}{\beta} B_{\beta z_1} + \frac{P}{4\beta}\left[C_{\beta z_2} - 2D_{\beta a} B_{\beta(a+z_2)} - C_{\beta a} A_{\beta(a+z_2)}\right]$; By trial and error M_{max} occurs at the location for which $z_1 = 775\ \text{m}$ and $z_2 = -1725\ \text{mm}$

$\beta z_1 = 0.7502$ and $B_{\beta z_1} = 0.3220$; $\beta z_2 = 1.6698$ and $C_{\beta z_2} = -0.2059$

$\beta(a + z_2) = 0.7502$ and $B_{\beta(a+z_2)} = 0.3220$, $A_{\beta(a+z_2)} = 0.6676$

$M_{max} = -\frac{80{,}000}{0.000968}(0.3220) + \frac{80{,}000}{4(0.000968)}[-0.2059 - 2(-0.0667)(0.3220) - (-0.1255)(0.6676)] = -28.25\ \text{kN.m}$

$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{28{,}250{,}000(69)}{17{,}070{,}000} = \underline{114.2\ \text{MPa at 775 mm from free end.}}$

10.44 $k = 129(0.300) = 38.7 \ N/mm^2$

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{38.7}{4(200{,}000)(95{,}300{,}000)}} = 0.000844 \ mm^{-1}$$

$$\beta L = 0.000844(3000) = 2.532$$

$$y_{max} = \frac{P\beta}{2k}\left[\frac{\cosh \beta L + \cos \beta L + 2}{\sinh \beta L + \sin \beta L}\right] = \frac{270{,}000(0.000844)}{2(38.7)}\left[\frac{6.329 - 0.820 + 2}{6.250 + 0.573}\right] = \underline{3.240 \ mm}$$

$$M_{max} = \frac{P}{4\beta}\left[\frac{\cosh \beta L - \cos \beta L}{\sinh \beta L + \sin \beta L}\right] = \frac{270{,}000}{4(0.000844)}\left[\frac{6.329 + 0.820}{6.250 + 0.573}\right] = 83.80 \ kN.m$$

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{83{,}800{,}000(152.5)}{95{,}300{,}000} = \underline{134.1 \ MPa}$$

10.45 $k = 38.7 \ N/mm^2; \quad \beta = 0.000844 \ mm^{-1}; \quad \beta L = 2.532$

For $\beta L = 2.0 \qquad y_{max} = 4.43\frac{P\beta}{2k} \qquad M_{max} = 1.08\frac{P}{4\beta}$

For $\beta L = 3.0 \qquad y_{max} = 4.00\frac{P\beta}{2k} \qquad M_{max} = 1.22\frac{P}{4\beta}$

Using a linear interpolation for $\beta L = 2.532$

$$y_{max} = 4.20\frac{P\beta}{2k} = \frac{4.20(270{,}000)(0.000844)}{2(38.7)} = \underline{12.37 \ mm}$$

$$M_{max} = 1.15\frac{P}{4\beta} = \frac{1.15(270{,}000)}{4(0.000844)} = 91.97 \ kN.m$$

$$\sigma_{max} = \frac{M_{max}c}{I_x} = \frac{91{,}970{,}000(152.5)}{95{,}300{,}000} = \underline{147.2 \ MPa}$$

10.46

$$u = \frac{pa^2}{Eh} = \frac{3(20)^2}{200{,}000(1)} = 0.0060 \ mm$$

$$k = \frac{p}{u} = \frac{Eh}{a^2} = \frac{20{,}000(1)}{(20)^2} = 500 \ N/mm^2$$

$$\beta = \sqrt[4]{\frac{3(1-\nu^2)}{a^2h^2}} = \sqrt[4]{\frac{3(1-0.29^2)}{(20)^2(1)^2}} = 0.2879 \ mm^{-1}$$

$$w = \frac{2ku}{\beta} = \frac{2(500)(0.0060)}{0.2879} = 20.84 \ N/mm$$

$$M_{max} = \frac{w}{4\beta} = \frac{20.84}{4(0.2879)} = 18.10 \ N.mm$$

$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{18.10(0.50)(12)}{(1)^4} = \underline{108.6 \ MPa}$$

10.47 The average pressure p_1 between ring and cylinder is given by the relation $p_1 = \frac{\sigma_\theta h}{a} = \frac{100(20)}{500} = 4.00$ MPa

The load per unit length around the cylinder applied by the ring is $20 p_1$. This load is applied to the cylinder by two line loads $w = \frac{20 p_1}{2} = \frac{20(4.00)}{2} = 40$ N/mm.

$$k = \frac{E h_{cyl}}{a_{cyl}^2} = \frac{72{,}000(5)}{495^2} = 1.469 \text{ N/mm}^2$$

$$\beta = \sqrt[4]{\frac{3(1-\nu^2)}{a_{cyl}^2 h_{cyl}^2}} = \sqrt{\frac{3(1-0.33^2)}{495^2 (5)^2}} = 0.02570 \text{ mm}^{-1}$$

Under center line of ring $z = 10$ mm and $\beta z = 0.02570(10) = 0.2570$

$A_{\beta z} = 0.9443$; $C_{\beta z} = 0.5517$

$$M_{max} = \frac{w}{4\beta}(2 C_{\beta z}) = \frac{40(2)(0.5517)}{4(0.02570)} = 429.3 \text{ N.mm}$$

$$\sigma_{zz(max)} = \frac{M_{max} c}{I} = \frac{429.3(2.5)(12)}{1(5)^3} = \underline{\pm 103.0 \text{ MPa}} \text{ (Tension on inside)}$$

$$u_{max} = -\frac{w\beta}{2k}(2A_{\beta z}) = -\frac{40(0.02570)(2)(0.9443)}{2(1.469)} = -0.6608 = a\,\epsilon_{\theta\theta}$$

$$\epsilon_{\theta\theta} = -\frac{0.6608}{495} = -0.001335 = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}\sigma_{zz} = \frac{\sigma_{\theta\theta}}{72{,}000} - \frac{0.33(103)}{72{,}000}$$

$$\sigma_{\theta\theta} = (-0.001335 + 0.000472)72{,}000 = \underline{-62.1 \text{ MPa}}$$

10.48 $w = 80$ N/mm; $\beta = 0.02570$ mm^{-1}; $k = 1.469$ N/mm^2; $\beta z = 0$

$$M_{max} = \frac{w}{4\beta} C_{\beta z} = \frac{80(1.0000)}{4(0.02570)} = 778.2 \text{ N.mm}$$

$$\sigma_{zz(max)} = \frac{M_{max} c}{I} = \frac{778.2(2.5)(12)}{(1)(5)^3} = \underline{\pm 186.8 \text{ MPa}} \text{ (Tension on inside)}$$

$$u_{max} = -\frac{w\beta}{2k} A_{\beta z} = -\frac{80(0.02570)(1.0000)}{2(1.469)} = -0.6998 \text{ mm} = a\,\epsilon_{\theta\theta}$$

$$\epsilon_{\theta\theta} = -\frac{0.6998}{495} = -0.001414 = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}\sigma_{zz} = \frac{\sigma_{\theta\theta}}{72{,}000} - \frac{0.33(186.8)}{72{,}000}$$

$$\sigma_{\theta\theta} = (-0.001414 + 0.000856)(72{,}000) = \underline{-40.2 \text{ MPa}}$$

10.49

$$\beta = \sqrt[4]{\frac{3(1-\nu^2)}{a^2h^2}} = \sqrt[4]{\frac{3(1-0.29^2)}{(2000)^2(10)^2}} = 0.009104 \text{ mm}^{-1}; \quad \frac{L}{2} = \frac{3\pi}{4\beta} = \frac{3\pi}{4(0.009104)} = 259 \text{ mm}$$

Cylinder

$$\sigma_{\theta\theta}(cyl) = \frac{p_1 a}{h} = \frac{2000p_1}{10} = 200\,p_1 \text{ (MPa)}$$

$$\sigma_{zz}(cyl) = \frac{p_1 a}{2h} = 100\,p_1 \text{(MPa)}$$

$$u_{cyl} = a\,\epsilon_{\theta\theta} = \frac{2000}{200{,}000}\left[200p_1 - 0.29(100p_1)\right]$$

$$= 1.710\,p_1 \text{ (mm)}$$

Sphere

$$\sigma_{\theta\theta(sph)} = \frac{p_1 a}{2h} = 100p_1 \text{ (MPa)}$$

$$\sigma_{zz(sph)} = \frac{p_1 a}{2h} = 100p_1 \text{ (MPa)}$$

$$u_{sph} = a\,\epsilon_{\theta\theta} = \frac{2000}{200{,}000}\left[100p_1 - 0.29(100p_1)\right]$$

$$= 0.710\,p_1 \text{ (mm}$$

Assume that the displacement u for zero bending moment is midway between u_{cyl} and u_{sph}.

$$u = \frac{u_{cyl} - u_{sph}}{2} = \frac{1.710p_1 - 0.710p_1}{2} = 0.500\,p_1 \text{ (mm)}$$

$$k = \frac{Eh}{a^2} = \frac{200{,}000(10)}{2000^2} = 0.500 \text{ N/mm}$$

$$u = 0.500\,p_1 = \frac{2w\beta}{k}; \quad w = \frac{0.500p_1(0.500)}{2(0.009104)} = \underline{13.73\,p_1 \text{ (N/mm)}}$$

M_{max} occurs at $\beta z = \frac{\pi}{4}$. $B_{\beta z} = 0.3224$ $\quad D_{\beta z} = 0.3224$

$$y = \frac{2w\beta}{k} D_{\beta z} = \frac{2(13.73p_1)(0.009104)(0.3224)}{0.500} = 0.161\,p_1 \text{ (mm)}$$ FROM Eq (10.36)

$$M_{max} = -\frac{w}{\beta} B_{\beta z} = -\frac{13.73p_1(0.3224)}{0.009104} = -486.2\,p_1 \text{ (N.mm)}$$ FROM Eq (10.38)

$$\sigma_{zz}(bending) = \frac{M_{max}c}{I} = \frac{486.2p_1(5)(12)}{(1)(10)^3} = \underline{29.2\,p_1 \text{(MPa)}}$$ (Tension on inside)

$$\sigma_{zz} = \sigma_{zz(cyl)} + \sigma_{zz(bending)} = 100p_1 + 29.2p_1 = \underline{129.2p_1 \text{ (MPa)}}$$

The actual displacement of the cylinder at the location where $\sigma_{zz} = 129.2\,p_1$ is equal to $(u_{cyl} - u + y)$

$$\epsilon_{\theta\theta} = \frac{u_{cyl} - u + y}{a} = \frac{1.710p_1 - 0.500p_1 + 0.161p_1}{2000} = 0.0006855p_1 = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}\sigma_{zz}$$

$$\sigma_{\theta\theta} = 0.0006855p_1(200{,}000) + 0.29(129.2p_1) = \underline{174.6p_1 \text{ (MPa)}}$$

At this location $\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{2} = \frac{174.6p_1 - (-p_1)}{2} = 87.8p_1$

At location in cylinder away from junction,

$$\tau_{max} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{2} = \frac{200p_1 - (-p_1)}{2} = 100.5\,p_1 \text{ (MPa)}$$

$$\text{Ratio} = \frac{87.8}{100.5} = 0.874$$

11.1 From Example 11.1, $E = 200$ GPa, $\nu = 0.29$, $a = 10$ mm, $b = 50$ mm, $p_1 = 300$ MPa, and $p_2 = 0$. By Eq. (11.25), the radial displacement as a function of r is

$$u = \frac{r}{E(b^2-a^2)}\left[(1-\nu)p_1 a^2 + \frac{(1+\nu)a^2b^2}{r^2}p_1\right] = 0.000044375\, r + \frac{0.2015625}{r} \qquad (a)$$

At the inner surface, $r = 10$ mm and Eq. (a) yields

$$u_{r=10\,mm} = 0.0206 \text{ mm}$$

At the outer surface, $r = 50$ mm and Eq. (a) yields

$$u_{r=50\,mm} = 0.00625 \text{ mm}$$

11.2 By Example 11.2, $E = 72$ GPa, $\nu = 0.33$, $a = 100$ mm, $b = 400$ mm, $p_1 = 150$ MPa, and $p_2 = 0$. The principal stresses are given by Eqs. (11.20), (11.21), and (11.22). For $p_2 = 0$ and $r = b = 400$ mm, these equations yield

$$\begin{aligned} \sigma_{rr} &= 0 \\ \sigma_{\theta\theta} &= p_1\frac{2a^2}{b^2-a^2} = 150\frac{2(100)^2}{400^2-100^2} = 20 \text{ MPa} \\ \sigma_{zz} &= p_1\frac{a^2}{b^2-a^2} = 150\frac{100^2}{400^2-100^2} = 10 \text{ MPa} \end{aligned} \qquad (a)$$

The maximum shear stress is, by Eqs. (a) and (2.39),

$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \frac{1}{2}(20-0) = 10 \text{ MPa} \qquad (b)$$

By Eq. (11.24), with $p_2 = P = 0$ and $r = b = 400$ mm, the increase in the outer diameter is

$$\Delta d = 2u_{r=b} = 2p_1\frac{a^2b(2-\nu)}{E(b^2-a^2)} = 300\frac{(100)^2(400)(2-0.33)}{72{,}000(400^2-100^2)}$$

or

$$\Delta d = 0.18\bar{5} \text{ mm}$$

$\boxed{11.3}$ (a) With $p_2 = 0$, Eqs. (11.20) and (11.21) reduce to

$$\sigma_{rr} = p_1 \frac{a^2(r^2 - b^2)}{r^2(b^2 - a^2)}, \qquad \sigma_{\theta\theta} = p_1 \frac{a^2(r^2 + b^2)}{r^2(b^2 - a^2)} \qquad (a)$$

For $r = a = 100$ mm, Eqs. (a) yield, with $p_1 = 200$ MPa, $a = 100$ mm, and $b = 200$ mm,

$$\sigma_{rr} = -200 \text{ MPa}, \qquad \sigma_{\theta\theta} = 333.3 \text{ MPa} \qquad (b)$$

Similarly, for $r = 150$ mm,

$$\sigma_{rr} = -51.85 \text{ MPa}, \qquad \sigma_{\theta\theta} = 185.18 \text{ MPa} \qquad (c)$$

and for $r = 200$ mm,

$$\sigma_{rr} = 0, \qquad \sigma_{\theta\theta} = 133.3 \text{ MPa} \qquad (d)$$

(b) By Eqs. (b), (c), and (d), we may plot σ_{rr} and $\sigma_{\theta\theta}$ (Fig. a).

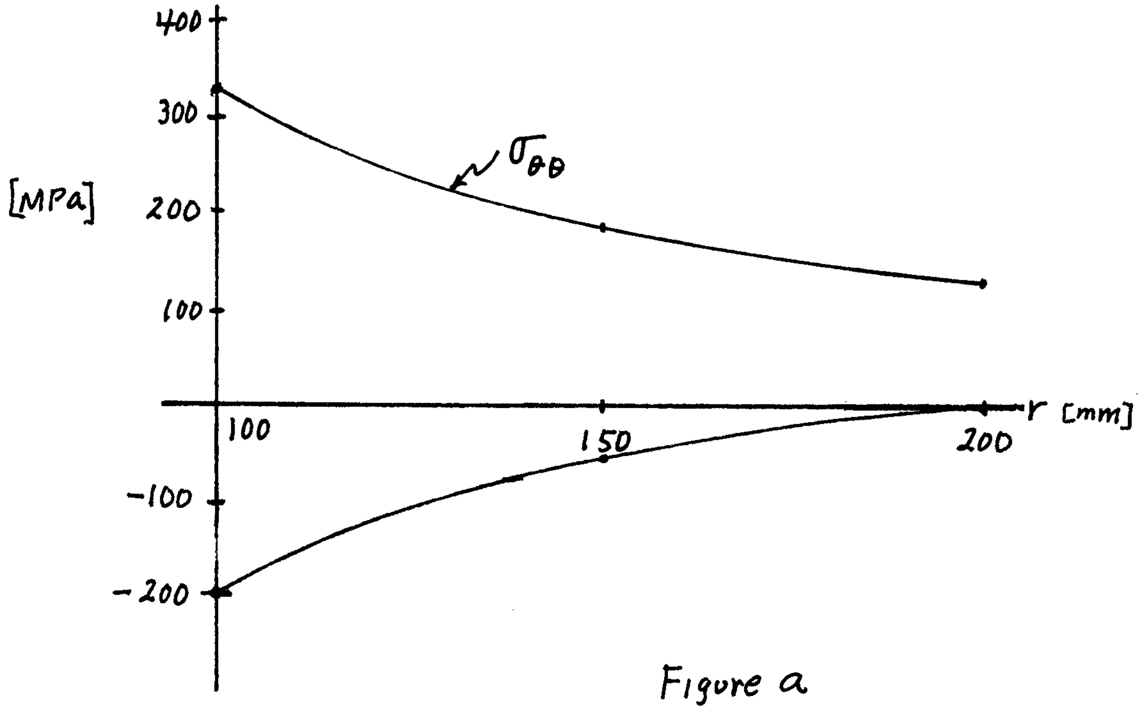


Figure a

11.4 By Eqs. (11.20)–(11.22), with $a = 100$ mm, $b = 250$ mm, $p_1 = 80.0$ MPa, and $p_2 = 0$, we have at $r = a = 100$ mm,

$$\sigma_{rr} = p_1 \frac{a^2 - b^2}{b^2 - a^2} = -p_1 = -80 \text{ MPa}, \quad \sigma_{\theta\theta} = p_1 \frac{a^2 + b^2}{b^2 - a^2} = 80 \frac{100^2 + 250^2}{250^2 - 100^2} = 110.5 \text{ MPa};$$

$$\sigma_{zz} = p_1 \frac{a^2}{b^2 - a^2} = 80 \frac{100^2}{250^2 - 100^2} = 15.2 \text{ MPa}$$

11.5 By the formulas of Problem 11.4, we may tabulate the stresses (Table P11.5)

Table P11.5 – Stresses in MPa

$r \rightarrow$	100 mm	125 mm	150 mm	175 mm	200 mm	225	250
σ_{rr}	−80.0	−45.7	−27.1	−15.9	−8.6	−3.6	0
$\sigma_{\theta\theta}$	110.5	76.2	57.6	46.3	39.0	34.1	30.5

11.6 By the data of Problems 11.4 and 11.6, we have $a = 100$ mm, $b = 250$ mm, $p_1 = 80.0$ MPa, $p_2 = 0$, $E = 200$ GPa, and $\nu = 0.29$. By Eq. (11.24), the radial displacements u_a and u_b at $r = a$ and $r = b$, respectively, are

$$u_a = \frac{100}{200{,}000(250^2 - 100^2)}\left[(1 - 0.58)(80)(100)^2 + (1.29)(80)(250)^2\right] = 0.0646 \text{ mm}$$

$$u_b = \frac{250}{200{,}000(250^2 - 100^2)}\left[(1 - 0.58)(80)(100)^2 + (1.29)(80)(100)^2\right] = 0.0326 \text{ mm}$$

By Eq. (11.15), with $P = \Delta T = 0$, the axial strain is

$$\epsilon_{zz} = \frac{(1 - 2\nu)p_1 a^2}{E(b^2 - a^2)} = \frac{0.42(80)(100)^2}{200{,}000(250^2 - 100^2)} = 0.000032$$

Hence, the elongation of a length $L = 1$ m $= 1000$ mm is

$$e_z = 1000\,\epsilon_{zz} = 0.032 \text{ mm}$$

Hence, the final dimensions are

$$a^* = 100.0646 \text{ mm}, \quad b^* = 250.0326 \text{ mm}, \quad L^* = 1000.032 \text{ mm}$$

11.7 By Eqs. (11.21) and (11.22), with $a = 10$ mm, $b = 20$ mm, $p_1 = 100$ MPa, and $p_2 = 40$ MPa, also $P = 0$, we have at $r = a = 10$ mm

$$\sigma_{\theta\theta} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} + \frac{a^2 b^2}{r^2(b^2 - a^2)}(p_1 - p_2) = \frac{100(10^2) - 40(20^2)}{20^2 - 10^2} + \frac{20^2(100 - 40)}{20^2 - 10^2} = 60.0 \text{ MPa}$$

$$\sigma_{zz} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} = \frac{100(10^2) - 40(20^2)}{20^2 - 10^2} = -20 \text{ MPa}$$

11.8 For the outer cylinder (Fig. a), at $r=c$, the increase in the radius is, by Eq. (11.25), with $p_1 = p_s$, $p_2 = 0$, $a = c$ and $b = b$,

$$u_{r=c} = \frac{c p_s}{E}\left(\frac{c^2+b^2}{b^2-c^2} + \nu\right) \qquad \text{(a)}$$

Similarly, for the inner cylinder (Fig. b), at $r=c$, the decrease in the radius is, by Eq. (11.25), with $p_1 = 0$, $p_2 = p_s$, $a = a$, and $b = c$,

$$u_{r=c} = \frac{c p_s}{E}\left(\frac{a^2+c^2}{c^2-a^2} - \nu\right) \qquad \text{(b)}$$

Adding Eqs. (a) and (b) and equating the result to δ, we obtain after some simplification

$$\frac{c p_s}{E}\left[\frac{2c^2(b^2-a^2)}{(b^2-c^2)(c^2-a^2)}\right] = \delta$$

or

$$p_s = \frac{E\delta}{c}\left[\frac{(b^2-c^2)(c^2-a^2)}{2c^2(b^2-a^2)}\right]$$

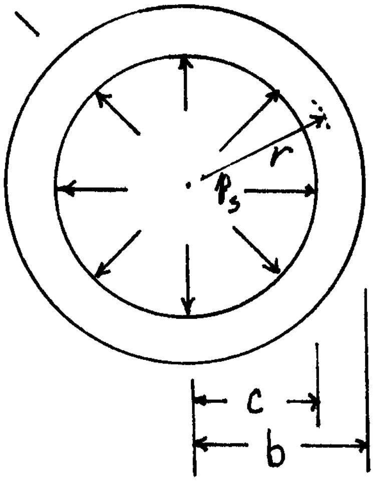


Figure a

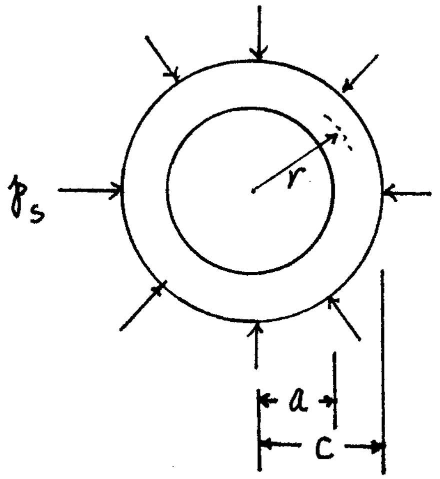


Figure b

11.9 By the result of Problem 11.8 and with $E = 200$ GPa, $a = 100$ mm, $c = 200$ mm, $b = 300$ mm, and $\delta/c = 0.001$, we have

$$p_s = \frac{E\delta}{c}\left[\frac{(b^2-c^2)(c^2-a^2)}{2c^2(b^2-a^2)}\right] = 200{,}000(0.001)\left[\frac{(300^2-200^2)(200^2-100^2)}{2(200)^2(300^2-100^2)}\right] = 46.875\text{MPa} \quad (a)$$

For the inner cylinder, $a = 100\text{ mm} \le r \le 200\text{ mm} = c$, with $p_1 = 0$, and $p_2 = p_s$, we have, by Eq. (11.21),

$$\sigma_{\theta\theta} = -p_s \frac{c^2}{c^2-a^2}\left(1+\frac{a^2}{r^2}\right) \quad (b)$$

Then, by Eqs. (a) and (b), we find

For $r = a = 100$ mm, $\sigma_{\theta\theta} = -125$ MPa

For $r = 150$ mm, $\sigma_{\theta\theta} = -90.28$ MPa

For the outer cylinder, $c = 200\text{ mm} \le r \le 300\text{ mm} = b$, with $p_1 = p_s$ and $p_2 = 0$, we have, by Eq. (11.21),

$$\sigma_{\theta\theta} = p_s \frac{c^2}{b^2-c^2}\left(1+\frac{b^2}{r^2}\right) \quad (c)$$

Then, by Eqs. (a) and (c), we obtain

For $r = 250$ mm, $\sigma_{\theta\theta} = 91.5$ MPa

For $r = b = 300$ mm, $\sigma_{\theta\theta} = 75$ MPa

11.10 By Fig. a, $a = 100$ mm, $c = 150$ mm, and $b = 200$ mm. Then, by the result given in Problem 11.8, with $\delta = 0.125$ mm,

$$p_s = \frac{E\delta}{c}\left[\frac{(b^2-c^2)(c^2-a^2)}{2c^2(b^2-a^2)}\right] = \frac{72000(0.125)}{150}\left[\frac{(200^2-150^2)(150^2-100^2)}{2(150^2)(200^2-100^2)}\right] = 9.722\text{ MPa} \quad (a)$$

First, let us determine the residual stress in the inner and outer cylinders at $r = c = 150$ mm. For the inner cylinder, with $p_1 = 0$ and $p_2 = p_s$, we find, by Eq. (11.21),

$$\left(\sigma_{\theta\theta}^{R}\right)_{\text{inner cyl.}} = -p_s \frac{c^2}{c^2-a^2}\left(1+\frac{a^2}{r^2}\right) \quad (b)$$

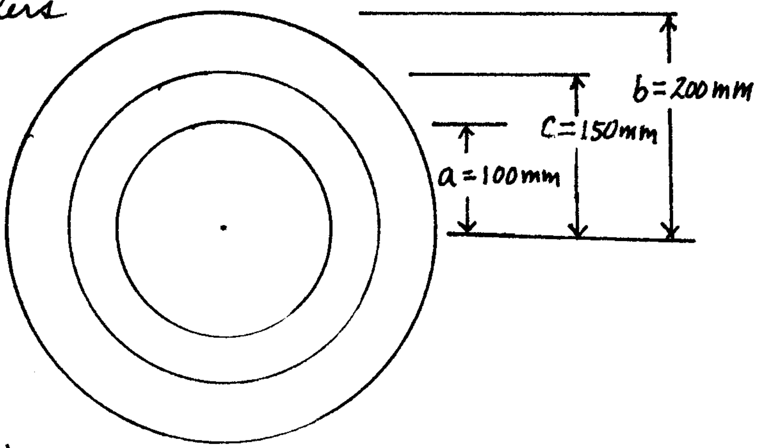


Figure a

Then, by Eqs. (a) and (b), for $r = c = 150$ mm,

$$\left(\sigma_{\theta\theta}^{R}\right)_{\text{inner cyl.}} = -9.722\frac{(150)^2}{(150^2-100^2)}\left(1+\frac{100^2}{150^2}\right) = -25.28\text{ MPa} \quad (c)$$

For the outer cylinder, with $p_1 = p_s$ and $p_2 = 0$, we find by Eq. (11.21)

$$\left(\sigma_{\theta\theta}^{R}\right)_{\text{outer cyl.}} = p_s \frac{c^2}{b^2-c^2}\left(1+\frac{b^2}{r^2}\right) \quad (d)$$

Then, by Eqs. (a) and (d), for $r = c = 150$ mm,

$$\left(\sigma_{\theta\theta}^{R}\right)_{\text{outer cyl.}} = 34.72\text{ MPa} \quad (e)$$

Next the stress $(\sigma_{\theta\theta})_{p_1}$ produced by the inner pressure $p_1 = 200$ MPa at $r = a = 100$ mm is given by

$$(\sigma_{\theta\theta})_{p_1} = p_1 \frac{a^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) \quad (f)$$

(Cont.)

$\boxed{11.10}$ (Cont.) With $r = c = 150$ mm, Eq. (f) yields

$$(\sigma_{\theta\theta})_{p_1} = 185.2 \text{ MPa} \qquad (g)$$

Hence, the net stress at $r = 150$ mm is:

For the inner cylinder, by Eqs. (c) and (g)

$$(\sigma_{\theta\theta})_{\text{inner cyl.}} = (\sigma_{\theta\theta})^{R}_{\text{inner cyl}} + (\sigma_{\theta\theta})_{p_1} = 159.9 \text{ MPa}$$

For the outer cylinder by Eqs. (e) and (g),

$$(\sigma_{\theta\theta})_{\text{outer cyl.}} = (\sigma^{R}_{\theta\theta})_{\text{outer cyl.}} + (\sigma_{\theta\theta})_{p_1} = 219.9 \text{ MPa}$$

$\boxed{11.11}$ Let $a = 40$ mm, $b = 120$ mm, $c = 60$ mm, and $\sigma^{R}_{\theta\theta}$ be the residual stress at $r = a$. Let Δr be the difference in radii $r = c = 60$ mm. Then with Eq. (11.21), $p_1 = 160$ MPa, and $p_2 = 0$, for $\sigma_{\theta\theta} = 130$ MPa,

$$\sigma_{\theta\theta} = 130 \text{ MPa} = \sigma^{R}_{\theta\theta} + p_1 \frac{b^2+a^2}{b^2-a^2} = \sigma^{R}_{\theta\theta} + 160\,\frac{120^2+40^2}{120^2-40^2} = \sigma^{R}_{\theta\theta} + 200$$

Therefore, the residual stress is, using Eq. (11.21),

$$\sigma^{R}_{\theta\theta} = 130 - 200 = -70 \text{ MPa} = -p_s \frac{c^2}{c^2-a^2}\left(1+\frac{a^2}{a^2}\right) = -3.6\, p_s$$

or $p_s = \frac{70}{3.6} = 19.44$ MPa. Then, by Eq. (11.25) [see also Example 11.3],

$$\Delta r = \frac{c}{E(b^2-c^2)}\left[(1-\nu)p_s c^2 + (1+\nu)p_s b^2\right] - \frac{c}{E(c^2-a^2)}\left[-(1-\nu)p_s c^2 - (1+\nu)p_s a^2\right]$$

$$= 0.0323 + 0.0368 = 0.0691 \text{ mm}$$

Hence, the outside diameter of the inner cylinder must be

$$\text{diameter} = 2(c+\Delta r) = 2(60.0691) = 120.138 \text{ mm}$$

11.12

$$\tau_{max} = \frac{\sigma_{\theta\theta(a)} - \sigma_{rr(a)}}{2} = \frac{\sigma_{\theta\theta(c)} - \sigma_{rr(c)}}{2}$$

$$p_1 \frac{b^2+a^2}{b^2-a^2} - p_s \frac{2c^2}{c^2-a^2} - (-p_1) = p_1 \frac{a^2c^2+a^2b^2}{c^2(b^2-a^2)} + p_s \frac{b^2+c^2}{b^2-c^2} - (-p_1) \frac{a^2b^2-a^2c^2}{c^2(b^2-a^2)} - (-p_s)$$

$$160 \frac{120^2+40^2}{120^2-40^2} - p_s \frac{2(60^2)}{60^2-40^2} + 160 = 160 \frac{40^2(60^2+120^2)}{60^2(120^2-40^2)} + p_s \frac{120^2+60^2}{120^2-60^2} + 160 \frac{40^2(120^2-60^2)}{60^2(120^2-40^2)} + p_s$$

$$p_s = 31.91 \text{ MPa}$$

$$\Delta r = 0.0691 \frac{31.91}{19.44} = 0.1134 \text{ mm}$$

$$\text{Diameter} = 2(60.1134) = \underline{120.227 \text{ mm}}$$

$$\tau_{max} = \frac{1}{2}\left[160 \frac{120^2+40^2}{120^2-40^2} - 31.91 \frac{2(60^2)}{60^2-40^2} + 160\right] = \underline{122.56 \text{ MPa}}$$

$$\sigma_{\theta\theta(a)} = 160 \frac{120^2+40^2}{120^2-40^2} - 31.91 \frac{2(60^2)}{60^2-40^2} = \underline{85.12 \text{ MPa}}$$

11.13

$$\sigma_{\theta\theta} = p_1 \frac{b^2+a^2}{b^2-a^2} = 60 \frac{80^2+20^2}{80^2-20^2} = \underline{68.0 \text{ MPa}}$$

11.14

$$\sigma_{\theta\theta(a)} = \sigma_{\theta\theta(c)}$$

$$p_1 \frac{b^2+a^2}{b^2-a^2} - p_s \frac{2c^2}{c^2-a^2} = p_1 \frac{a^2(c^2+b^2)}{c^2(b^2-a^2)} + p_s \frac{b^2+c^2}{b^2-c^2}$$

$$60 \frac{80^2+20^2}{80^2-20^2} - p_s \frac{2(40^2)}{40^2-20^2} = 60 \frac{20^2(40^2+80^2)}{40^2(80^2-20^2)} + p_s \frac{80^2+40^2}{80^2-40^2}$$

$$p_s(2.6667+1.6667) = 68.0 - 20.0$$

$$p_s = 11.08 \text{ MPa}$$

$$\sigma_{\theta\theta(a)} = 60 \frac{80^2+20^2}{80^2-20^2} - 11.08 \frac{2(40^2)}{40^2-20^2} = \underline{38.5 \text{ MPa}}$$

$$\Delta r = \frac{c}{E(b^2-c^2)} [(1-\nu) p_s c^2 + (1+\nu) p_s b^2] - \frac{c}{E(c^2-a^2} [-(1-\nu) p_s c^2 - (1+\nu) p_s a^2]$$

$$= \frac{40}{103{,}000(80^2-40^2)} [0.80(11.08)(40^2) + 1.20(11.08)(80^2) - \frac{40}{103{,}000(40^2-20^2)} [-0.80(11.08)(40^2) - 1.20(11.08)(20^2)]$$

$$= 0.01434 \text{ mm}$$

$$\text{Diameter} = 2(40.01434) = \underline{80.0287 \text{ mm}}$$

11.15 By Eq. (11.25), with $c_o = 100$ mm, $c_i = 100.125$ mm, $a = 0$, $b = 300$ mm, $E = 200 \times 10^3$ N/mm², $\nu = 0.3$, we have (see also Example 11.3)

$$\frac{100}{200\times10^3(90000-10000)}\left[(0.7)(10{,}000)+(1.3)(90{,}000)\right]p_s - \frac{100.125}{200\times10^3(100.125)^2}\left[-(1.3)(100.125)^2\right]p_s$$

$= 0.125$ mm. Therefore, $p_s = 87.67$ MPa. This pressure is an internal pressure for the outer cylinder and an external pressure for the shaft.

Thus, for the shaft, by Eqs. (11.20), (11.21) and (11.15), with $a = 0$, $b = 100.125$, $p_1 = 0$, $p_2 = 87.67 \text{ MPa} = p_s$, $\sigma_{rr} = \sigma_{\theta\theta} = -p_s = -87.67$ MPa, and $\sigma_{zz} = 0$. Hence, the principal stresses (ordered) are $\sigma_1 = 0$, $\sigma_2 = \sigma_3 = -87.67$ MPa

For the cylinder, by Eqs. (11.20), (11.21) and (11.15), with $a = c_o = 100$ mm, $b = 300$ mm, $p_1 = p_s$, $p_2 = 0$, and $r = a = c_o = 100$ mm, $\sigma_{rr} = -p_s = -87.67$ MPa, $\sigma_{\theta\theta} = 1.25 p_s = 109.59$ MPa, and $\sigma_{zz} = 0$. Therefore, the principal stresses (ordered) are

$$\sigma_1 = 109.59 \text{ MPa}, \quad \sigma_2 = 0, \quad \sigma_3 = -87.67 \text{ MPa}.$$

11.16

(a) The critical stress occurs at $r=a$.

$$\sigma_{rr}=-p_1;\quad \sigma_{\theta\theta}=p_1\frac{b^2+a^2}{b^2-a^2}-p_2\frac{2b^2}{b^2-a^2};\quad \tau_{max}=\frac{\sigma_{\theta\theta}-\sigma_{rr}}{2}=\underline{(p_1-p_2)\frac{b^2}{b^2-a^2}}$$

(b) Part (a) is valid as long as $\sigma_{zz}\le\sigma_{\theta\theta}$ or $-\sigma_{zz}\ge\sigma_{rr}$

$$\sigma_{zz}=\frac{p_1a^2-p_2b^2}{b^2-a^2}+\frac{P}{\pi(b^2-a^2)}=\sigma_{\theta\theta}=\frac{p_1(b^2+a^2)}{b^2-a^2}-\frac{2p_2b^2}{b^2-a^2};\quad P=\pi b^2(p_1-p_2)$$

$$-\sigma_{zz}=-\frac{p_1a^2-p_2b^2}{b^2-a^2}-\frac{P}{\pi(b^2-a^2)}=\sigma_{rr}=-p_1;\quad P=\pi b^2(p_2-p_1)$$

$$\underline{\pi b^2(p_2-p_1)\le P\le\pi b^2(p_1-p_2)}$$

11.17 From Problem 11.16 with $p_1=p_Y$

$$\tau_{max}=\tau_Y=(p_Y-p_2)\frac{b^2}{b^2-a^2}$$

$$\underline{p_Y=\frac{\tau_Y(b^2-a^2)}{b^2}+p_2}$$

11.18

$$\sigma_{rr}=-p_1;\quad \sigma_{\theta\theta}=p_1\frac{b^2+a^2}{b^2-a^2};\quad \sigma_{zz}=\frac{\sigma_{\theta\theta}+\sigma_{rr}}{2}=p_1\frac{a^2}{b^2-a^2}$$

$$\tau_{oct}=\frac{1}{3}\sqrt{(\sigma_{\theta\theta}-\sigma_{rr})^2+(\sigma_{\theta\theta}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{rr})^2}$$

$$=\frac{p_1}{3(b^2-a^2)}\sqrt{(b^2+a^2+b^2-a^2)^2+(b^2+a^2-a^2)^2+(a^2+b^2-a^2)^2}$$

$$=\underline{\frac{\sqrt{2}\,p_1b^2}{\sqrt{3}(b^2-a^2)}}$$

11.19

$p_Y=SF\,p_1=1.75(140)=245$ MPa

$$\tau_{max}=\frac{Y}{2}=p_Y\frac{b^2}{b^2-a^2}$$

$$b^2Y-a^2Y=p_Yb^2(2)$$

$$b^2=\frac{a^2Y}{Y-2p_Y}=\frac{40^2(600)}{600-2(245)}=8727$$

$b=93.4$ mm

Diameter $=2b=2(93.4)=$ 186.8 mm

11.20

$$\tau_{oct(max)} = \frac{\sqrt{2}\,Y}{3} = \frac{\sqrt{2}\,p_Y b^2}{\sqrt{3}(b^2-a^2)}$$

$$b^2Y - a^2Y = \sqrt{3}\,p_Y b^2$$

$$b = \sqrt{\frac{a^2Y}{Y-\sqrt{3}\,p_Y}} = \sqrt{\frac{40^2(600)}{600-\sqrt{3}(245)}} = 73.9 \text{ mm}$$

$$\text{Diameter} = 2b = 2(73.9) = \underline{147.8 \text{ mm}}.$$

11.21 Yielding is iniated at inner radius where $r=a$.

$$\sigma_{rr} = -p_1 = -30.0 \text{ MPa}; \quad \sigma_{\theta\theta} = p_1\frac{b^2+a^2}{b^2-a^2} = 30\,\frac{80^2+60^2}{80^2-60^2} = 107.1 \text{ MPa}$$

$$\sigma_{zz} = p_1\frac{a^2}{b^2-a^2} + \frac{P}{\pi(b^2-a^2)} = 30\,\frac{60^2}{80^2-60^2} + \frac{650{,}000}{\pi(80^2-60^2)} = 112.5 \text{ MPa}$$

At initiation of yielding the loads and hence the stresses are increased by the factor of safety SF.

(a)
$$\tau_{max} = \frac{Y}{2} = SF\,\frac{\sigma_{max}-\sigma_{min}}{2}$$

$$SF = \frac{Y}{\sigma_{max}-\sigma_{min}} = \frac{280}{112.5-(-30.0)} = \underline{1.96}$$

(b)
$$\tau_{oct(max)} = \frac{\sqrt{2}\,Y}{3} = \frac{SF}{3}\sqrt{(\sigma_{\theta\theta}-\sigma_{rr})^2+(\sigma_{zz}-\sigma_{rr})^2+(\sigma_{zz}-\sigma_{\theta\theta})^2}$$

$$SF = \frac{\sqrt{2}(280)}{\sqrt{(107.1+30.0)^2+(112.5+30.0)^2+(112.5-107.1)^2}} = \underline{2.00}$$

11.22 Use maximum principal stress theory of failure.

$$\sigma_u = SF\,\sigma_{\theta\theta} = SF\,p_1\frac{b^2+a^2}{b^2-a^2}$$

$$p_1 = \frac{\sigma_u}{SF}\,\frac{b^2-a^2}{b^2+a^2} = \frac{160}{3.00}\,\frac{30^2-15^2}{30^2+15^2} = \underline{32.0 \text{ MPa}}$$

11.23 (a) Let $a=50$mm, $b=150$mm, and $c=75$mm. Due to internal pressure $SF(p_1)$, the stresses at $r=a$ and $r=c$ are found to be

$$\sigma_{rr(a)} = -SF\,p_1 = -1.85p_1 \qquad \sigma_{rr(c)} = 1.85p_1\frac{50^2(75^2-150^2)}{75^2(150^2-50^2)} = -0.694\,p_1$$

$$\sigma_{\theta\theta(a)} = 1.85p_1\frac{150^2+50^2}{150^2-50^2} = 2.313\,p_1 \qquad \sigma_{\theta\theta(c)} = 1.85p_1\frac{50^2(75^2+150^2)}{75^2(150^2-50^2)} = 1.156\,p_1$$

Superimposed on the above stresses the shrinking pressure p_s causes residual stresses $\sigma^R_{rr(a)}$ and $\sigma^R_{\theta\theta(a)}$ at $r=a$ and $\sigma^R_{rr(c)}$ and $\sigma^R_{\theta\theta(c)}$ at $r=c$.

(cont.)

11.23 Continued

Pressure p_s is an external pressure for the inner cylinder.

$\sigma^R_{rr(a)} = 0$ $\qquad$ $\sigma^R_{\theta\theta(a)} = -2p_s \frac{75^2}{75^2-50^2} = -3.600\,p_s$

Pressure p_s is an internal pressure for the outer cylinder.

$\sigma^R_{rr(c)} = -p_s$ $\qquad$ $\sigma^R_{\theta\theta(c)} = p_s \frac{150^2+75^2}{150^2-75^2} = 1.667\,p_s$

The actual stresses at $r=a$ and $r=c$ are

$\sigma_{rr(a)} = -1.85\,p_1$ $\qquad$ $\sigma_{rr(c)} = -0.694\,p_1 - p_s$

$\sigma_{\theta\theta(a)} = 2.313\,p_1 - 3.600\,p_s$ $\qquad$ $\sigma_{\theta\theta(c)} = 1.156\,p_1 + 1.667\,p_s$

The maximum shearing stress yield condition states

$$\tau_{max} = \frac{Y}{2} = \frac{\sigma_{\theta\theta(a)} - \sigma_{rr(a)}}{2} = \frac{\sigma_{\theta\theta(c)} - \sigma_{rr(c)}}{2} = \frac{4.163p_1 - 3.600p_s}{2} = \frac{1.85p_1 + 2.667p_s}{2}$$

$p_s = 0.3691\,p_1$

$$\frac{Y}{2} = \frac{700}{2} = \frac{4.163\,p_1 - 3.600(0.3691p_1)}{2}$$

$p_1 = \underline{247.0 \text{ MPa}}$; $\quad p_s = 0.3691(247.0) = \underline{91.2 \text{ MPa}}$

(b) $$\Delta r = \frac{75}{200{,}000(150^2-75^2)}\left[0.71(91.2)(75^2)+1.29(91.2)(150^2)\right] - \frac{75}{200{,}000(75^2-50^2)}\left[-0.71(91.2)(75^2)-1.29(91.2)(50^2)\right]$$

$= 0.1459$; $\quad$ Diameter $= 2(75.1459) = \underline{150.292 \text{ mm}}$

11.24 Let $a = 50$ mm, $b = 150$ mm, and $c = 75$ mm. Without residual stresses

$\sigma_{\theta\theta(a)} = p_1 \frac{150^2+50^2}{150^2-50^2} = 1.25\,p_1$ $\qquad$ $\sigma_{\theta\theta(c)} = p_1 \frac{50^2(75^2+150^2)}{75^2(150^2-50^2)} = 0.625\,p_1$

The shrinking pressure p_s causes circumfrential residual stress $\sigma^R_{\theta\theta(a)}$ at $r=a$ and $\sigma^R_{\theta\theta(c)}$ at $r=c$. Pressure p_s is an external pressure for the inner cylinder and an internal pressure for the outer cylinder.

$\sigma^R_{\theta\theta(a)} = -p_s \frac{2(75^2)}{75^2-50^2} = -3.600\,p_s$ $\qquad$ $\sigma^R_{\theta\theta(c)} = p_s \frac{150^2+75^2}{150^2-75^2} = 1.667\,p_s$

The maximum principal stress criterion of failure states

$\sigma_{max} = \sigma_u = 480 = 1.25\,p_1 - 3.6\,p_s = 0.625\,p_1 + 1.667\,p_s$; $\quad p_s = 0.1187\,p_1$

$480 = 1.25\,p_1 - 3.6(0.1187\,p_1)$

$p_1 = \underline{583.5} \text{ MPa}$; $\quad p_s = 0.1187(583.5) = \underline{69.3 \text{ MPa}}$

11.25

$$p_P = 2\tau_Y \ln\frac{b}{a} = 2\,\frac{460}{2}\ln\frac{210}{90} = \underline{389.8\ \text{MPa}}$$

11.26

(a) $p_1 = \frac{p_P}{SF} = \frac{389.8}{3.00} = \underline{129.9\ \text{MPa}}$

(b) $p_Y = \tau_Y\,\frac{b^2-a^2}{b^2} = \frac{460}{2}\,\frac{210^2-90^2}{210^2} = 187.8\ \text{MPa}$

$$SF = \frac{p_Y}{p_1} = \frac{187.8}{129.9} = \underline{1.45}$$

11.27

$$\tau_{Y1} = \frac{Y_1}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.9\ \text{MPa}; \quad \tau_{Y2} = \frac{Y_2}{\sqrt{3}} = \frac{600}{\sqrt{3}} = 346.4\ \text{MPa}$$

$$p_P = 2\tau_{Y1}\ln\frac{b_1}{a_1} + 2\tau_{Y2}\ln\frac{b_2}{a_2} = 2(230.9)\ln\frac{30}{20} + 2(346.4)\ln\frac{60}{30} = \underline{667.5\ \text{MPa}}$$

11.28

(a) $\tau_Y = \frac{Y}{2} = \frac{450}{2} = 225\ \text{MPa}$; $p_P = 2(225)\ln\frac{40}{20} = \underline{311.9\ \text{MPa}}$; $\sigma_{\theta\theta} = 2(225)\left(1-\ln\frac{40}{20}\right) = 138.1\ \text{MPa}$

(b) Unloading linearly from p_P gives theoretical residual circumferential stress $\sigma_{\theta\theta(T)}^{R}$.

$$\sigma_{\theta\theta(T)}^{R} = 138.1 - 311.9\,\frac{40^2+20^2}{40^2-20^2} = -381.7\ \text{MPa}$$

It is assumed that the actual residual circumferential stress is 50% of the theoretical.

$$\sigma_{\theta\theta}^{R} = 0.5(-381.7) = \underline{-190.9\ \text{MPa}}$$

(c) For pressure $SF(p_1)$ with zero residual stresses,

$$\sigma_{rr} = -1.80\,p_1 \qquad \sigma_{\theta\theta} = 1.80\,p_1\,\frac{40^2+20^2}{40^2-20^2} = 3.000\,p_1$$

Adding these stresses to residual stresses gives

$$\sigma_{rr} = -1.80\,p_1 \qquad \sigma_{\theta\theta} = 3.000\,p_1 - 190.9$$

which can be substituted into the maximum shearing stress yield condition to obtain

$$\tau_{max} = \frac{Y}{2} = \frac{\sigma_{\theta\theta}-\sigma_{rr}}{2} = \frac{450}{2} = \frac{3.000\,p_1 - 190.9 - (-1.80\,p_1)}{2}; \quad p_1 = \underline{133.5\ \text{MPa}}$$

Without the residual stresses, the maximum shearing stress yield condition gives

$$\tau_{max} = \frac{450}{2} = \frac{3.000\,p_1 - (-1.80\,p_1)}{2}; \quad p_1 = \underline{93.8\ \text{MPa}}$$

11.29

$$\sigma_{\theta\theta} = \frac{0.0000117(200{,}000)(100)}{2(1-0.29)\ln\frac{250}{100}}\left[1-\ln\frac{250}{100}-\frac{100^2(250^2+100^2)}{100^2(250^2-100^2)}\ln\frac{250}{100}\right]=\underline{-212.5\text{ MPa}}$$

$$\sigma_{rr}=\underline{0};\quad \sigma_{zz}=\sigma_{\theta\theta}+\sigma_{rr}=\underline{-212.5\text{ MPa}}$$

11.30

$$\text{(a) }\sigma_{rr}=0;\quad \sigma_{\theta\theta}=\sigma_{zz}=-212.5\frac{T_0}{100}=-2.125\,T_0\ (\text{MPa})$$

$$\tau_{max}=\frac{Y}{2}=\frac{\sigma_{max}-\sigma_{min}}{2}=\frac{500}{2}=\frac{0-(-2.125T_0)}{2};\quad T_0=\underline{235.3\,^\circ\text{C}}$$

$$\text{(b) }\tau_{oct(max)}=\frac{\sqrt{2}(500)}{3}=\frac{1}{3}\sqrt{(0+2.125T_0)^2+(-2.125T_0+2.125T_0)^2+(-2.125T_0-0)^2}$$

$$T_0=\underline{235.3\,^\circ\text{C}}$$

11.31

$$\sigma_{rr}=\underline{-100.0\text{ MPa}}$$

$$\sigma_{\theta\theta}=-2.125T_0+p_1\frac{b^2+a^2}{b^2-a^2}=-2.125(50)+100\frac{250^2+100^2}{250^2-100^2}=\underline{31.8\text{ MPa}}$$

$$\sigma_{zz}=-2.125T_0+p_1\frac{a^2}{b^2-a^2}=-2.125(50)+100\frac{100^2}{250^2-100^2}=\underline{-87.2\text{ MPa}}$$

11.32 For $p_1=p_2=0$ and $T_0=70\,^\circ\text{C}$

$$\sigma_{rr}=0$$

$$\sigma_{\theta\theta}=\sigma_{zz}=\frac{0.000020(96{,}500)(70)}{2(1-0.35)\ln\frac{75}{35}}\left[1-\ln\frac{75}{35}-\frac{35^2(75^2+35^2)}{35^2(75^2-35^2)}\ln\frac{75}{35}\right]=-129.4\text{ MPa}$$

(a) With internal pressure p_1 and $T_0=70\,^\circ\text{C}$

$$\sigma_{rr}=-p_1$$

$$\sigma_{\theta\theta}=-129.4+p_1\frac{75^2+35^2}{75^2-35^2}=-129.4+1.557p_1$$

$$\sigma_{zz}=-129.4+p_1\frac{35^2}{75^2-35^2}=-129.4+0.278p_1$$

$$\tau_{max}=\frac{Y}{2}=\frac{\sigma_{max}-\sigma_{min}}{2}=\frac{240}{2}=\frac{-129.4+1.557p_1-(-p_1)}{2};\quad p_1=\underline{144.5\text{ MPa}}$$

(b) With external pressure p_2 and $T_0=70\,^\circ\text{C}$

$$\sigma_{rr}=0$$

$$\sigma_{\theta\theta}=-129.4-2p_2\frac{75^2}{75^2-35^2}=-129.4-2.557p_2$$

$$\sigma_{zz}=-129.4-p_2\frac{75^2}{75^2-35^2}=-129.4-1.278p_2$$

(cont.)

11.32 Continued

$$\tau_{max} = \frac{Y}{2} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{240}{2} = \frac{0-(-129.4-2.557p_2)}{2}; \quad p_2 = \underline{43.2\ \text{MPa}}$$

(c) With internal pressure p_1 and $T_0 = 0$

$\sigma_{rr} = -p_1$

$\sigma_{\theta\theta} = 1.557\,p_1$

$\sigma_{zz} = 0.278\,p_1$

$$\tau_{max} = \frac{Y}{2} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{240}{2} = \frac{1.557p_1 - (-p_1)}{2}; \quad p_1 = \underline{93.9\ \text{MPa}}$$

With external pressure p_2 and $T_0 = 0$

$\sigma_{rr} = 0$

$\sigma_{\theta\theta} = -2.557\,p_2$

$\sigma_{zz} = -1.278\,p_2$

$$\tau_{max} = \frac{Y}{2} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{240}{2} = \frac{0-(-2.557p_2)}{2}; \quad p_2 = \underline{93.9\ \text{MPa}}$$

11.33 By Eq. (11.54), the maximum stress is $\sigma_{\theta\theta}$ or

$$(\sigma_{\theta\theta})_{max} = \frac{3+\nu}{4}\rho b^2 \omega^2 \left[1 + \frac{1-\nu}{3+\nu}\frac{a^2}{b^2}\right] \qquad (a)$$

With $a = 150$ mm, $b = 300$ mm, $\rho = 7200$ kg/m^3, $E = 70$ GPa, $\nu = 0.25$, and $\sigma_u = 170$ MPa, Eq. (a) yields

$$(\sigma_{\theta\theta})_{max} = \sigma_u = 170\times10^6 = \frac{3.25}{4}(7200)(0.30)^2\omega^2\left[1 + \frac{0.75}{3.25}\,\frac{150^2}{300^2}\right]$$

or

$$556.850\,\omega^2 = 170\times10^6$$

Hence,

$$\omega = 552.5 \text{ rad/s} = 5276 \text{ rpm}$$

11.34 Since the wheel is bonded to the rigid steel shaft at $r = a$ and no forces act at $r = b$,

$$u = 0 \quad \text{at} \quad r = a = 100\text{ mm} \qquad (a)$$

$$\sigma_{rr} = 0 \quad \text{at} \quad r = b = 400\text{ mm} \qquad (b)$$

By Eq. (11.47), the general solution for u is, with $T = 0$,

$$u = -\frac{(1-\nu^2)}{8E}\rho r^3 \omega^2 + C_1 r + \frac{C_2}{r} \qquad (c)$$

Since $a = 100$ mm, $b = 400$ mm, $\rho = 2000\text{ kg/m}^3$, $E = 12\text{ GPa}$, $\nu = 0.32$, Eqs. (a) and (c) yield

$$C_1 + 100\, C_2 = 1.87 \times 10^{-10}\, \omega^2 \qquad (d)$$

By Eq. (11.49), the general solution for σ_{rr} is

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[-\frac{(3+\nu)(1-\nu^2)}{8E}\rho r^2 \omega^2 + (1+\nu)C_1 - \frac{(1-\nu)}{r^2}C_2\right] \qquad (e)$$

For $r = b = 400$ mm, Eqs. (b) and (e) yield

$$C_1 - 3.2196\overline{96}\, C_2 = 7.52533 \times 10^{-9}\omega^2 \qquad (f)$$

The solution of Eqs. (d) and (f) is

$$C_1 = 7.29643 \times 10^{-9}\omega^2, \quad C_2 = -7.10943 \times 10^{-11}\omega^2 \qquad (g)$$

Then, by Eqs. (11.49), (11.50), and (g), we obtain

$$\sigma_{rr} = \omega^2(-830.00 r^2 + 128.76 + 0.64631 r^{-2})$$

$$\sigma_{\theta\theta} = \omega^2(-490.00 r^2 + 128.76 - 0.64631 r^{-2}) \qquad (h)$$

By Eqs. (h), the maximum values of $\sigma_{\theta\theta}$ and σ_{rr} are

(Cont)

11.34

$$(\sigma_{rr})_{max} = 185.09\,\omega^2 \text{ at } r = a = 100\text{ mm}$$

$$(\sigma_{\theta\theta})_{max} = 93.17\,\omega^2 \text{ at } r = 190.6\text{ mm} \qquad (i)$$

Hence,

$$(\sigma_{\theta\theta})_{max} < (\sigma_{rr})_{max}$$

Therefore,

$$(S.F)(\sigma_{rr})_{max} = \sigma_u = 20\text{ MPa} \qquad (j)$$

Equation (j) determines the maximum allowable rotational speed. So, by the first of Eqs. (i) and Eq. (j), with S.F = 2, the allowable rotational speed is

$$\omega = 232.44\text{ rad/s} = 2220\text{ rpm}$$

11.35 The boundary conditions are

$$u = 0 \text{ at } r = a \qquad (a)$$

$$\sigma_{rr} = 0 \text{ at } r = b \qquad (b)$$

By Eq. (11-47), with $T = 0$, the general solution for u is

$$u = -\frac{(1-\nu^2)}{8E}\rho r^3\omega^2 + C_1 r + C_2/r \qquad (c)$$

Hence, for $r = a$, Eqs. (a) and (c) yield

$$a^2C_1 + C_2 = \frac{(1-\nu^2)}{8E}\rho a^4\omega^2 \qquad (d)$$

By Eq. (11.49), the general solution for σ_{rr} is

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[-\frac{(3+\nu)(1-\nu^2)}{8E}\rho r^2\omega^2 + (1+\nu)C_1 - \frac{(1-\nu)}{r^2}C_2\right] \quad (e)$$

For $r = b$, Eqs. (b) and (e) yield

$$(1+\nu)\,b_1^2\,C_1 - (1-\nu)C_2 = \frac{(3+\nu)(1-\nu^2)}{8E}\rho b^4\omega^2 \qquad (f)$$

The solution of Eqs. (d) and (f) is

(cont.)

11.35 cont.

$$C_1 = \frac{(1-\nu^2)}{8E}\rho b^2\omega^2\left[\frac{(1-\nu)a^4 + (3+\nu)b^4}{(1-\nu)a^2b^2 + (1+\nu)b^4}\right]$$

$$C_2 = \frac{(1-\nu^2)}{8E}\rho b^2\omega^2\left[\frac{(1+\nu)a^4 - (3+\nu)a^2b^2}{(1-\nu)a^2 + (1+\nu)b^2}\right]$$

11.36 (a) Since $\sigma_{zz} = 0$, the maximum shear stress criterion yields

$$\tau_{max} = \frac{Y}{2} = \text{maximum of } \left|\frac{\sigma_{rr} - \sigma_{zz}}{2}\right| \text{ or } \left|\frac{\sigma_{\theta\theta} - \sigma_{zz}}{2}\right|, \qquad (a)$$

$$\text{or } \left|\frac{\sigma_{\theta\theta} - \sigma_{rr}}{2}\right| = \text{maximum of } \left|\frac{\sigma_{rr}}{2}\right|, \left|\frac{\sigma_{\theta\theta}}{2}\right|, \left|\frac{\sigma_{\theta\theta} - \sigma_{rr}}{2}\right|.$$

By Example 11.7, Eq. (g)

$$(\sigma_{rr})_{max} = (\sigma_{\theta\theta})_{max} = \frac{3+\nu}{8}\rho b^2\omega^2 \qquad (b)$$

Hence, by Eqs. (a) and (b), with $\omega = \omega_Y$,

$$Y = \frac{3+\nu}{8}\rho b^2\omega_Y^2 \; ; \quad \omega_Y = \sqrt{\frac{8Y}{(3+\nu)\rho b^2}} \qquad (c)$$

(b) When the angular speed increases beyond ω_Y, a plastic zone develops at $r = 0$ and spreads radially outward to $r = r_p$. Since in the elastic region $\sigma_{\theta\theta} > \sigma_{rr}$ [see Eqs. (f) of Example 11.7], at $r = r_p$ yielding is governed by the condition $Y = |\sigma_{\theta\theta} - \sigma_{zz}| = |\sigma_{\theta\theta}|$ or $Y = \sigma_{\theta\theta}$ since $\sigma_{\theta\theta}$ is positive. So $\sigma_{\theta\theta} = Y$ in the plastic region $0 \le r \le r_p$. But σ_{rr} is not known in the plastic region. However, σ_{rr} can be determined by the equilibrium condition [Eq. (11.45)], with $\sigma_{\theta\theta} = Y$. Hence,

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\sigma_{rr} = \frac{1}{r}\frac{d}{dr}(r\sigma_{rr}) = \frac{Y}{r} - \rho r\omega^2$$

Integration yields

$$\sigma_{rr} = Y - \frac{1}{3}\rho r^2\omega^2 + \frac{C}{r} \qquad (d)$$

(Cont.)

11.36 cont where C is the constant of integration. Since the plastic region includes $r=0$, $C=0$ for bounded stress. Then, Eq. (d) reduces to

$$\sigma_{rr} = Y - \frac{1}{3}\rho r^2 \omega^2 \qquad (e)$$

When the disk is fully plastic $r = b = r_p$. Then, since $\sigma_{rr} = 0$ at $r = b$, Eq. (e) yields

$$\omega = \omega_p = \sqrt{\frac{3Y}{\rho b^2}} \qquad (f)$$

By Eqs. (c) and (f),

$$\frac{\omega_p}{\omega_Y} = \sqrt{\frac{3(3+\nu)}{8}} \qquad (g)$$

(c) To determine the residual stress that remain in the disk, after it comes to rest, we subtract the elastic stresses computed with $\omega = \omega_p$ from the plastic stresses. So by Eqs (f) of Example 11.7, the elastic stresses are

$$\begin{aligned}(\sigma_{rr})_E &= \frac{3+\nu}{8}\rho b^2 \omega_p^2 \left(1 - \frac{r^2}{b^2}\right)\\ (\sigma_{\theta\theta})_E &= \frac{3+\nu}{8}\rho b^2 \omega_p^2 \left[1 - \frac{(1+3\nu)}{(3+\nu)}\frac{r^2}{b^2}\right]\end{aligned} \qquad (h)$$

The plastic stresses are

$$\begin{aligned}(\sigma_{rr})_p &= Y - \frac{1}{3}\rho r^2 \omega_p^2\\ (\sigma_{\theta\theta})_p &= Y\end{aligned} \qquad (i)$$

So, by Eqs. (h) and (i), the residual stresses are

$$(\sigma_{rr})_R = (\sigma_{rr})_p - (\sigma_{rr})_E = Y - \frac{\rho\omega_p^2}{24}\left[3(3+\nu)b^2 - (1+3\nu)r^2\right]$$

$$(\sigma_{\theta\theta})_R = (\sigma_{\theta\theta})_p - (\sigma_{\theta\theta})_E = Y - \frac{\rho\omega_p^2}{8}\left[(3+\nu)b^2 - (1+3\nu)r^2\right]$$

or substituting for ω_p by Eq. (f), we have

(cont.)

11.36 cont.

$$(\sigma_{rr})_R = -\frac{(1+3\nu)Y}{8}\left(1-\frac{r^2}{b^2}\right)$$

$$(\sigma_{\theta\theta})_R = -\frac{(1+3\nu)Y}{8}\left(1-\frac{3r^2}{b^2}\right)$$

Note that with $\sigma_{zz}=0$ and $0 \le \nu \le 0.5$),

$$\left|(\sigma_{rr})_R - \sigma_{zz}\right|_{\substack{max \\ (r=0)}} = \frac{1+3\nu}{8}Y < Y$$

$$\left|(\sigma_{\theta\theta})_R - \sigma_{zz}\right|_{\substack{max \\ (r=0)}} = \frac{1+3\nu}{8}Y < Y$$

$$\left|(\sigma_{\theta\theta})_R - (\sigma_{rr})_R\right|_{\substack{max \\ (r=b)}} = \frac{1+3\nu}{4} < Y$$

Hence, upon unloading further yielding does not occur.

11.37 (a) With $T = T_0(1-\frac{r}{b})$, Eq. (11.46) yields, with $\omega = 0$,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = (1+\nu)\alpha T_0 \frac{d}{dr}\left(1-\frac{r}{b}\right) \qquad (a)$$

Integration of Eq. (a) yields

$$u = \frac{(1+\nu)\alpha T_0}{6}\left(3r - \frac{2r^2}{b}\right) + C_1 r + \frac{C_2}{r} \qquad (b)$$

Since $u=0$ at $r=0$, $C_2=0$. Then, by the first of Eqs. (11.44) and (b),

$$\sigma_{rr}^T = \frac{E}{1-\nu^2}\left[(1-\nu^2)\alpha T_0\left(\frac{r}{3b}-\frac{1}{2}\right)+(1+\nu)C_1\right] \qquad (c)$$

By the boundary condition $\sigma_{rr}=0$ at $r=b$, Eq. (c) yields

$$C_1 = \frac{1}{6}(1-\nu)\alpha T_0 \qquad (d)$$

Then, by Eqs. (c) and (d),

$$\sigma_{rr}^T = \frac{E\alpha T_0}{3}\left(\frac{r}{b}-1\right) \qquad (e)$$

(cont.)

11.37 cont. and by Eqs. (b), (d), and the second of Eqs. (11.44),

$$\sigma_{\theta\theta}^{T} = \frac{E\alpha T_0}{3}\left(\frac{2r}{b} - 1\right) \qquad (f)$$

(b) Superposing the elastic stresses due to ω [see Eqs. (f), Example 11.7] on the stresses due to T [Eqs. (e) and (f) above], we obtain

$$\begin{aligned} \sigma_{rr} &= \frac{3+\nu}{8}\rho b^2\omega^2\left(1 - \frac{r^2}{b^2}\right) + \frac{E\alpha T_0}{3}\left(\frac{r}{b} - 1\right) \\ \sigma_{\theta\theta} &= \frac{3+\nu}{8}\rho b^2\omega^2\left[1 - \frac{(1+3\nu)}{(3+\nu)}\frac{r^2}{b^2}\right] + \frac{E\alpha T_0}{3}\left(\frac{r}{b} - 1\right) \end{aligned} \qquad (g)$$

at r = 0, Eqs. (g) yield

$$\sigma_{rr} = \frac{3+\nu}{8}\rho b^2\omega^2 - \frac{E\alpha T_0}{3}$$

$$\sigma_{\theta\theta} = \frac{3+\nu}{8}\rho b^2\omega^2 - \frac{E\alpha T_0}{3}$$

Hence, at r = 0, both $\sigma_{\theta\theta}$ and σ_{rr} are decreased due to T

at r = b, Eqs. (g) yield

$$\sigma_{rr} = 0$$

$$\sigma_{\theta\theta} = \frac{1-\nu}{4}\rho b^2\omega^2 + \frac{E\alpha T_0}{3}$$

Hence, at r = b, σ_{rr} remains unchanged (it is zero at the boundary r = b). However, $\sigma_{\theta\theta}$ is increased by the term $E\alpha T_0/3$.

11.38 Since in the elastic state, before yielding occurs, $\sigma_{\theta\theta} > \sigma_{rr}$ for $r \neq 0$ (see Example 11.7), yielding occurs when $\sigma_{\theta\theta} = Y$. Hence, in the plastic region, $0 \le r \le r_P$,

$$\sigma_{\theta\theta} = Y \qquad (a)$$

but σ_{rr} is unknown in the plastic region; it may be determined by the equilibrium condition, Eq. (11.45), with $\sigma_{\theta\theta} = Y$. Thus,

$$\frac{1}{r}\frac{d}{dr}(r\sigma_{rr}) = \frac{Y}{r} - \rho r \omega^2 \qquad (b)$$

Integration of Eq. (b) yields

$$\sigma_{rr} = Y - \frac{1}{3}\rho r^2 \omega^2 + \frac{C}{r} \qquad (c)$$

where C is a constant of integration. Since σ_{rr} must be bounded at $r = 0$, $C = 0$ in Eq. (c). Then, in the plastic region, by Eqs. (a) and (c),

$$\sigma_{rr} = Y - \frac{1}{3}\rho r^2 \omega^2, \quad \sigma_{\theta\theta} = Y \qquad (d)$$

In the elastic region, $r_P \le r \le b$, the stresses are given by Eqs. (11.49) and (11.50), which for simplicity may be written as

$$\begin{aligned} \sigma_{rr} &= -\frac{3+\nu}{8}\rho r^2 \omega^2 + C_3 - \frac{1}{r^2}C_4 \\ \sigma_{\theta\theta} &= -\frac{1+3\nu}{8}\rho r^2 \omega^2 + C_3 + \frac{1}{r^2}C_4 \end{aligned} \qquad (e)$$

where C_3 and C_4 are constants. Hence, by Eqs. (d) and (e), with the continuity conditions $\sigma_{rr}(r^+) = \sigma_{rr}(r^-)$, $\sigma_{\theta\theta}(r^+) = \sigma_{\theta\theta}(r^-)$ at $r = r_P$, we obtain

(Cont.)

11.38 cont.

$$-\frac{3+\nu}{8}\rho r_p^2\omega^2+C_3-\frac{1}{r_p^2}C_4 = Y-\frac{1}{3}\rho r_p^2\omega^2$$
$$-\frac{1+3\nu}{8}\rho r_p^2\omega^2+C_3+\frac{1}{r_p^2}C_4 = Y \qquad (f)$$

Equations (f) are two equations in terms of three unknowns C_3, C_4, and r_p. A third equation is obtained from the condition that in the elastic region $r_p \le r \le b$ $\sigma_{rr}=0$ at $r=b$. Thus, by the first of Eqs. (e), we have

$$-\frac{3+\nu}{8}\rho b^2\omega^2+C_3-\frac{1}{b^2}C_4=0 \qquad (g)$$

Solving Eqs. (f) for C_3 and C_4 in terms of r_p, we find

$$C_3 = Y+\frac{1+3\nu}{12}\rho\omega^2 r_p^2$$
$$C_4 = \frac{1+3\nu}{24}\rho\omega^2 r_p^4 \qquad (h)$$

Then substitution of Eqs. (h) into Eq. (g), yields, after some simplification,

$$\left(\frac{r_p}{b}\right)^4-2\left(\frac{r_p}{b}\right)^2+\frac{24}{1+3\nu}\left(\frac{3+\nu}{8}-\frac{Y}{\rho b^2\omega^2}\right)=0$$

Hence, by the quadratic rule and simplification, we find

$$\left(\frac{r_p}{b}\right)^2 = 1-\sqrt{\frac{8}{1+3\nu}\left[\frac{3Y}{\rho b^2\omega^2}-1\right]} \qquad (i)$$

Then, with the numerical values $\nu=0.29$, $\rho=7850$ kg/mm³, $Y=620$ MPa, and $b=0.400$ m, we obtain

$$r_p = 0.400\sqrt{1-\sqrt{\frac{6.3353\times10^6}{\omega^2}-4.27807}} \quad [\mathrm{m}] \qquad (j)$$

As a check, note that by Eq. (i) or Eq. (j)

(cont.)

11.38 cont.

$$r_p = 0, \text{ for } \omega = \omega_Y = \sqrt{\frac{8Y}{(3+\nu)\rho b^2}} = 1095.59 \text{ rad/s}$$

and $$r_b = b = 0.400 \text{ m}, \text{ for } \omega = \omega_P = \sqrt{\frac{3Y}{\rho b^2}} = 1216.92 \text{ rad/s}$$

11.39 (a) With $T = T_0 \frac{r}{b}$, Eq. (11.47) yields

$$u = -\frac{(1-\nu^2)}{8E}\rho r^3 \omega^2 + \frac{(1+\nu)}{3b} T_0 r^2 + C_1 r + \frac{C_2}{r} \qquad (a)$$

Since $u = 0$ at $r = 0$, $C_2 = 0$. Then, by the first of Eqs. (11.44) and (a),

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu \frac{du}{dr}\right) - \frac{E\alpha T_0}{(1-\nu)b} r$$

or

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[-\frac{(1-\nu^2)}{8E}(3+\nu)\rho r^2 \omega^2 - \frac{(1-\nu^2)\alpha T_0}{3b} r + (1+\nu) C_1\right] \qquad (b)$$

also, since $\sigma_{rr} = 0$ at $r = b$, Eq. (b) yields

$$C_1 = \frac{(1-\nu)(3+\nu)\rho b^2 \omega^2}{8E} + \frac{(1-\nu)\alpha T_0}{3} \qquad (c)$$

Then, by Eqs. (b) and (c),

$$\sigma_{rr} = \frac{3+\nu}{8}\rho\omega^2 (b^2 - r^2) + \frac{E\alpha T_0}{3}\left(1 - \frac{r}{b}\right)$$

Similarly, by Eqs. (a), (c), and the second of Eqs. (11.44),

$$\sigma_{\theta\theta} = \frac{\rho\omega^2}{8}\left[(3+\nu)b^2 - (1+3\nu)r^2\right] + \frac{E\alpha T_0}{3}\left(1 - \frac{2r}{b}\right)$$

(b) By Eqs. (a) and (c), the increase of the radius b is, with $r = b$,

$$u = \frac{(1-\nu)}{4E}\rho b^3 \omega^2 + \frac{2}{3}\alpha T_0 b$$

11.40 In the elastic state before yielding occurs, $\sigma_{\theta\theta} > \sigma_{rr}$ [see Eqs. (11.49) and (11.50)]. Hence, yielding first occurs when $\sigma_{\theta\theta} = Y$. Hence, in the plastic region $a \le r \le r_p$, where r_p denotes the location of the plastic-elastic interface,

$$\sigma_{\theta\theta} = Y \tag{a}$$

but σ_{rr} is unknown in the plastic region. The stress σ_{rr} in the plastic region may be determined by the equilibrium condition [Eq. (11.45)], with $\sigma_{\theta\theta} = Y$. Thus,

$$\frac{1}{r}\frac{d}{dr}(r\sigma_{rr}) = \frac{Y}{r} - \rho r \omega^2 \tag{b}$$

Integration of Eq. (b) yields

$$\sigma_{rr} = Y - \frac{1}{3}\rho r^2 \omega^2 + \frac{C}{r} \tag{c}$$

where C is a constant of integration. Since $\sigma_{rr} = 0$ at $r = a$, we obtain, by Eq. (c),

$$C = \frac{1}{3}\rho a^3 \omega^2 - aY \tag{d}$$

Equations (a), (c), and (d) yield for the plastic region,

$$\begin{aligned} (\sigma_{rr})_p &= \left(1 - \frac{a}{r}\right)Y - \frac{1}{3}\rho\omega^2\left(r^2 - \frac{a^3}{r}\right) \\ (\sigma_{\theta\theta})_p &= Y \end{aligned} \tag{e}$$

In the elastic region, $r_p \le r \le b$, the stresses are given by Eqs. (11.49) and (11.50), which for simplicity may be written

$$\begin{aligned} (\sigma_{rr})_E &= -\frac{3+\nu}{8}\rho r^2 \omega^2 + C_3 - \frac{1}{r^2}C_4 \\ (\sigma_{\theta\theta})_p &= -\frac{3+\nu}{8}\rho r^2 \omega^2 + C_3 + \frac{1}{r^2}C_4 \end{aligned} \tag{f}$$

Then, by Eqs. (e) and (f), for continuity of stresses at $r = r_p$, we have

(cont.)

11.40 cont. $(\sigma_{rr})_E = (\sigma_{rr})_P$ and $(\sigma_{\theta\theta})_E = (\sigma_{\theta\theta})_P$ or

$$-\frac{3+\nu}{8}\rho r_P^2\omega^2 + C_3 - \frac{1}{r_P^2}C_4 = Y\left(1-\frac{a}{r_P}\right) - \frac{1}{3}\rho\omega^2\left(r_P^2 - \frac{a^3}{r_P}\right) \qquad (g)$$
$$-\frac{1+3\nu}{8}\rho r_P^2\omega^2 + C_3 + \frac{1}{r_P^2}C_4 = Y$$

Also in the elastic region at $r = b$, $\sigma_{rr} = 0$, or by the first of Eqs. (f)

$$-\frac{3+\nu}{8}\rho b^2\omega^2 + C_3 - \frac{1}{b^2}C_4 = 0 \qquad (h)$$

With $a = 0.10$ m, $b = 0.40$ m, $\rho = 7850$ kg/m^3, $Y = 620$ MPa, $\nu = 0.29$, and $r_P = \frac{1}{2}(a+b) = 0.25$ m, Eqs (g) and (h) are three equations in the three unknowns ω, C_3, and C_4. Substitution of numerical values into Eqs. (g) and (h) and solving for ω yields

$$\omega = 930.85 \text{ rad/s}$$

This value of ω compares to $\omega_Y = 769.52$ rad/s and to $\omega_P = 1062.21$ rad/s.

12.1 By Eq. (12.15), for a pinned-end column,

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (a)$$

where by Table A.1, Appendix A

$$E = 200\ \text{GPa} \qquad (b)$$

and by Table B.1, Appendix B,

$$I_x = \frac{\pi d^4}{64} = \frac{\pi (0.05)^4}{64} = 3.068 \times 10^{-7}\ \text{m}^4 \qquad (c)$$

Hence, with $P_{cr} = 133$ kN, Eqs. (a), (b), and (c) yield

$$L = \sqrt{\frac{\pi^2 EI}{P_{cr}}} = \sqrt{\frac{\pi^2 (200\times 10^9)(3.068\times 10^{-7})}{133 \times 10^3}} = 2.134\ \text{m}$$

12.2 (a) By Eq. (12.15), for a pinned-end column, with a safety factor of 2 and $P_{allow.} = 65$ kN, we have

$$P_{cr} = 2P_{allow.} = 2(65) = \frac{\pi^2 EI}{L^2} \qquad (a)$$

For A36 steel, E = 200 GPa, and since L = 2.2 m, Eq. (a) yields

$$I_{min} = \frac{2(65)(10^3)(2.2)^2}{\pi^2 (200 \times 10^9)} = 3.19 \times 10^{-7}\ \text{m}^4 \qquad (b)$$

For an equal-leg angle section (Fig. a),

$$A = bt + (b-t)t = 2bt - t^2$$

$$A\bar{x} = (bt)\frac{b}{2} + (b-t)(t)\left(\frac{t}{2}\right)$$

$$= \frac{1}{2}\left(b^2 t + bt^2 - t^3\right)$$

or

$$\bar{x} = \bar{y} = \frac{b^2 t + bt^2 - t^3}{2(2bt - t^2)} \qquad (c)$$

also,

$$I_{xx} = I_{yy} = \frac{1}{12}bt^3 + bt\left(\bar{y} - \frac{t}{2}\right)^2 + \frac{1}{12}t(b-t)^3 + (b-t)t\left[\frac{1}{2}(b-t) + t - \bar{y}\right]^2$$

$$(d)$$

$$I_{xy} = bt\left(\bar{x} - \frac{b}{2}\right)\left(\bar{y} - \frac{t}{2}\right) + (b-t)(t)\left(\bar{x} - \frac{t}{2}\right)\left[\bar{y} - t - \frac{1}{2}(b-t)\right]$$

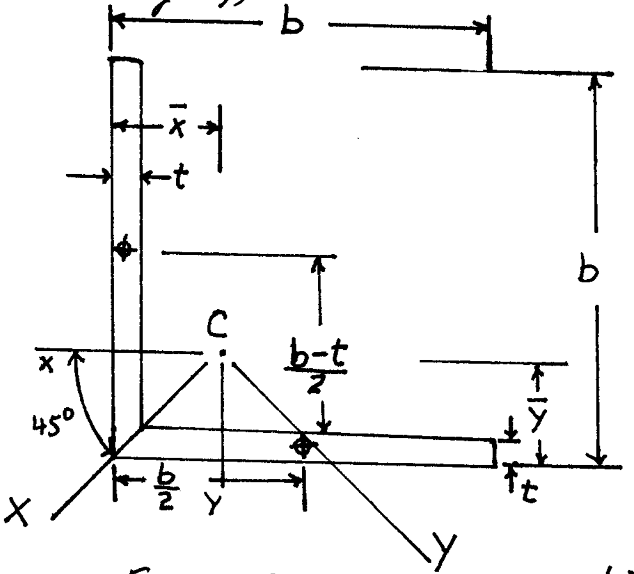


Figure a

(cont.)

12.2 cont.

By Eqs. (B.13), Appendix B, the principal moments of inertia are (Fig. a)

$$I_{XX} = \tfrac{1}{2}(I_{xx}+I_{yy})+I_{xy}$$
$$I_{YY} = \tfrac{1}{2}(I_{xx}+I_{yy})-I_{xy} \qquad \text{(e)}$$

By Eqs. (c), (d), and (e) and Table C.3, Appendix C, we find by trial and error that the equal-leg angle section L76×76×12.7 ($b = 76$ mm and $t = 12.7$ mm) is satisfactory. That is

$$I_{xx} = I_{yy} = 0.915\times10^{-6}\ \text{m}^4$$
$$I_{xy} = -0.528\times10^{-6}\ \text{m}^4$$
$$I_{XX} = (0.915-0.528)\times10^{-6} = 3.87\times10^{-7}\ \text{m}^4$$
$$I_{YY} = (0.915+0.528)\times10^{-6} = 1.443\times10^{-6}\ \text{m}^4$$

So, the minimum moment of inertia is $I_{XX} = 3.87\times10^{-7}\ \text{m}^4$, which is slightly larger than the required minimum $I_{min} = 3.19\times10^{-7}\ \text{m}^4$. As a check, by Table C.3,

$$I_{XX}(=I_{ZZ}) = Ar^2 = (1770\ \text{mm}^2)(14.8\ \text{mm})^2 = 3.877\times10^{-7}\ \text{m}^4.$$

(b) By Table A.1, Appendix A, $Y = 250$ MPa for A36 steel. By Eq. (a), $P_{cr} = 130$ kN, and by Table C.3, $A = 1.770\times10^{-3}\ \text{m}^2$. Hence,

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{130\times10^3}{1.770\times10^{-3}} = 73.45\ \text{MPa} < Y = 250\ \text{MPa}$$

Therefore, the column fails by buckling.

12.3 By Table A.1, Appendix A, for white oak, $E = 13.8$ GPa. With a safety factor of 4 and with $I = \frac{1}{12}b^4$ for a square cross section ($b \times b$), Eq. (12.15) yields, with $L = 7.5$ m and a load of 80 kN,

$$P_{cr} = 4(80 \times 10^3) = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (13.8 \times 10^9)(b^4/12)}{(7.5)^2}$$

or

$$b = 0.19956 \text{ m}$$

(b) Since $b = 0.19956$ m, the area A of the cross section is $A = b^2 = 0.0398 \text{ m}^2$, and

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{4(80 \times 10^3)}{0.0398} = 8.035 \text{ MPa}$$

Hence, since the proportional limit is $\sigma_{PL} = 26$ MPa, $\sigma_{PL} > \sigma_{cr}$ and the use of the Euler formula is justified.

12.4 (a) The box section section formed by two L 102 × 89 × 9.5 angle sections is shown in Fig. a. By Fig. a,

$$I_{xx} = \tfrac{1}{12}(98.5)(102)^3 - \tfrac{1}{12}(79.5)(83)^3 = 4.9227 \times 10^{-6} \text{ m}^4$$

$$I_{yy} = \tfrac{1}{12}(102)(98.5)^3 - \tfrac{1}{12}(83)(79.5)^3 = 4.6479 \times 10^{-6} \text{ m}^4$$

Therefore, $I_{min} = I_{yy} = 4.6479 \times 10^{-6} \text{ m}^4$

also, by Fig. a, the cross section area is

$$A = (102)(98.5) - (83)(79.5) = 3.4485 \times 10^{-3} \text{ m}^2$$

Hence, $r = \sqrt{I_{min}/A} = 0.0367$ m.

So, the slenderness ratio is

$l/r = 6/0.0367 = 163.4$

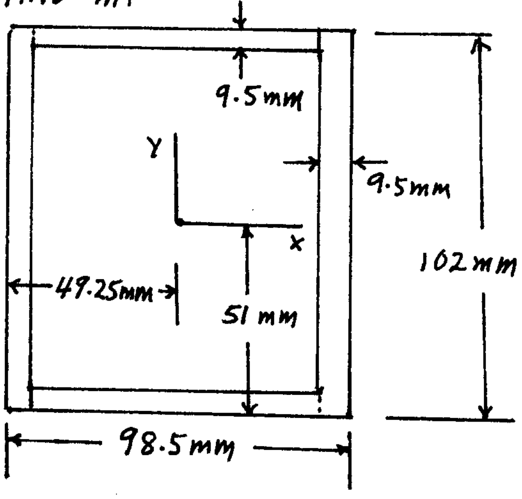


Figure a

(b) The critical stress is $\sigma_{cr} = \dfrac{\pi^2 E}{(l/r)^2}$

or $\sigma_{cr} = \dfrac{\pi^2 (200 \times 10^9)}{(163.4)^2} = 73.90$ MPa.

Since $Y = 250$ MPa, $\sigma_{cr} < Y$. Therefore, the use of Euler's equation is justified.

12.5

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi b a^3}{64}\cdot\frac{4}{\pi a b}} = \frac{a}{4} = \frac{50}{4} = 12.5$$

$$\frac{L}{r} = \frac{2000}{12.5} = 160$$

$$P_{cr} = \frac{\pi^2 E A}{(L/r)^2} = \frac{\pi^2 (11,000)(\pi)(150)(50)}{4\,(160)^2} = 24,980\ \text{N}$$

$$P = \frac{P_{cr}}{SF} = \frac{24.98}{1.5} = 16.65\ \text{kN}$$

12.6

For the square, $A = b^2 = 2000\ \text{mm}^2$; $b = 44.72\ \text{mm}$

$$I = \frac{1}{12} b^4 = \frac{(44.72)^4}{12} = 333,300\ \text{mm}^4$$

$$P_{cr} = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 (72,000)(333,300)}{(750)^2} = 421\ \text{kN}$$

For the circle, $A = \frac{\pi D^2}{4} = 2000\ \text{mm}^2$; $D = 50.46\ \text{mm}$

$$I = \frac{\pi D^4}{64} = \frac{\pi (50.46)^4}{64} = 318,200\ \text{mm}^4$$

$$P_{cr} = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 (72,000)(318,200)}{(750)^2} = 402\ \text{kN}$$

For the hollow circle, $A = \frac{\pi}{4}(D^2 - 30^2) = 2000\ \text{mm}^2$; $D = 58.71\ \text{mm}$

$$I = \frac{\pi}{64}(58.71^4 - 30^4) = 543,400\ \text{mm}^4$$

$$P_{cr} = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 (72,000)(543,400)}{(750)^2} = 686\ \text{kN}$$

12.7

(a) Let A_1 = area of hole
$= 2500\ \text{mm}^2$

Let A_2 = area of solid
$= 100^2 - 50^2 = 7500\ \text{mm}^2$

Let A = area bounded by outer perimeter $= 10{,}000\ \text{mm}^2$

$$\therefore A_2 y_0 = A \times 50 - A_1 \times 55$$
$$= 10000 \times 50 - 2500 \times 55$$
$$= 362500\ \text{mm}^3$$

$$\therefore y_0 = \frac{362500}{7500} = 48.33\ \text{mm}$$

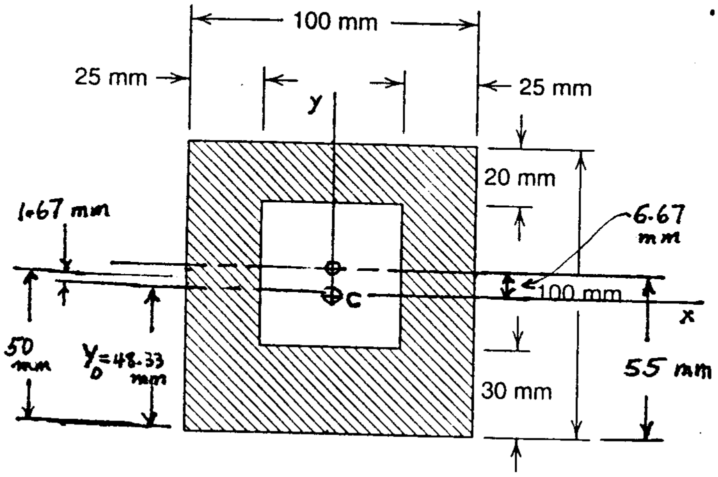


Figure P12.7

By the figure,

$$I_x = \frac{1}{12}(100)(100)^3 + (100)(100)(1.67)^2 - \frac{1}{12}(50)(50)^3 - (50)(50)(6.67)^2$$
$$= 7{,}729{,}166.7\ \text{mm}^4 \quad \therefore r_x = \sqrt{\frac{I_x}{A_2}} = 32.10$$

$$I_y = \frac{1}{12}(100)(100)^3 - \frac{1}{12}(50)(50)^3 = 7{,}812{,}500\ \text{mm}^4 \quad \therefore r_y = \sqrt{\frac{I_y}{A_2}} = 32.27$$

Hence, $r_{min} = r_x$ and $(L/r_x)_{max} = \frac{2000}{32.10} = 62.31$

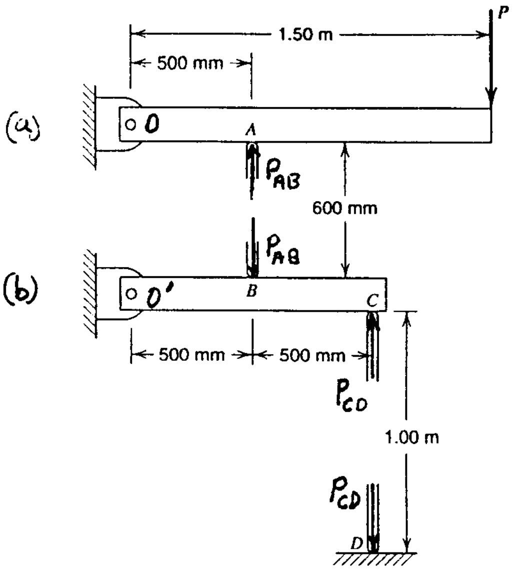
(b) Assuming the column to be slender, the pin-ended column Euler formula yields

$$P_{cr} = \frac{\pi^2 E A_2}{(L/r_x)^2} = \frac{\pi^2 (72\times10^9)(7500)(\frac{1}{10^6})}{(62.31)^2}$$

or

$$P_{cr} = 1373\ \text{kN}$$

12.8



By Fig. (a),

$$\sum M_0 = 1500P - 500P_{AB} = 0$$

$$P_{AB} = 3P$$

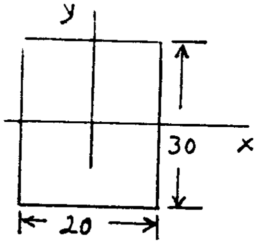
By Fig. (b),

$$\sum M_{0'} = 1000P_{CD} - 500P_{AB} = 0$$

$$\therefore \quad P_{CD} = \frac{1}{2}P_{AB} = \frac{3}{2}P$$

Therefore, the load on column AB is 3P, and the load on column CD is $\frac{3}{2}P$.

Consider axes (x, y), Fig. (c).



(c)

$$I_x = \frac{1}{12}(20)(30)^3 = 45{,}000 \text{ mm}^4$$

$$I_y = \frac{1}{12}(30)(20)^3 = 20{,}000 \text{ mm}^4 = I_{min}$$

For column AB, by Eq. (12.2),

$$3P = P_{cr(AB)} = \frac{\pi^2 E I_y}{(L_{AB})^2} = \frac{\pi^2(72000)(20{,}000)}{(600)^2} = 39.48 \text{ kN}$$

$$\therefore \; P = 13.16 \text{ kN}$$

For column CD,

$$\frac{3}{2}P = P_{cr(CD)} = \frac{\pi^2 E I_y}{(L_{CD})^2} = \frac{\pi^2(72000)(20{,}000)}{(1000)^2} = 14.21 \text{ kN}$$

$$\therefore \quad P = 9.47 \text{ kN}$$

Therefore, the critical load for the system is

$$P = P_{cr} = 9.47 \text{ kN}$$

Thus, column CD buckles before column AB.

12.9 $I = \frac{\pi d^4}{64} = \frac{\pi(10)^4}{64} = 490.9\ mm^4$

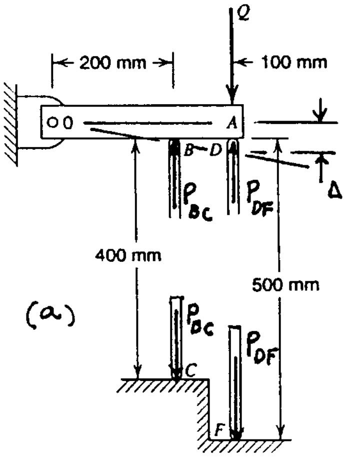
$A = \frac{\pi d^2}{4} = \frac{\pi(10)^2}{4} = 78.54\ mm^2$

For column BC, $E_S = 200$ GPa

For Column DF, $E_A = 72$ GPa

By Fig. , $\Sigma M_0 = 300Q - 300P_{DF} - 200P_{BC} = 0$

or, $1.5P_{DF} + P_{BC} = 1.5Q$ (a)



(a)

Under the action of load Q, bar OA rotates and causes column DF to compress an amount Δ, and column BC to compress an amount $\frac{2}{3}\Delta$, Fig. a. Therefore, the stress in Column DF due to Δ is

$$\sigma_{DF} = \frac{P_{DF}}{A} = E_A\epsilon_{DF} = E_A\frac{\Delta}{L_{DF}} \quad (b)$$

and the stress in column BC is

$$\sigma_{BC} = \frac{P_{BC}}{A} = E_S\epsilon_{BC} = E_S\frac{\frac{2}{3}\Delta}{L_{BC}} = E_S\frac{2\Delta}{3L_{BC}} \quad (c)$$

or

$$P_{DF} = E_A\frac{(\Delta)(A)}{L_{DF}}\ ;\quad P_{BC} = \frac{2E_S(\Delta)(A)}{3L_{BC}} \quad (d)$$

Substitution of Eqs.(d) into Eq.(a) yields

$$1.5Q = \left(\frac{3E_A}{2L_{DF}} + \frac{2E_S}{3L_{BC}}\right)(\Delta)(A) = \left(\frac{3}{2}\,\frac{72\times10^3}{500} + \frac{2}{3}\,\frac{200\times10^3}{400}\right)(78.54)\Delta$$

where Δ is in mm and Q is in Newtons. Therefore $\Delta = 3.477\times10^{-5}Q$. Substitution of Δ into Eqs.(d) yields

$$P_{DF} = 0.393Q \text{ and } P_{BC} = 0.910Q$$

Thus, for column BC, $P_{cr} = 0.910Q = \pi^2E_SI/L_{BC}^2$; or $\pi^2(200{,}000)(490.9)/[400^2\times0.910] = Q$; $Q = 6.655$ kN. For Column DF, $P_{cr} = 0.393Q = \pi^2E_AI/L_{DF}^2$; $\pi^2(72{,}000)(490.9)/[500^2\times0.393] = Q$. ∴ $Q = 3.550$ kN. Therefore, column DF collapses first, when $Q = Q_{cr} = 3.550$ kN

12.10

$$A = \frac{\pi}{4}(d_o^2 - d_i^2) = \frac{\pi}{4}(101.6^2 - 90.1^2) = 1731.4 \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{4}(101.6^4 - 90.1^4) = 1{,}995{,}544 \text{ mm}^4$$

The critical load for the pipe is

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (200{,}000)(1{,}995{,}544)}{(5000)^2} = 157.56 \text{ kN}.$$

Therefore, the initial load of 80 kN does not cause the column to buckle. However, since body AB is fixed, a temperature increase ΔT causes an additional load $P_T = EA\alpha(\Delta T)$ or

$$P_T = (200{,}000) \times 11.7 \times 10^{-6} \times 1731.4 (\Delta T) = 4.05 (\Delta T) \text{ kN}.$$

Hence

$$P_{cr} = P_T + 80 = 80 + 4.05(\Delta T) = 157.56 \text{ kN}$$

or $\Delta T = 19.1°C$. Therefore, the temperature at which the column buckles is $T = 20 + 19.1 = 39.1°C$.

12.11

$$A = \frac{\pi}{4}(d_o^2 - d_i^2) = \frac{\pi}{4}(104^2 - 100^2) = 640.9 \text{ mm}^2$$

$$I = \frac{\pi}{64}(d_o^2 - d_i^2) = \frac{\pi}{64}(104^4 - 100^4) = 833{,}798 \text{ mm}^4$$

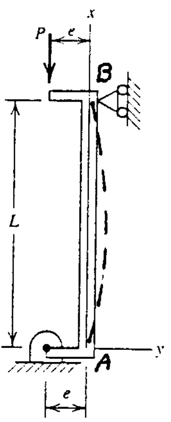
(a) Let the pressure in the vessel be p. Therefore, the axial load in the pressure vessel is

$P = \frac{\pi}{4} d_i^2 p = \frac{\pi}{4}(100)^2 p = 7854\, p$. The critical axial load for the vessel is $P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (200{,}000)(833{,}798)}{(9000)^2}$

or $P_{cr} = 20319 \text{ N} = 7854 p$. Hence, $p = 2587 \text{ kN/m}^2 = 2.587 \frac{\text{N}}{\text{mm}^2}$

(b) At buckling, $\sigma_{average} = \frac{P_{cr}}{A} = \frac{20319}{640.9} = 31.7 \text{ MPa}$

12.12 The buckled form (dashed line) is shown in Fig (a). The free-body diagram of section AB is shown in Fig. (b). The moment at section x is given by (see Fig. c.)



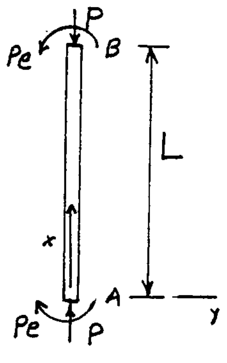
(a)

$$\Sigma M_A = Pe + M(x) + Py = 0$$

or $$M(x) = -Py - Pe = EIy''$$

∴ $$EIy'' + Py = -Pe$$ or

$$y'' + k^2 y = -k^2 e\ ;\ k^2 = \frac{P}{EI} \quad (a)$$



(b)

The solution of Eq.(a) is

$$y = A\sin kx + B\cos kx - e \quad (b)$$

The boundary conditions are

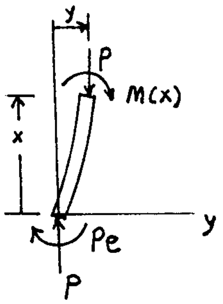
$$y = 0 \quad \text{for } x = 0, L \qquad (c)$$

Therefore, Eqs.(b) and (c) yield

$$A = e(1-\cos kL)/\sin kL\ ,\ B = e$$

Hence,

$$y(x) = e\left[\frac{1-\cos kL}{\sin kL}\cdot\sin kx + \cos kx - 1\right] \quad (d)$$



(c)

As P increases, the deflection $y(x)$ of the column increases. When attains values for which $kL = \pi, 2\pi, ---, m\pi$, y becomes infinitely large, since $\sin kL$ goes to zero as kL goes to $\pi, 2\pi, ---, n\pi$. $\sin kL$ first becomes zero when $kL = \pi$. Therefore, $k^2 = \frac{\pi^2}{L^2} = \frac{P_{cr}}{EI}$; or

$$P_{cr} = \frac{\pi^2 EI}{L^2}\ .$$

12.13 The free-body diagram of AB (Fig. a) is shown in Fig. b. Summation of moments about point A (Fig. b) yields

$$\Sigma M_A = Pe + Pe - BL = 0; \text{ or } B = \frac{2Pe}{L}$$

$$\Sigma F_y = B - A = 0 \therefore A = B = \frac{2Pe}{L}.$$ The moment at section x is given by (see Fig. c) $\Sigma M_A = -Pe + M(x) + Py + \frac{2Pe}{L}x = 0$

or $$M(x) = -Py + Pe\left(1 - \frac{2x}{L}\right) = EIy''$$

Hence, $$y'' + k^2 y = k^2 e\left(1 - \frac{2x}{L}\right) \quad (a)$$

where $k^2 = P/EI$. The solution of Eq(a) is

$$y(x) = A \sin kx + B \cos kx + e\left(1 - \frac{2x}{L}\right) \quad (b)$$

The boundary conditions are

$$y = 0, \text{ for } x = 0, L \quad (c)$$

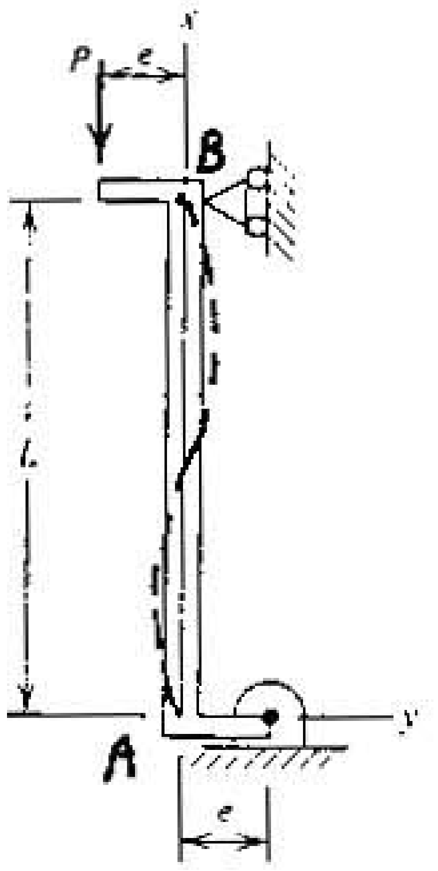
Equations (b) and (c) yield

$$A = \frac{e(1 + \cos kL)}{\sin kL}, \quad B = -e$$

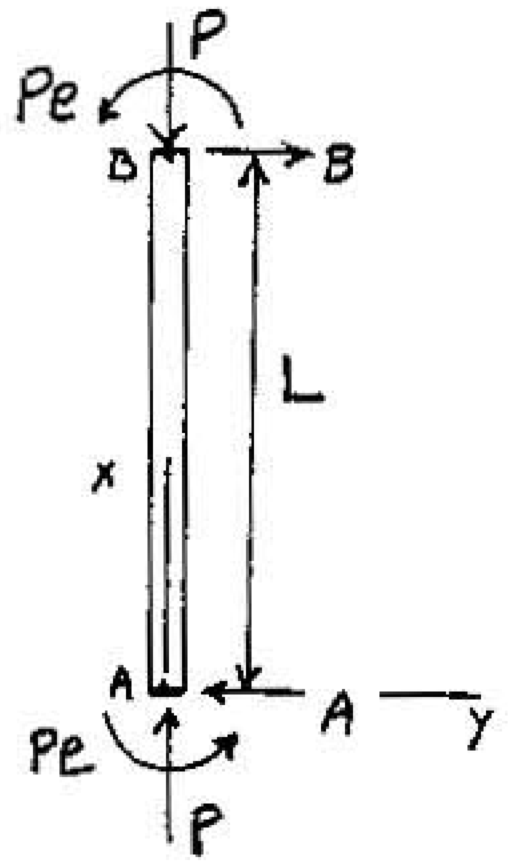
Therefore,

$$y = e\left[\frac{1 + \cos kL}{\sin kL} \sin kx - \cos kx + 1 - \frac{2x}{L}\right] \quad (d)$$

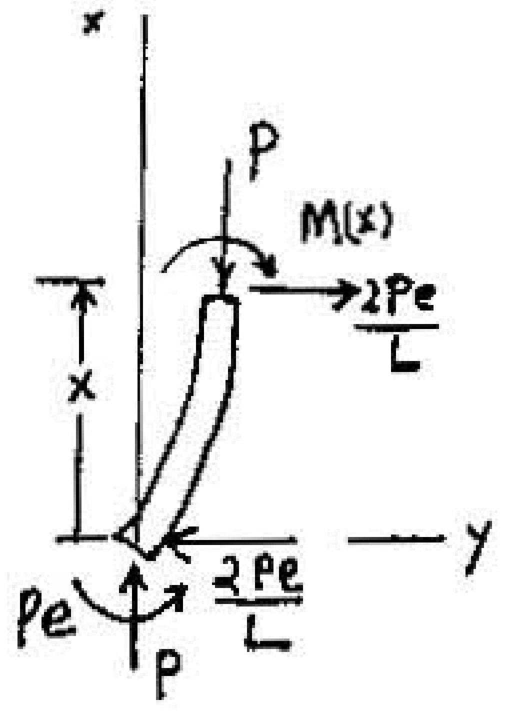
As in Prob. 12.12, the condition $y \to \infty$ gives the buckling loads, for $\sin kL = 0$; or $kL = n\pi$, $n = 1, 2, \ldots$. However, for $n = 1, 3, 5, \ldots$ the ratio [Eq.(d)] $\frac{1 + \cos kL}{\sin kL} = \frac{0}{0} \to 0$ by L'Hospital's rule. Hence, the critical load occurs for $n = 2$. Then, $kL = 2\pi$ and $P_{cr} = \frac{\pi^2 EI}{(L/2)^2} = \frac{4\pi^2 EI}{L^2}$, and the column buckles into two lobes (Fig. a). For n = 4, the column buckles into four lobes, etc.



(a)



(b)



(c)

2.14 (a) For buckling in the x direction, the column buckles into two lobes (Fig. a). Hence, by Eq. (12.17), with $E = 72 GPa$ and $L = 1.8$ m, $P_2 = 4\dfrac{\pi^2 E I_y}{L^2}$. By Fig. a, $I_y = \frac{1}{12}(100)(38)^3 = 4.5727\times10^{-7} m^4$. Hence,

$$P_2 = 4\,\frac{\pi^2(72\times10^9)(4.5727\times10^{-7})}{(1.8)^2} = 401.16\,kN \quad (a)$$

For buckling in the y direction, the column buckles in one lobe (Fig. b). Hence, by Eq. (12.15), $P_1 = \dfrac{\pi^2 E I_x}{L^2}$, where by Fig. b,

$$I_x = \tfrac{1}{12}(38)(100)^3 = 3.1667\times10^{-6}\,m^4$$

So,

$$P_1 = \frac{\pi^2(72\times10^9)(3.1667\times10^{-6})}{(1.8)^2} = 694.54\,kN$$

Therefore since $P_2 < P_1$, the buckling load is

$$P_{cr} = P_2 = 401.16\ kN$$

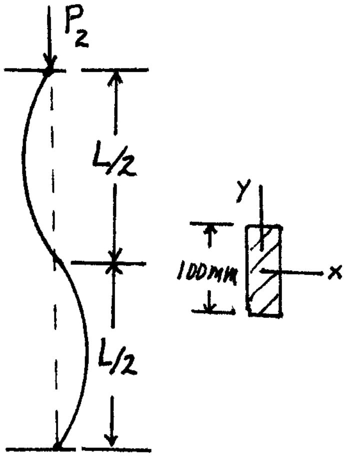


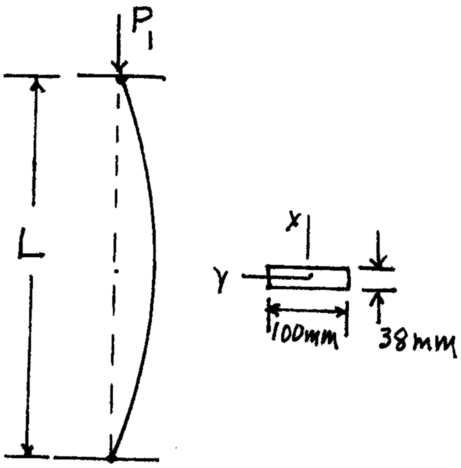
Figure a

(b) Since the cross-sectional area is $A = 38(100) = 3800\,mm^2 = 0.0038\,m^2$, the stress at buckling is

$$\sigma_{cr} = \frac{P_2}{A} = 105.57\ MPa$$

The yield stress for 7075-T6 aluminum is $Y = 500$ MPa. So, the proportional limit is

$$\sigma_{PL} = 0.9Y = 450\,MPa$$



Figure

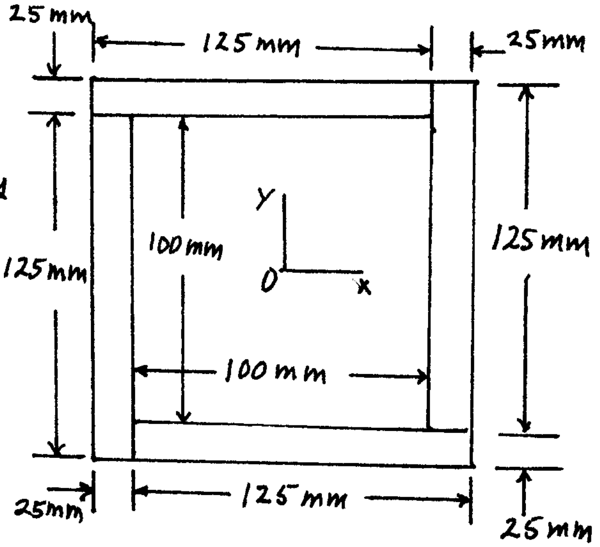
Hence, $\sigma_{cr} < \sigma_{PL}$.

Therefore, the use of Euler's equation is justified.

12.15 (a) The cross section of the column is shown in Fig. a. By Fig. a,

$$I_{xx} = I_{yy} = \frac{1}{12}(150)^4 - \frac{1}{12}(100)^4 = 33.854 \times 10^{-6}\ \text{m}^4 \quad (a)$$

The buckled shape is shown in Fig. b. The forced boundary conditions are



$$y = y' = 0 \ \text{at}\ x = 0 \qquad (b)$$
$$y = 0 \quad \text{at}\ x = L$$

The natural boundary condition is

$$EI\,y'' = 0 \ \text{at}\ x = L \quad (c)$$

The general solution is

$$y = A\sin kx + B\cos kx + Cx + D \quad (d)$$

By Eqs. (b), (c), and (d), we obtain

$$B + D = 0, \text{ or } D = -B$$
$$Ak + C = 0, \text{ or } C = -Ak \qquad (e)$$
$$A\sin kL + B\cos kL + CL + D = 0$$
$$A\sin kL + B\cos kL = 0, \text{ or } B = -A\tan kL$$

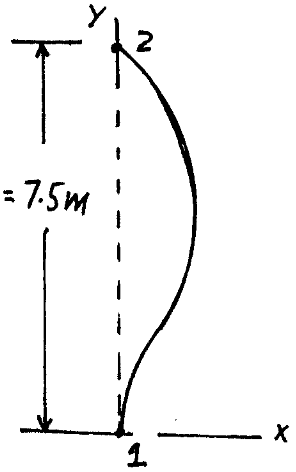


Figure b

Substituting for B, C, and D in the third of Eqs. (e), we find in terms of A,

$$A(\tan kL - kL = 0.$$

or since $A \neq 0$,

$$\tan kL = kL \quad (f)$$

The smallest nonzero solution of Eq. (f) is $kL = 4.493$, or

$$k = \frac{4.493}{7.5} = 0.60\ \text{m}^{-1} \quad (g)$$

(Cont.)

12.15 cont. Therefore, with $E = 12.5$ GPa (see Table A.1) and Eqs. (a) and (g),

$$P_{cr} = k^2 EI = (0.60)^2(12.5\times10^9)(33.854\times10^{-6}) = 152.34 \text{ kN} \quad (h)$$

(b) By Fig. a, the cross-sectional area of the column is

$$A = 4(125)(25) = 0.0125 \text{ m}^2 \quad (i)$$

So by Eqs. (h) and (i), $\sigma_{cr} = P_{cr}/A = 12.19$ MPa. Therefore, for the use of the Euler equation to be justified, we must have

$$\sigma_{PL} > \sigma_{cr} = 12.19 \text{ MPa}.$$

12.16 By Fig. a, the area of the cross section of the column is $A = (100)(150) = 0.015 \text{ m}^2$. Its length is $L = 6.0$ m. Also,

$$I_x = \tfrac{1}{12}(100)(150)^3 = 28.125\times10^{-6} \text{ m}^4$$
$$I_y = \tfrac{1}{12}(150)(100)^3 = 12.50\times10^{-6} \text{ m}^4$$

Hence, for buckling about the x axis, the column is pinned. Therefore, with $E = 8.3$ GPa,

$$(P_{cr})_x = \frac{\pi^2 E I_x}{L^2} = \frac{\pi^2(8.3\times10^9)(28.125\times10^{-6})}{(6)^2}$$

or $(P_{cr})_x = 64.0$ kN

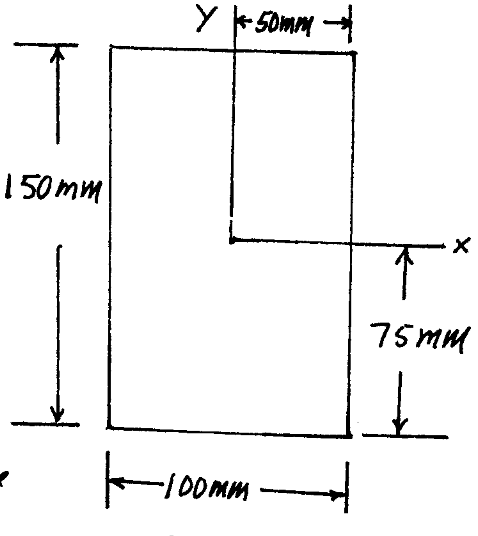


Figure a

For deflection in the x direction (buckling about the y axis), the column ends are constrained, with an effective length factor $K = 0.60$. Therefore, by the Euler equation, the buckling load is

$$(P_{cr})_y = \frac{\pi^2 E I_y}{(L_{eff})^2} = \frac{\pi^2(8.3\times10^9)(12.5\times10^{-6})}{(0.60\times6)^2} = 79.01 \text{ kN}$$

(cont.)

12.16 cont. Therefore, since $(P_{cr})_y > (P_{cr})_x$, the critical buckling load for the column is

$$P_{cr} = (P_{cr})_x = 64.0 \, kN$$

(b) The critical stress is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{64\times10^3}{0.015} = 4.27 \, MPa$$

Since the proportional limit 20.7 MPa is greater than 4.27 MPa, the Euler equation is applicable

12.17 This problem is similar to Problem 12.15. Here, we solve it by using Table 12.1. By Table 12.1, the effective length of a fixed-pinned column is 0.70L. Hence, the critical buckling stress is

$$\sigma_{cr} = \frac{\pi^2 E I}{(0.70L)^2 A} \qquad (a)$$

The cross-sectional area of the column (Fig. a) is

$$A = 4(125)(25 = 0.0125 \, m^2 \qquad (b)$$

also,

$$I_{xx} = I_{yy} = 33.854\times10^{-6} \, m^4 \qquad (c)$$

With $E = 13.8$ GPa and $L = 7.6$ m, Eqs. (a), (b), and (c) yield

$$\sigma_{cr} = \frac{\pi^2(13.8\times10^9)(33.854\times10^{-6})}{(0.70\times7.6)^2(0.0125)} = 13.03 \, MPa$$

Hence, the minimum required proportional limit stress is

$$\sigma_{PL} > 13.03 \, MPa$$

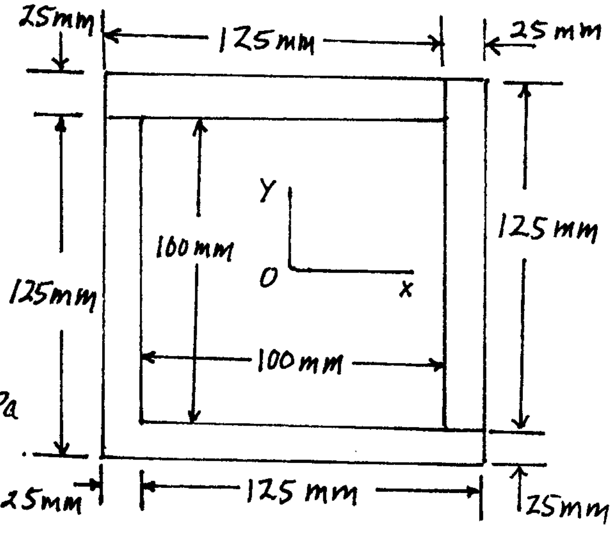


Figure b

12.18

(a) $P_{cr} = \dfrac{\pi^2 EI}{\left(\frac{L}{2}\right)^2} = \dfrac{\pi^2 E\left(\frac{\pi d^4}{64}\right)}{\left(\frac{L}{2}\right)^2} = \dfrac{\pi^3 E d^4}{16 L^2}$. For a safety factor of 2.00, the column must be designed to carry a load of 2(40 kN) = 80 kN. Therefore,

$$P_{cr} = 80\,\text{kN} = \frac{\pi^3 d^4 E}{16 L^2} \text{ or } d^4 = \frac{(80{,}000)(16)(2.5)^2}{\pi^3(200\times 10^9)} = 12.90\times 10^{-7}\,\text{m}^4$$

or $d^4 = 12.90\times 10^{-7}\,\text{m}^4 = 12.90\times 10^5\,\text{mm}^4$

$\therefore\ d = 33.70$ mm.

(b) $A = \dfrac{\pi d^2}{4} = \dfrac{\pi(33.70)^2}{4} = 892.0\ \text{mm}^2$

In part (a), the column was designed to buckle at $P_{cr} = 80$ kN. To buckle elastically at 80 kN, the column material must have a minimum proportional limit of $\sigma_{PL} = \dfrac{80000}{892} = 89.7$ MPa

12.19 By Fig. P12.19, $A = 150\times 150 - 125\times 100 = 10{,}000\ \text{mm}^2$

$$I_x = \frac{1}{12}\left[150\times 150^3 - 125\times 100^3\right] = 31{,}770{,}833\ \text{mm}^4$$

$$I_y = \frac{2}{12}(25)(150)^3 + \frac{1}{12}(100)(25)^3 = 14{,}192{,}708\ \text{mm}^4$$

For buckling in the y-direction, the critical load is given by $k_n L = 2n\pi$, with $n=1$ [See Example 12.2, Eq. (d)]. Therefore,

$$P_{cr(y)} = \frac{\pi^2 E I_x}{(L/2)^2} = \frac{\pi^2(72000)(31{,}770{,}833)}{(9000/2)^2} = 1114.9\ \text{kN}$$

For buckling in the x-direction (with two lobes), the critical load is given by $k_n L = 2n\pi$, with $n=2$. Therefore,

$$P_{cr(x)} = \frac{16\pi^2 E I_y}{L^2} = \frac{16(\pi)^2(72000)(14{,}192{,}708)}{(9000)^2} = 1992.2\ \text{kN}.$$

$\therefore P_{cr} = P_{cr(y)} = 1114.9$ kN. (b) Thus, $\sigma_{PL} = \dfrac{P_{cr}}{A} = \dfrac{1114.9}{10{,}000} = 111.5$ MPa, minimum.

12.20 By Fig. a

$A = 150 \times 150 - 125 \times 100 = 0.01\ m^2$

$I_x = \frac{1}{12}(150 \times 150^3) - \frac{1}{12}(125)(100)^3 = 3.177 \times 10^{-5}\ m^4$

$I_y = \frac{2}{12}(25)(150)^3 + \frac{1}{12}(100)(25)^3 = 1.4193 \times 10^{-5}\ m^4$

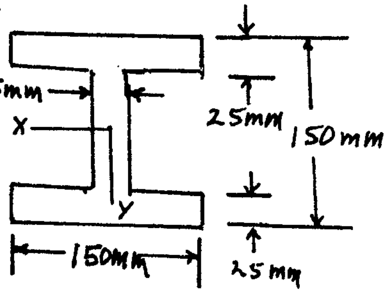


Figure a

For deflection in the y direction (buckling about the x axis), the effective length is L. Hence,

$$(P_{cr})_x = \frac{\pi^2 E I_x}{L^2} = \frac{\pi^2 (72 \times 10^9)(3.177 \times 10^{-5})}{(9.0)^2}$$

or

$$(P_{cr})_x = 278.7\ kN$$

For deflection in the x direction (buckling about the y axis, with two lobes), Fig. b, the effective length is L/2. Therefore,

$$(P_{cr})_y = \frac{\pi^2 E I_y}{(L/2)} = \frac{\pi^2 (72 \times 10^9)(1.4193 \times 10^{-5})}{(4.5)^2} = 498.1\ kN$$

Since $(P_{cr})_y > (P_{cr})_x$, the critical buckling load is

$$P_{cr} = (P_{cr})_x = 278.7\ kN$$

(b) The critical buckling stress is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{278.7}{0.01} = 27.87\ MPa$$

Hence, the proportional limit σ_{PL} must be such that

$$\sigma_{PL} \geq \sigma_{cr} = 27.87\ MPa$$

for the column to buckle elastically.

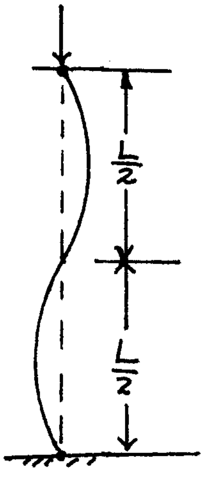
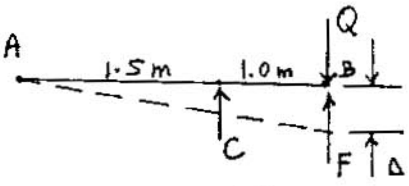


Figure b

12.21 Under the action of load Q, the bar at point B moves downward a distance Δ and at point C a distance $0.6\,\Delta$, Fig. (a). Summation of moments about point A yields



(a)

$$Q = F + 0.6C \qquad (a)$$

where in terms of Δ,

$$F = EA\frac{\Delta}{L} = \frac{(72000)}{5000}\left(\frac{\pi \times 100^2}{4}\right)\Delta = (14.4)(7854)\Delta \;= 113{,}098\,\Delta$$

$$C = \frac{EA(0.6\Delta)}{L} = 67{,}859\,\Delta\,, \qquad \Delta \text{ in mm.} \qquad (b)$$

The critical load for column FH is $\left(I = \frac{\pi d^4}{4} = \frac{\pi \times 100^2}{4} = 4{,}908{,}738 \text{ mm}^4\right)$

$$F_{cr} = P_{cr(FH)} = \frac{\pi^2 EI}{(0.7L)^2} = \frac{\pi^2(72{,}000)(4{,}908{,}738)}{(0.7\times 5000)^2} = 284{,}638 \text{ N} \qquad (c)$$

Therefore, by Eq. (b), at this load, $\Delta = 284{,}638/113{,}098 = 2.52$ mm

Similuly, the critical load for column CD is

$$C_{cr} = P_{cr(CD)} = \frac{\pi^2 EI}{L^2} = 139{,}528 \text{ N} \qquad (d)$$

Therefore, by Eq. (b), at this load, $\Delta = 139{,}528/67{,}859 = 2.06$ mm.

Consequently, Column CD buckles before column FH.

By Eqs. (a) and (b), with $\Delta = 2.06$ mm, the least value of Q that will cause one of the columns to buckle (column CD) is

$$Q = F + 0.6C = (113098 + 0.6 \times 67859)(2.06)$$

or $\quad Q = 316.8$ kN

12.22 (a) By Prob. 12.21, F and C, and therefore Q, are proportional to **E**. Also, P_{cr} is proportional to E. Therefore, $\Delta = 2.06$ mm for buckling of column CD. Also, $Q = F + 0.6C$, where $F = (113098\Delta)\frac{200}{72} = 647.2$ kN, and $C = (67859\,\Delta)\frac{200}{72} = 388.3$ kN

$$\therefore \quad Q = 647.2 + 0.6 \times 388.3 = 880.2 \text{ kN}$$

(b) Since both columns have the same cross-sectional area ($A = 7854$ mm^2), the largest stress is in column FD. Therefore, the minimum proportional limit σ_{PL} to ensure that the columns buckle elastically is

$$\sigma_{PL} = \frac{F}{A} = \frac{647.2}{7854} = 82.4 \text{ MPa}$$

12.23 By Fig. P12.23 b, $A = (100)(100) - (50)(75) = 6250$ mm^2

$$I_x = \frac{1}{12}\left[(100)(100)^3 - (50)(75)^3\right] = 6{,}575{,}521 \text{ mm}^4 = I_{min}.$$

$$I_y = \frac{1}{12}\left[(100)(100)^3 - (75)(50)^3\right] = 7{,}552{,}083 \text{ mm}^4 = I_{max}.$$

By Fig. P12.23 a, the force in each column is

$$P = \frac{Q}{2\cos 45^\circ} = \frac{Q}{\sqrt{2}} \qquad (a)$$

(a) For buckling in the plane, the columns are pinned-clamped. Therefore, for in-plane buckling

$$P_{cr} = \frac{Q}{\sqrt{2}} = \frac{\pi^2 E I_x}{(0.7L)^2} = \frac{\pi^2(200{,}000)(6{,}575{,}521)}{(0.7 \times 5000)^2} = 1060 \text{ kN}$$

or $Q = 1499$ kN

(b) For buckling out of the plane, the columns are

(Cont.)

12.23 cont. Clamped-free. Therefore, for out-of-plane buckling, the effectively length is 2L and

$$P_{cr} = \frac{Q}{\sqrt{2}} = \frac{\pi^2 E I_y}{(2L)} = \frac{\pi^2(200,000)(7,552,083)}{(2\times 5000)^2} = 149.1\ \text{kN}$$

or $Q = 210.9$ kN. Consequently, buckling occurs first by buckling out-of-plane for $Q = 210.8$ kN.

12.24 (a) Since $t = 0.24$ mm and $b = 12.0$ mm, $t/b = 0.02$. By Fig. 12.11, $t/b = 0.02$ is in the region of local buckling.

(b) By Fig. 12.11, with $t/b = 0.02$,

$$\sigma_{cr} = 11.4\ \text{MPa}$$

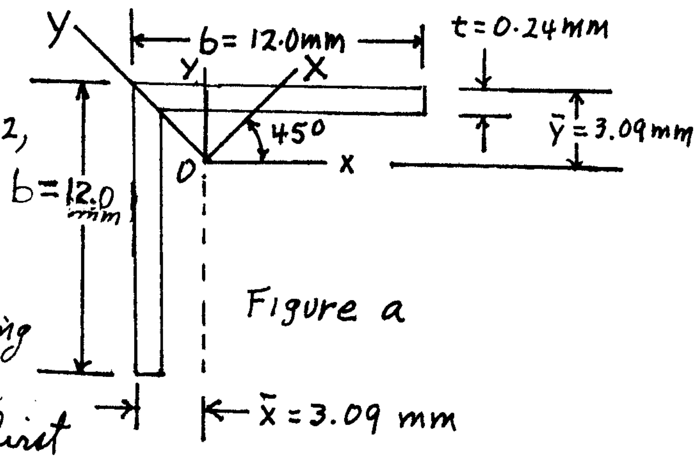


Figure a

(c) To calculate the buckling load and hence, the Euler buckling stress, we must first calculate the axis about which the column buckles. By Fig. a,

$$A = 12(0.24) + (12 - 0.24)(0.24) = 5.7024\ \text{mm}^2 = 5.7024\times 10^{-6}\ \text{m}^2$$

$$A\bar{x} = 12(0.24)(6) + (12-0.24)(0.24)(0.12) = 17.6187\ \text{mm}^2 = 1.76187\times 10^{-5}\ \text{m}^2$$

or $\bar{x} = 3.09\ \text{mm} = \bar{y}$, by symmetry.

Hence,

$$I_{xx} = \frac{1}{12}(12)(0.24)^3 + 12(0.24)(3.09 - 0.12)^2 + \frac{1}{12}(0.24)(12.0 - 0.24)^3 + (12.0 - 0.24)(0.24)(3.03)^2$$

or

$$I_{xx} = 83.86\ \text{mm}^4 = I_{yy}\ \text{by symmetry.}$$

Also, by Fig. a,

$$I_{xy} = (12.0)(0.24)(6 - 3.09)(3.09 - 0.12) + (12.0 - 0.24)(0.24)(-3.09 + 0.12)(-6.12 + 309)$$

or

$$I_{xy} = 50.29\ \text{mm}^4$$

(cont.)

12.24 cont. The principal axes X, Y are at $\theta = 45°$ from the x, y axes (Fig. a). By Eq. (B.13), the minimum principal area moment of inertia is I_{YY}. Thus,

$$I_{YY} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

or

$$I_{YY} = 33.57 \text{ mm}^4 = 33.57 \times 10^{-12} \text{ m}^4$$

Hence,

$$(\sigma_{cr})_{Euler} = \frac{P_{cr}}{A} = \frac{\pi^2 E I_{YY}}{AL^2} = \frac{\pi^2 (74.5 \times 10^9)(33.57 \times 10^{-12})}{(5.702 \times 10^{-6})(0.559)^2}$$

or

$$(\sigma_{cr})_{Euler} = 13.85 \text{ MPa, compared to } \sigma_{cr} = 11.4 \text{ MPa}$$

12.25 $A = bt + (b-t)(t) = 12(0.635) + 11.365(0.635) = 14.8368 \text{ mm}^2$

Then, by Fig. 12.11 and symmetry,

$$\bar{x} = \bar{y} = \frac{12(0.635)(0.3175) + 11.365(0.635)(0.635 + 5.6825)}{14.8368} = 3.2360 \text{ mm}$$

$$I_{xx} = I_{yy} = \frac{0.635(12)^3}{12} + 12(0.635)(6 - 3.2360)^2 + \frac{11.365(0.635)^3}{12} + 11.365(0.635)(3.236 - 0.3175)^2$$

$$= 211.367 \text{ mm}^4$$

$$I_{xy} = 12(0.635)(6 - 3.2360)(0.3175 - 3.2360) + 11.365(0.635)(3.2360 - 6.3175)(3.236 - 0.3175)$$

$$= -126.372 \text{ mm}^4$$

$$I_{min} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right) + I_{xy}^2} = I_{xx} - |I_{xy}| = 84.995 \text{ mm}^4$$

Hence,

$$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{84.995}{14.8368}} = 2.393 \text{ mm}; \quad \frac{\ell}{r} = \frac{559}{2.393} = 233.6$$

$$\sigma_{average} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(\ell/r)^2} = \frac{\pi^2 (74{,}500)}{(233.6)^2} = 13.47 \text{ MPa}.$$

The experimentally determined stress is, by Fig. 12.11, with $t/b = \frac{0.635}{12} = 0.0529$, is $\sigma_{EXP} \approx 12.72$ MPa compared to 13.47 MPa. The experiment stress is approximately 5.57% lower than the average stress.

12.26 (a) The stress-strain diagram is shown in Fig. a.

(b) By plotting the slope of the stress-strain diagram, we obtain the tangent-modulus curve as a function of stress (Fig. a).

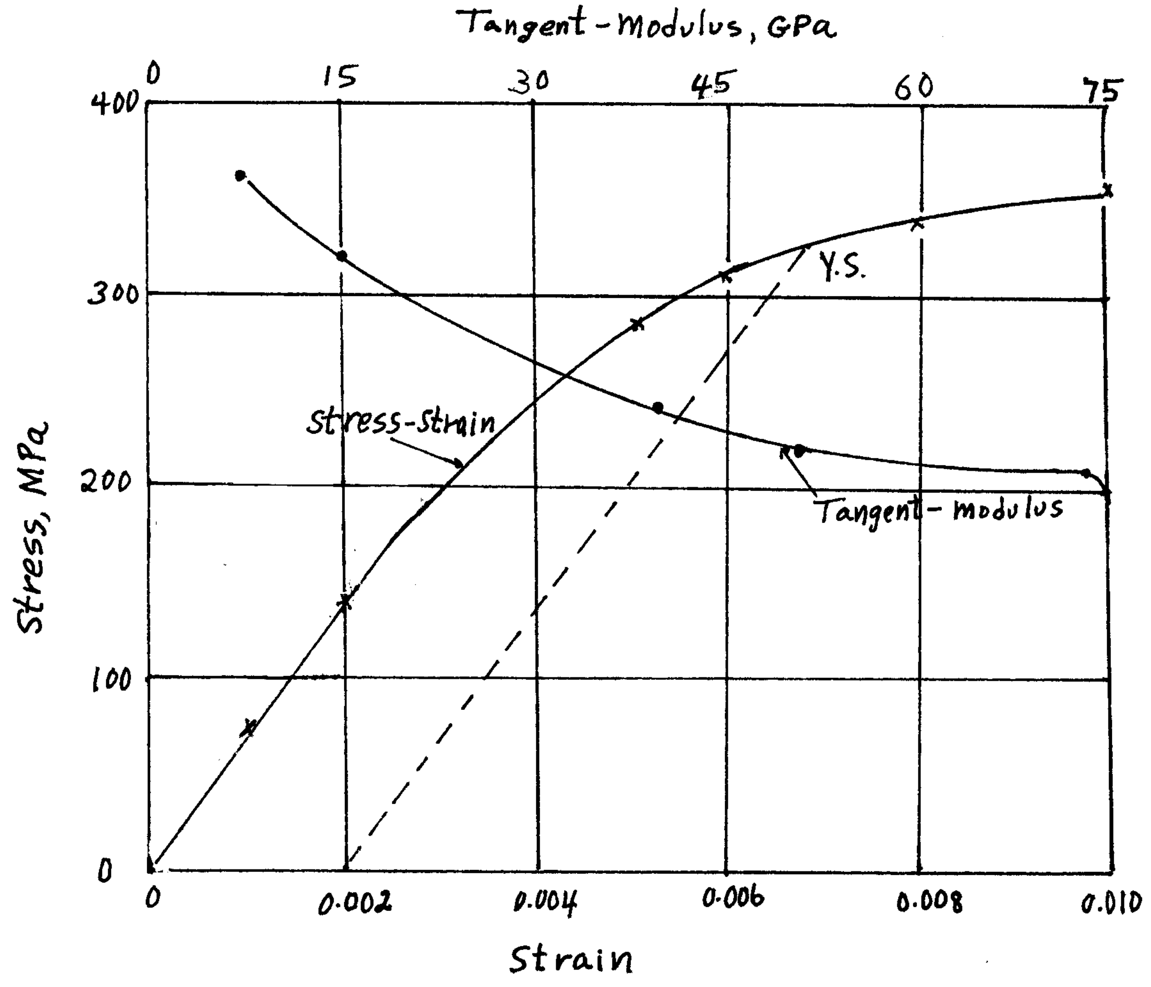


Figure a

(c) The rectangular cross section of the column is shown in Fig. B. By Fig. b,

$A = 12(12.5) = 150\ mm^2 = 1.50 \times 10^{-4}\ m^2$

$I_{xx} = \frac{1}{12}(12.5)(12)^3 = 1.8 \times 10^{-9}\ m^4$

$I_{yy} = \frac{1}{12}(12)(12.5)^3 = 1.953 \times 10^{-9}\ m^4$

Since $I_{yy} > I_{xx}$, the column buckles about the x axis.

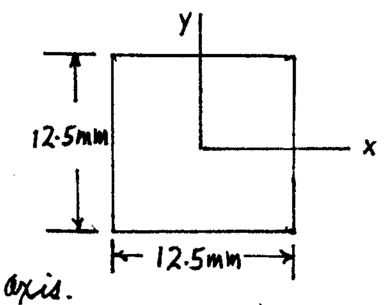


Figure b

(Cont.)

12.26 cont. Since the ends are clamped, $K = 0.50$ (Table 12.1). Hence, the value of KL/r_x may be calculated as follows:

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{1.8 \times 10^{-9}}{1.5 \times 10^{-4}}} = 0.00346\,m = 3.46\,mm$$

Then, since $L = 250$ mm,

$$\frac{KL}{r_x} = \frac{0.5(250)}{3.46} = 36.13$$

Hence, by Figs 12.14 or Fig. 12.15, $\sigma_{cr} \approx 260.9$ MPa and $P_{cr} = \sigma_{cr} A \approx 39.14$ kN.

12.27 By Fig. a,

$$A = 50(32) = 1600\,mm^2 = 1.6 \times 10^{-3}\,m^2$$

$$I_x = \frac{1}{12}(32)(50)^3 = 3.33 \times 10^{-7}\,m^4$$

$$I_y = \frac{1}{12}(50)(32)^3 = 1.365 \times 10^{-7}\,m^4$$

Since $I_x > I_y$, the column buckles about the y axis. Since the ends are pinned, $K = 1.0$ (Table 12.1). Hence, the value of KL/r_y is determined as follows:

$$r_y = \sqrt{\frac{I_y}{A}} = 0.00924\,m = 9.24\,mm, \text{ and}$$

since $L = 500$ mm,

$$\frac{KL}{r_y} = 54.13$$

Then, by Fig. 12.14a or Fig. 12.15,

$$\sigma_{cr} \approx 213\ MPa$$

Therefore,

$$P_{cr} = \sigma_{cr} A \approx 340\ kN$$

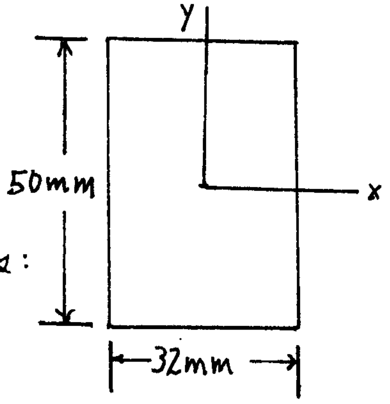


Figure a

12.28 By Fig. a,

$$A = 2(20)(5) + 5(25) = 325\text{ mm}^2 = 3.25\times10^{-4}\text{ m}^2$$

$$A\bar{x} = 2(5)(25)\left(\tfrac{25}{2}\right) + 15(5)\left(\tfrac{5}{2}\right) = 3312.5\text{ m}^3$$

Hence,

$$\bar{x} = \frac{3312.5}{325} = 10.19\text{ mm} = 0.01019\text{ m}$$

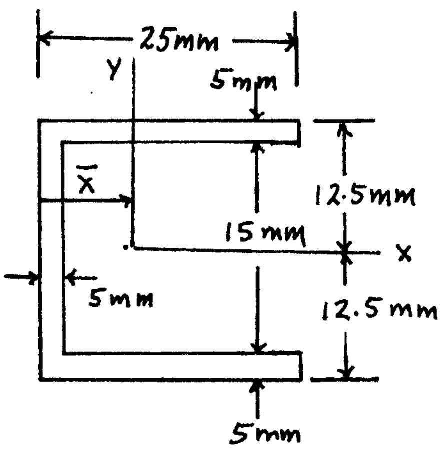


Figure a

Then, by Fig. a,

$$I_x = \tfrac{2}{12}(25)(5)^3 + 2(25)(5)(10)^3 + \tfrac{1}{12}(5)(15)^3$$

$$= 26{,}927.1\text{ mm}^4 = 2.6927\times10^{-8}\text{ m}^4$$

$$I_y = \tfrac{1}{12}(15)(5^3) + 15(5)(10.19-2.5)^2$$

$$+ \tfrac{2}{12}(5)(25)^3 + 2(5)(25)(12.5-10.19)^2$$

$$= 18{,}946.3\text{ mm}^4 = 1.8946\times10^{-8}\text{ m}^4$$

Since $I_x > I_y$, buckling occurs about the y axis. Since the column ends are fixed, $K = 0.50$. also,

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{1.8946\times10^{-8}}{3.25\times10^{-4}}} = 0.007635\text{ m} = 7.635\text{ mm}$$

with $L = 450$ mm,

$\frac{KL}{r_y} = 29.47$. So, with $KL/r_y = 29.47$, by Fig. 12.15 (or Fig 12.14a), we find $\sigma_{cr} \approx 287$ MPa. Hence with a safety factor of 2,

$$P_{cr} = \sigma_{cr}A \approx 287\times10^6(3.25\times10^{-4}) = 93.26\text{ kN}$$

or

$$P_{design} = \frac{P_{cr}}{2} = 46.63\text{ kN}$$

12.29 (a) The stress-strain diagram is plotted in Fig. a.

(b) By plotting the slope of the stress-strain diagram as a function of stress, the tangent modulus curve may be plotted (Fig. a).

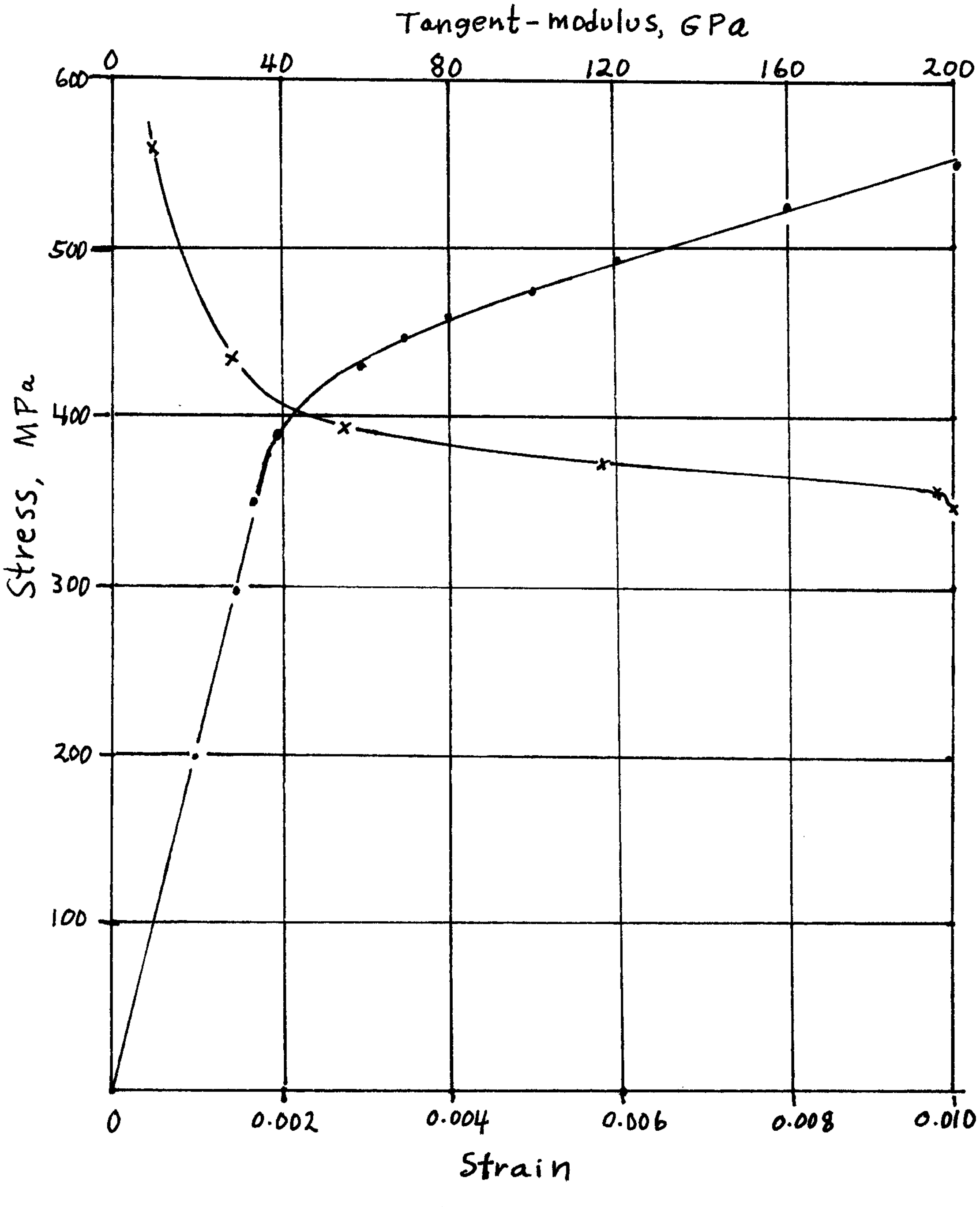


Figure a

(Cont.)

12.29 cont. The rectangular cross section of the column is shown in Fig. b. By Fig. b,

$$A = 100(125) = 12500 \text{ mm}^2 = 0.0125 \text{ m}^2$$

$$I_x = \frac{1}{12}(100)(125)^2 = 16.276\times10^6 \text{mm}^4 = 1.6276\times10^{-5} \text{m}^4$$

$$I_y = \frac{1}{12}(125)(100)^2 = 10.4167\times10^6 \text{mm}^4 = 1.0417\times10^{-5} \text{m}^4$$

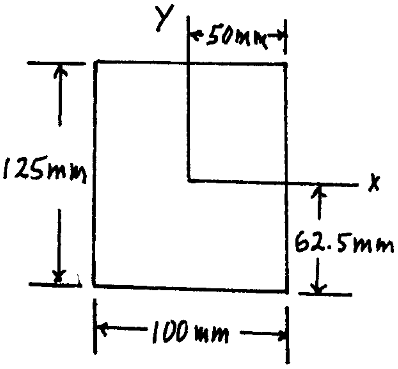


Figure b

Since $I_x > I_y$, buckling occurs about the y axis. Therefore, the least radius of gyration of the cross section is

$$r_y = \sqrt{\frac{I_y}{A}} = 28.87 \text{ mm} = 0.02887 \text{ m}$$

Then, with $L = 1.5$ m and $K = 0.5$ (clamped ends),

$$\frac{KL}{r_y} = \frac{0.5(1.5)}{0.02887} = 25.98 \qquad \text{(a)}$$

Let, as a trial, $E_T = 20$ GPa. Corresponding to $E_T = 20$ GPa, by the stress - strain diagram (Fig. a), $\sigma_T \approx 470$ MPa. By the right side of Eq. (12.45) and Eq. (a), we have

$$\frac{\pi^2 E_T}{(KL/r_y)^2} = \frac{\pi^2(20\times10^9)}{(25.98)^2} = 292.4 \text{MPa} < 470 \text{MPa}.$$

as a second trial, let $E_T = 32$ GPa. Then, by the stress-strain diagram (Fig. a), $\sigma_T \approx 430$ MPa and

$$\frac{\pi^2 E_T}{(KL/r_y)^2} = \frac{\pi^2(32\times10^9)}{(25.98)^2} \approx 468 \text{ MPa} > 430 \text{MPa}.$$

as a third trial, take $E_T = 30$ GPa. Then, $\sigma_T \approx 437$ MPa and

$$\frac{\pi^2 E_T}{(KL/r_y)^2} = \frac{\pi^2(30\times10^9)}{(25.98)^2} = 438.7 \text{ MPa} \approx 437 \text{ MPa},$$ which

is reasonably close. Hence,

$$P_T = \sigma_T A = (437\times10^6)(1.25\times10^{-2}) = 5.4625 \text{ MN}$$

12.30

$\frac{\ell}{r} = 41.57$. Assume that $E_T = 44{,}500$ MPa in Fig. E 12.3 and read $\sigma_T = 254$ MPa.

$$\sigma_T = \frac{P_T}{A} = \frac{\pi^2 E_T}{(\ell/r)^2} = \frac{\pi^2(44{,}500)}{(41.57)^2} = 254 \text{ MPa (checks)}$$

$$P_T = \sigma_T A = 254(25)^2 = \underline{159 \text{ kN}}$$

12.31 Assume that $E_T = 50{,}000$ MPa in Fig. E 12.3 and read $\sigma_T = 252$ MPa.

$$\sigma_T = \frac{P_T}{A} = \frac{SF(P)}{A}\;;\; A = \frac{SF(P)}{\sigma_T} = \frac{\pi D^2}{4} = \frac{2.00(810{,}000)}{252}$$

$D = 90.47$ mm

But $P_T = \frac{\pi^2 E_T I}{\ell^2} = SF(P)$

$$I = \frac{SF(P)\,\ell^2}{\pi^2 E_T} = \frac{\pi D^4}{64} = \frac{2.00(810{,}000)(1000^2)}{\pi^2(50{,}000)}\;;\; D = \underline{90.43 \text{ mm}} \text{ (checks)}$$

12.32 From $310 = A(0.004306)^n$ and $370 = A(0.00600)^n$ we obtain

(a) $\frac{370}{310} = \left(\frac{0.00600}{0.004306}\right)^n = 1.19355 = 1.3934^n$

$n = 0.5334$ and $A = 5667$

(b) $345 = 5667\,\epsilon^{0.5334}$

$$\epsilon = \left(\frac{345}{5667}\right)^{1.8748} = 0.005262$$

For coordinates $\sigma = 345$ MPa and $\epsilon = 0.005262$

$$E_T = \frac{d\sigma}{d\epsilon} = An\epsilon^{n-1} = \frac{5667(0.5334)}{0.005262^{0.4666}} = 34{,}970 \text{ MPa}$$

$$r = \sqrt{\frac{I}{\text{area}}} = \sqrt{\frac{60(40)^3}{12(60)(40)}} = 11.55 \text{ mm}$$

$$\ell = r\sqrt{\frac{\pi^2 E_T}{P_T/\text{area}}} = 11.55\sqrt{\frac{\pi^2(34{,}970)}{345}} = 365 \text{ mm}$$

Since the column has fixed ends, $L = 2\ell = 2(365) = \underline{730 \text{ mm}}$

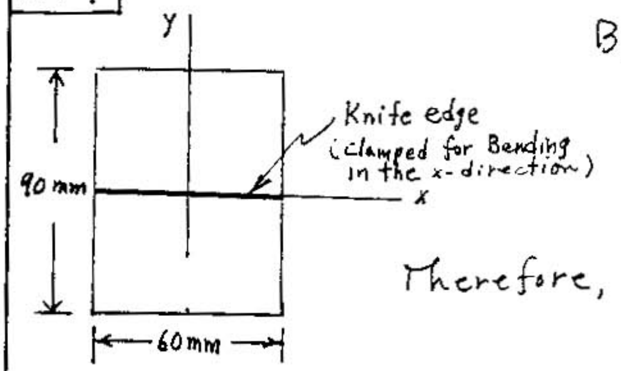
12.33

E_T varies from $E_T = E = 72{,}000$ MPa to $E_T = 38{,}400$ MPa.

$$\frac{\ell}{r} = \sqrt{\frac{\pi^2 E_T}{\sigma_{PL}}} = \sqrt{\frac{\pi^2 E_T}{310}} = 0.1784\sqrt{E_T}$$

$$\left.\frac{\ell}{r}\right|_{max} = 0.1784\sqrt{72{,}000} = \underline{47.9}\;;\; \left.\frac{\ell}{r}\right|_{min} = 0.1784\sqrt{38{,}400} = \underline{35.0}$$

12.34



(a)

By Fig. a, $I_x = \frac{1}{12}(60)(90)^3 = 3{,}645{,}000\ mm^4$

$I_y = \frac{1}{12}(90)(60)^3 = 1{,}620{,}000\ mm^4$

Area: $A = (90)(60) = 5400\ mm^2$

Therefore, $r_x = \sqrt{\frac{I_x}{A}} = 25.98\ mm$

$r_y = \sqrt{\frac{I_y}{A}} = 17.32$

and $\frac{L}{r_x} = \frac{1000}{25.98} = 38.49$; $\frac{L}{r_y} = 57.74$

For buckling about the x-axis (toward the 60 mm side)

$K_x = 1$, $\therefore \frac{K_x L}{r_x} = 38.49$

For buckling about the y-axis (toward the 90 mm side)

$K_x = 0.70$, $\therefore \frac{K_y L}{r_y} = 0.7 \times 57.74 = 40.42$.

By Fig. 12.14, the buckling stress for buckling about the x-axis (with $\frac{K_x L}{r_x} = 38.49$) is $\frac{P_{cr}}{A} \approx 260\ MPa$.

Therefore, $P_{cr} = 260A = (260)(5400) = 1404\ kN$

For buckling about the y-axis (with $\frac{K_y L}{r_y} = 40.42$),

$\frac{P_{cr}}{A} \approx 250\ MPa$. Therefore, $P_{cr} = (250)(5400) = 1350\ kN$.

Hence, the critical load is $P = 1350\ kN$.

12.35

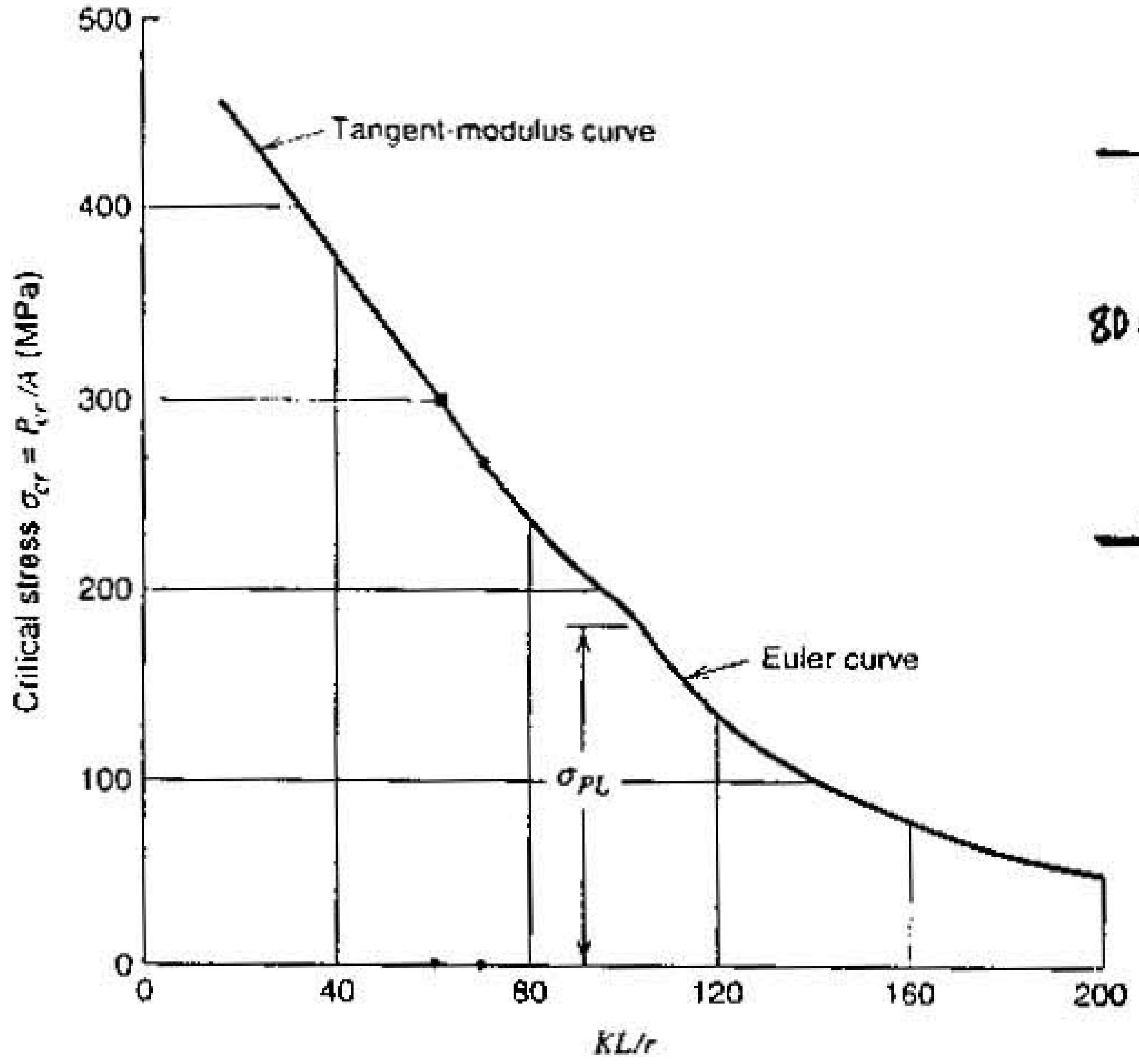


Figure b

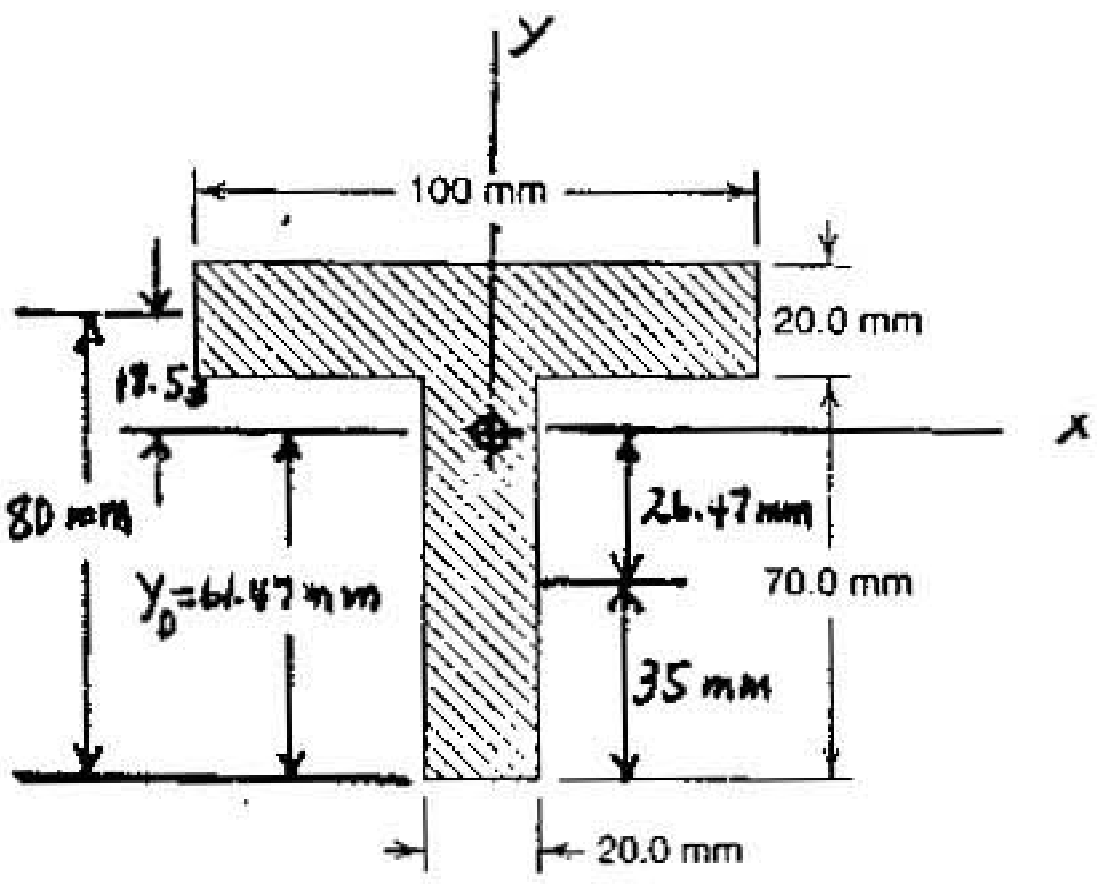


Figure a

By Fig. a,

$A = \text{Area} = 100\times20 + 70\times20 = 3400 \text{ mm}^2$

$A y_0 = (20)(100)(80) + (20)(70)(35)$

$y_0 = \dfrac{160000 + 49{,}000}{3400} = 61.47 \text{ mm}$

Also by Fig. a, we have $I_x = \frac{1}{12}(100)(20)^3 + (100)(20)(18.53)^2 + \frac{1}{12}(20)(70)^3 + (20)(70)(26.47)^2 = 2{,}305{,}980 \text{ mm}^4$ and $I_y = \frac{1}{12}(20)(100)^3 + \frac{1}{12}(70)(20)^3 = 1{,}713{,}333 \text{ mm}^4$. Therefore, we have

$$r_x = \sqrt{\frac{I_x}{A}} = 26.04 \quad \text{and} \quad r_y = \sqrt{\frac{I_y}{A}} = 22.45. \quad \text{Hence}$$

$$K_x \frac{L}{r_x} = 0.7\left(\frac{2200}{26.04}\right) = 59.14 \quad \text{and} \quad K_y \frac{L}{r_y} = 0.7\left(\frac{2200}{22.45}\right) = 68.60.$$

Now for buckling about the x-axis, $\frac{K_x L}{r_x} = 59.14$, Fig. b yields $\frac{P_{cr}}{A} \approx 300 \text{MPa} = \frac{2.5 P_x}{A}$; or $P_x = \frac{(300)(3400)}{2.5} = 408 \text{ kN}$.

For buckling about the y-axis, $\frac{K_y L}{r_y} = 68.60$, Fig. b yields $\frac{P_{cr}}{A} \approx 267 \text{ MPa} = \frac{2.5 P_y}{A}$; or $P_y = \frac{(267)(3400)}{2.5} = 363 \text{ kN}$.

Hence the critical design load (with factor of safety 2.5) is $P_{Design} = 363 \text{ kN}$.

12.36 By Figure a

$A = \text{Area} = (100)(95) - (60)(80) = 4700 \text{ mm}^2$

$I_x = \frac{1}{12}\left[(100)(95)^3 - (80)(60)^3\right] = 5{,}704{,}792 \text{ mm}^4$

$I_y = \frac{1}{12}\left[(95)(100)^3 - (60)(80)^3\right] = 5{,}356{,}667 \text{ mm}^4$

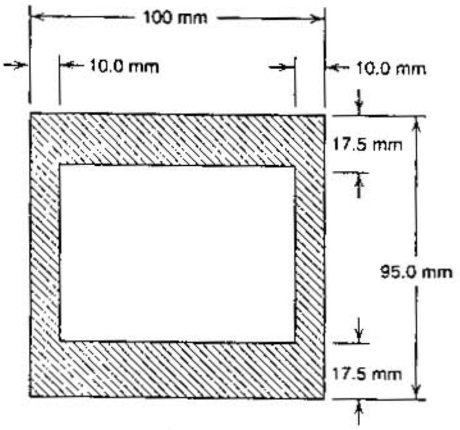


Figure a

Hence,

$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{5{,}704{,}792}{4700}} = 34.84$

$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{5{,}356{,}667}{4700}} = 33.76$

$\frac{L}{r_x} = \frac{2200}{34.84} = 63.1 \; ; \; \frac{L}{r_y} = \frac{2200}{33.76} = 65.2$

Also, $K = 0.70$. Therefore, for buckling about the x-axis,

$K\frac{L}{r_x} = 0.7 \times 63.1 = 44.2$, and by Fig. 12.14a, $\frac{P_{cr}}{A} \approx 236$ MPa or

$\frac{1.8 P_x}{A} = 236$ MPa. Therefore, $P_x = \frac{(236)(4700)}{1.8} = 616.2$ kN.

Similarly for buckling about the y-axis, $\frac{KL}{r_y} = 0.70 \times 65.2 = 45.6$.

Then by Fig. 12.14a $\frac{P_{cr}}{A} \approx 227.5 \text{ MPa} = \frac{1.8 P_y}{A}$ or $P_y = \frac{(227.5)(4700)}{1.8}$

$= 594.0$ kN. Hence, the design load is $P_y = P_{design} = 594$ kN

12.37 The cross section of the column is shown in Fig. a.

By Fig. a, Area $= A = 50 \times 75 = 3750 \text{ mm}^2$

$I_x = \frac{1}{12}(50)(75)^3 = 1{,}757{,}812 \text{ mm}^4$

$I_y = \frac{1}{12}(75)(50)^3 = 781{,}250 \text{ mm}^4$

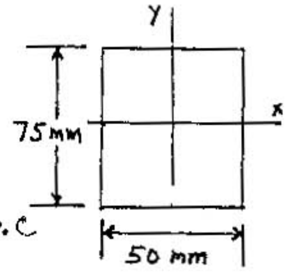


Figure a

Since $I_y < I_x$, buckling will occur in the plane of Fig. c

By Fig. b, $\Sigma F_y = AB + CD - Q = 0$; $Q = AB + CD$

$\Sigma M_Q = 200\,AB - 400\,CD = 0$; $AB = 2CD$

Therefore, $Q = 3CD$, $CD = \frac{Q}{3}$ and $AB = \frac{2}{3}Q$

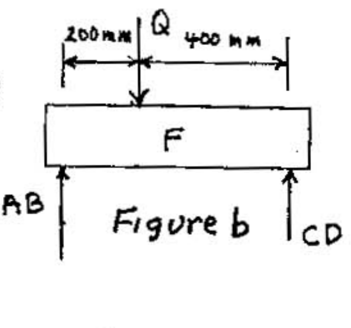


Figure b

By Fig. d, we find $E = 70$ GPa for stress from 0 to 140 MPa, and $E = 35$ GPa for stress greater than 140 MPa.

(Continued)

12.37 (continued)

For the fixed-fixed column AB, the buckling load is $P_{cr(AB)} = \frac{\pi^2 E I_y}{(\frac{L}{2})^2}$. The corresponding axial stress is, for $E = 70$ GPa, $\sigma = \frac{P_{cr(AB)}}{A} = \frac{\pi^2 E I_y}{A(\frac{L}{2})^2} = \frac{\pi^2(70{,}000)(781{,}250)}{(3750)(\frac{1400}{2})^2} = 293.7$ MPa.

Since $293.7 > 140$, the modulus $E = 35$ GPa must be used.

Therefore, for $E = 35$ GPa, $\sigma = \frac{\pi^2(35{,}000)(781{,}250)}{(3750)(\frac{1400}{2})^2} = 146.9 \text{ MPa} > 140$.

Hence, $P_{cr(AB)} = \sigma A = (146.9)(3750) = 550.76$ kN, and the corresponding value of Q is $\frac{3}{2}(550.76) = 826.1$ kN

For the pinned-pinned column CD, Fig. c, $P_{cr(CD)} = \frac{\pi^2 E I_y}{L^2}$ and $\sigma = \frac{\pi^2 E I_y}{A(L)^2}$. For $E = 70$ GPa,

$\sigma = \frac{\pi^2(70{,}000)(781{,}250)}{(3750)(1400)^2}$

$= 73.43$ MPa.

Since $73.43 < 140$, column CD will buckle elastically. The critical load for column CD is

$P_{cr(CD)} = (73.43)(3750)$

$= 275.38$ kN.

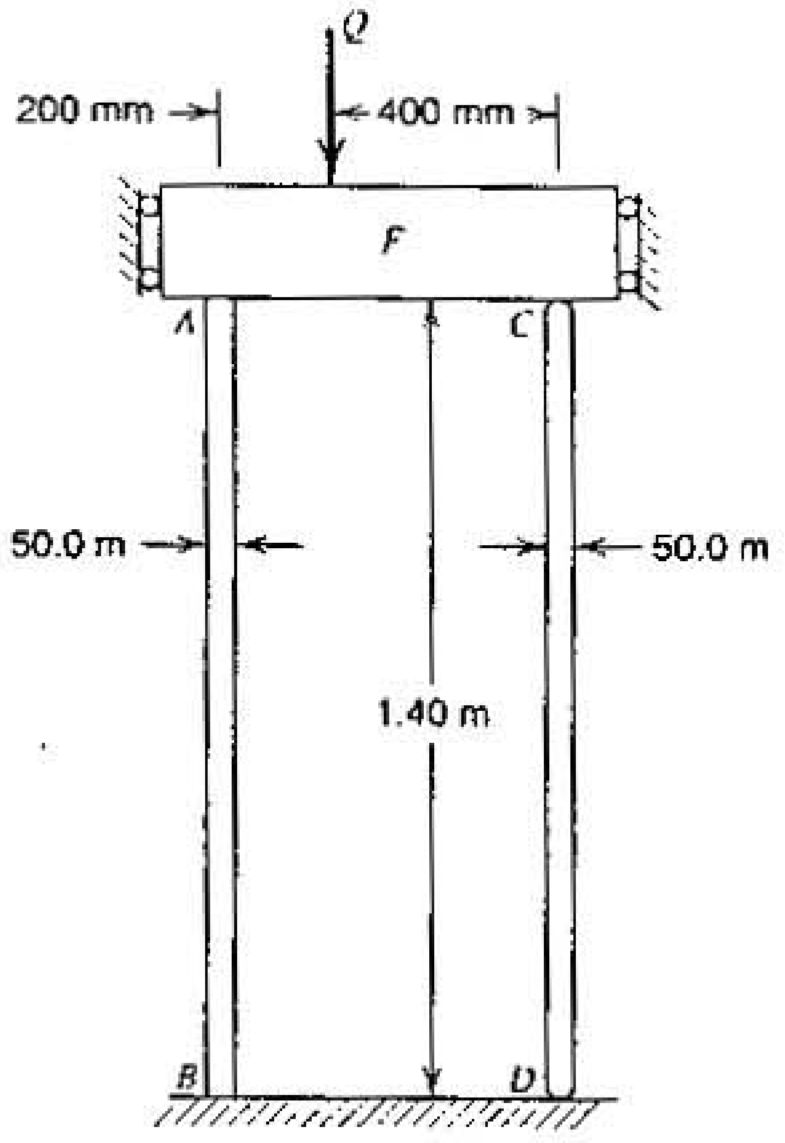


Figure c

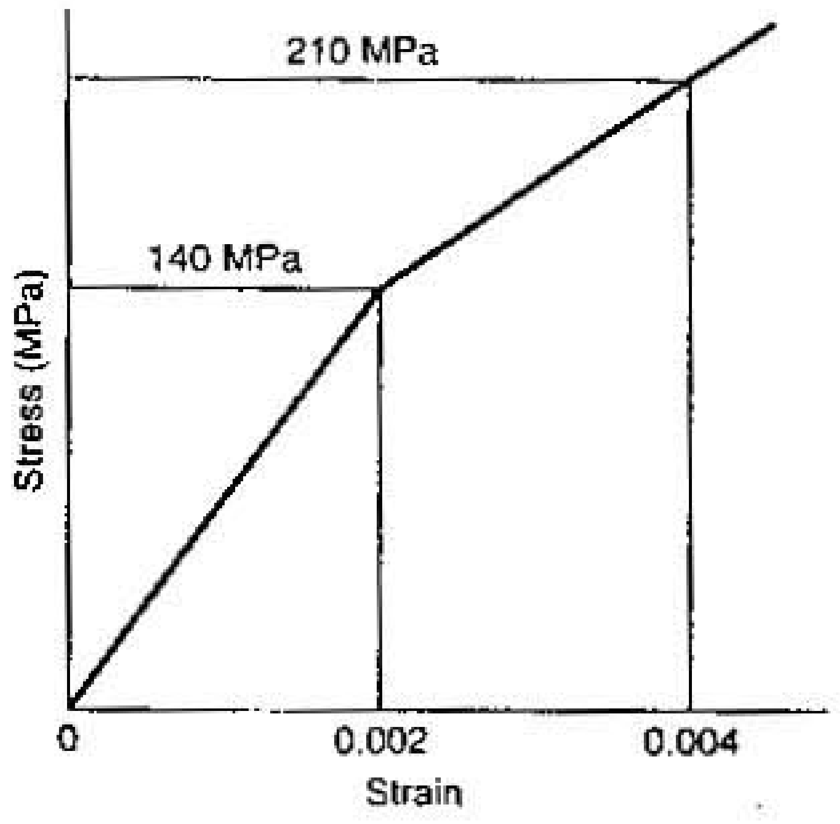


Figure d

The corresponding value of Q is $3(275.38) = 826.1$ kN.

Hence, both columns buckle when $Q = 826.1$ kN; Column AB buckles inelastically and column CD buckles elastically.

13.1 The boundary conditions are

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0 \text{ for } x = 0, a$$
$$w = 0 \text{ and } \frac{\partial^2 w}{\partial y^2} = 0 \text{ for } y = 0, b \qquad (1)$$

Equations (1) may be satisfied by taking

$$w = w_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \qquad (2)$$

where w_0 is a constant that is chosen to satisfy Eq.(13.55)

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_0}{D} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \qquad (3)$$

$$\frac{w_0 \pi^4}{a^4} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + 2\frac{w_0 \pi^4}{a^2 b^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{w_0 \pi^4}{b^4} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} = \frac{p_0}{D} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$w_0 = \underline{\frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)}} = w_{max} \text{ at } x = \frac{a}{2} \text{ and } y = \frac{b}{2}$$

Moments M_{xx} and M_{yy} are given by Eqs.(13.54)

$$M_{xx} = -D(w_{xx} + \nu w_{yy}) = \frac{\pi^2 D w_0 (b^2 + \nu a^2)}{a^2 b^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$M_{yy} = -D(w_{yy} + \nu w_{xx}) = \frac{\pi^2 D w_0 (a^2 + \nu b^2)}{a^2 b^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

The maximum values for M_{xx} and M_{yy} occurs at $x = \frac{a}{2}$ and $y = \frac{b}{2}$.

$$M_{xx(max)} = \frac{\pi^2 D(b^2 + \nu a^2)}{a^2 b^2} \frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \underline{\frac{p_0 a^2 b^2 (b^2 + \nu a^2)}{\pi^2 (a^4 + 2a^2b^2 + b^4)}}$$

$$M_{yy(max)} = \frac{\pi^2 D(a^2 + \nu b^2)}{a^2 b^2} \frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \underline{\frac{p_0 a^2 b^2 (a^2 + \nu b^2)}{\pi^2 (a^4 + 2a^2b^2 + b^4)}}$$

The Kirchhoff shear forces are given by Eqs.(13.54)

$$V_x = -D[w_{xxx} + (2-\nu) w_{xyy}] = \frac{\pi^3 D w_0 [b^2 + (2-\nu)a^2]}{a^3 b^2} \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$V_y = -D[w_{yyy} + (2-\nu) w_{yxx}] = \frac{\pi^3 D w_0 [a^2 + (2-\nu)b^2]}{a^2 b^3} \sin\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

The maximum value for V_x occurs for $x = 0, a$ and $y = \frac{b}{2}$.

$$V_{x(max)} = \frac{\pi^3 D[b^2 + (2-\nu)a^2]}{a^3 b^2} \frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \underline{\frac{p_0 a b^2 [b^2 + (2-\nu)a^2]}{\pi(a^4 + 2a^2b^2 + b^4)}}$$

The maximum value for V_y occurs for $x = \frac{a}{2}$ and $y = 0, b$

$$V_{y(max} = \frac{\pi^3 D[a^2 + (2-\nu)b^2]}{a^2 b^3} \frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \underline{\frac{p_0 a^2 b [a^2 + (2-\nu)b^2]}{\pi(a^4 + 2a^2b^2 + b^4)}}$$

13.2 The boundary conditions are

$$w=0 \text{ and } \frac{\partial^2 w}{\partial x^2}=0 \text{ for } x=0, a$$
$$w=0 \text{ and } \frac{\partial^2 w}{\partial y^2}=0 \text{ for } y=0, a \tag{1}$$

Equations (1) may be satisfied by taking

$$w = w_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \tag{2}$$

where w_0 is a constant that is chosen to satisfy Eq. (13.55).

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_0}{D}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} \tag{3}$$

$$\frac{w_0 m^4\pi^4}{a^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} + 2\frac{w_0 m^2 n^2\pi^4}{a^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} + \frac{w_0 n^4\pi^4}{a^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} = \frac{p_0}{D}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

$$w_0 = \frac{p_0 a^4}{\pi^4 D(m^4+2m^2n^2+n^4)}$$

The locations with maximum deflections occur for values of x such that $x=a(4M-3)/2m$ (M includes all integers such that $(4M-3)/2<m$) and for values of y such that $y=a(4N-3)/2n$ (N are integers such that $(4N-3)/2<n$).

$$w_{max} = w_0 = \frac{p_0 a^4}{\pi^4 D(m^4+2m^2n^2+n^4)}$$

Moments M_{xx} and M_{yy} are given by Eqs. (13.54).

$$M_{xx} = -D(w_{xx}+\nu w_{yy}) = \frac{\pi^2 D w_0(m^2+\nu n^2)}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

$$M_{yy} = -D(w_{yy}+\nu w_{xx}) = \frac{\pi^2 D w_0(n^2+\nu m^2)}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

Maximum values for M_{xx} and M_{yy} occur at same locations as w_{max}.

$$M_{xx(max)} = \frac{p_0 a^2(m^2+\nu n^2)}{\pi^2(m^4+2m^2n^2+n^4)}$$

$$M_{yy(max)} = \frac{p_0 a^2(n^2+\nu m^2)}{\pi^2(m^4+2m^2n^2+n^4)}$$

The Kirchhoff shears are given by Eqs. (13.54)

$$V_x = -D[w_{xxx}+(2-\nu)w_{xyy}] = \frac{\pi^3 D w_0[m^3+(2-\nu)mn^2]}{a^3}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

$$V_y = -D[w_{yyy}+(2-\nu)w_{xxy}] = \frac{\pi^3 D w_0[n^3+(2-\nu)m^2n]}{a^3}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a}$$

$$V_{x(max)} = \frac{p_0 a[m^3+(2-\nu)mn^2]}{\pi(m^4+2m^2n^2+n^4)} \text{ for } \begin{array}{l} x=0, \frac{Ma}{m} \text{ for all } M\le m \\ y=\frac{a(4N-3)}{2n} \text{ when } \frac{4N-3}{2}<n \end{array}$$

$$V_{y(max)} = \frac{p_0 a[n^3+(2-\nu)m^2n]}{\pi(m^4+2m^2n^2+n^4)} \text{ for } \begin{array}{l} x=\frac{a(4M-3)}{2m} \text{ when } \frac{4M-3}{2}<m \\ y=0, \frac{Na}{n} \text{ for all } N\le n \end{array}$$

13.3 The boundary conditions are

$$w=0 \text{ and } \frac{\partial^2 w}{\partial x^2}=0 \text{ for } x=0, a \qquad (1)$$

$$w=0 \text{ and } \frac{\partial^2 w}{\partial y^2}=0 \text{ for } y=0, b$$

Equations (1) may be satisfied by taking

$$w = w_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (2)$$

where w_0 is a constant that is chosen to satisfy Eq. (13.55).

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_0}{D}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (3)$$

$$\frac{w_0 m^4\pi^4}{a^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + 2\frac{w_0 m^2 n^2 \pi^4}{a^2 b^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \frac{w_0 n^4 \pi^4}{b^4}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = \frac{p_0}{D}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$w_0 = \frac{p_0 a^4 b^4}{\pi^4(m^4b^4 + 2m^2n^2a^2b^2 + n^4a^4)D}$$

The locations with maximum deflections occur for values of x such that $x = a(4M-3)/2m$ (M inclues all integers such that $(4M-3)/2 < m$) and for values of y such that $y = b(4N-3)/2n$ (N are integers such that $(4N-3)/2 < n$).

$$w_{max} = \underline{\frac{p_0 a^4 b^4}{\pi^4 D(m^4b^4 + 2m^2n^2a^2b^2 + n^4a^4)}} = w_0$$

Moments M_{xx} and M_{yy} are given by Eqs. (13.54).

$$M_{xx} = -D(w_{xx} + \nu w_{yy}) = \frac{\pi^2 D w_0 (m^2b^2 + \nu n^2 a^2)}{a^2b^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_{yy} = -D(w_{yy} + \nu w_{xx}) = \frac{\pi^2 D w_0 (n^2a^2 + \nu m^2 b^2)}{a^2b^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Maximum values for M_{xx} and M_{yy} occur at same locations as w_{max}.

$$M_{xx(max)} = \underline{\frac{p_0 a^2 b^2 (m^2b^2 + \nu n^2a^2)}{\pi^2(m^4a^4 + 2m^2n^2a^2b^2 + n^4b^4)}}$$

$$M_{yy(max)} = \underline{\frac{p_0 a^2 b^2 (n^2a^2 + \nu m^2b^2)}{\pi^2(m^4a^4 + 2m^2n^2a^2b^2 + n^4b^4)}}$$

The Kirchhoff shears are given by Eqs. (13.54).

$$V_x = -D[w_{xxx} + (2-\nu)w_{xyy}] = \frac{\pi^3 D w_0 [m^3b^2 + (2-\nu)mn^2a^2]}{a^3b^2}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$V_y = -D[w_{yyy} + (2-\nu)w_{xxy}] = \frac{\pi^3 D w_0 [n^3a^2 + (2-\nu)m^2nb^2]}{a^2b^3}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$V_{x(max)} = \underline{\frac{p_0 ab^2[m^3b^2 + (2-\nu)mn^2a^2]}{\pi(m^4a^4 + 2m^2n^2a^2b^2 + n^4b^4)}} \quad \text{for} \quad \begin{aligned} &x = 0, \frac{Ma}{m} \text{ for all } M \le m \\ &y = \frac{b(4N-3)}{2n} \text{ when } \frac{4N-3}{2} < n \end{aligned}$$

$$V_{y(max)} = \underline{\frac{p_0 a^2 b[n^3a^2 + (2-\nu)m^2nb^2]}{\pi(m^4a^4 + 2m^2n^2a^2b^2 + n^4b^4)}} \quad \text{for} \quad \begin{aligned} &x = \frac{a(4M+3)}{2m} \text{ when } \frac{4M-3}{2} < m \\ &y = 0, \frac{Nb}{n} \text{ for all } N \le n \end{aligned}$$

13.4 The twisting moment M_{xy} is given by Eqs. (13.54).

$$M_{xy} = -(1-\nu)D\,w_{xy} \qquad \text{where} \qquad w = \frac{p_0 a^4}{4\pi^4 D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$= -\frac{p_0 a^2(1-\nu)}{4\pi^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}$$

$$\left|M_{xy(max)}\right| = \frac{p_0 a^2(1-\nu)}{4\pi^2} \quad \text{for } x=0,a \text{ and } y=0,a$$

The maximum shearing stress is given by Eqs.(13.35).

$$\sigma_{xy} = \frac{12zM_{xy}}{h^3} = -\frac{3p_0 a^2 z(1-\nu)}{\pi^2 h^3}\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}$$

$$\left|\sigma_{xy(max)}\right| = \underline{\frac{3p_0 a^2(1-\nu)}{2\pi^2 h^2}} \quad \text{for } x=0,a \text{ and } y=0,a$$

13.5 The stress components are given by Eqs.(13.35)

$$\sigma_{xx} = \frac{12zM_{xx}}{h^3} = \underline{\frac{3p_0 a^2(1+\nu)z}{\pi^2 h^3}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}$$

$$\sigma_{yy} = \frac{12zM_{yy}}{h^3} = \underline{\frac{3p_0 a^2(1+\nu)z}{\pi^2 h^3}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}$$

$$\sigma_{xy} = \frac{12zM_{xy}}{h^3} = \underline{-\frac{3p_0 a^2(1-\nu)z}{\pi^2 h^3}\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}}$$

13.6 (a) The load $p(x,y)$ is given by the relation

$$p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{1}$$

and the displacement $w(x,y)$ is given by the relation

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{2}$$

In Eq.(2) each W_{mn} is a constant that must be chosen to satisfy Eq. (13.55).

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{3}$$

The left and right sides of Eq.(3) contain an infinite number of terms. Corresponding terms on each side of the equation are identical with Eq.(3) of Problem 13.3 Therefore,

$$\pi^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left(\frac{m^4}{a^4}+2\frac{m^2n^2}{a^2b^2}+\frac{n^4}{b^4}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = \frac{1}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};\quad \text{Thus}$$

$$W_{mn} = \frac{A_{mn}}{\pi^4 D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \tag{4}$$

13.6 Continued

(b) In order to calculate any particular coefficient $A_{m'n'}$ of the series for the right side of Eq.(1), we multiply both sides of Eq.(1) by $\sin\frac{n'\pi y}{b}dy$ and integrate from 0 to b noting that $\int_0^b \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}dy = 0$ when $n \neq n'$ and $\int_0^b \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}dy = \frac{b}{2}$ when $n = n'$.

$$\int_0^b p(x,y)\sin\frac{n'\pi y}{b}dy = \frac{b}{2}\sum_{m=1}^{\infty} A_{mn'}\sin\frac{m\pi x}{a} \qquad (5)$$

Now multiply both sides of Eq.(5) by $\sin\frac{m'\pi x}{a}dx$ and integrate from 0 to a. We obtain

$$\int_0^a\int_0^b p(x,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dx\,dy = \frac{ab}{4}A_{m'n'}$$

Since $p(x,y) = p_0$, we obtain for each $A_{m'n'}$

$$A_{m'n'} = \frac{4p_0}{ab}\int_0^a\int_0^b \sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dx\,dy = \frac{16p_0}{\pi^2 m'n'} \qquad (6)$$

(c) Substitue Eq.(6) into Eq.(4) and finally into Eq.(2) to obtain

$$w(x,y) = \frac{16p_0}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (7)$$

(d) For the uniform load p_0 the deflection surface must be symmetrical with respect to axes $x = \frac{a}{2}$ and $y = \frac{b}{2}$; therefore, all even number terms for m and n in Eq.(7) must vanish. The maximum deflection occurs at the center of the plate

$$w_{max} = \frac{16p_0}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}(-1)^{\frac{m+n-2}{2}} \qquad (8)$$

This series converges extremely fast. Using only the first term for a square (a=b) plate we obtain

$$w_{max} \approx \frac{4p_0a^4}{\pi^2 D} = 0.0042\frac{p_0a^4}{D}$$

13.7 $\alpha = \frac{b}{a} = \frac{1}{2} = 0.5$; $\sigma_w = \frac{Y}{SF} = \frac{280}{2} = 140 = \frac{6M}{h^2}$; See Table 13.1.

$$M = \frac{pb^2}{12(1+\alpha^4)} = \frac{0.270(1000)^2}{12(1+0.5^4)} = 21{,}180\ \text{N.mm} = \frac{140h^2}{6};\quad h = \sqrt{\frac{21{,}180(6)}{140}} = \underline{30.1\ \text{mm}}$$

$$w_{max} = \frac{0.032(1-\nu^2)}{1+\alpha^4}\frac{pb^4}{Eh^3} = \frac{0.032(1-0.29^2)}{1+0.5^4}\frac{0.270(1000)^4}{200{,}000(30.1)^3} = \underline{1.37\ \text{mm}}$$

13.8 $p_Y = SF(p) = 2.00(0.270) = 0.540$ MPa ; See Table 13.1.

$$M_{bc} = \frac{pb^2}{8(3+4\alpha^4)} = \frac{0.540(1000)^2}{8[3+4(0.5)^4]} = 20{,}770 \text{ N.mm}$$

$$M_{ac} = 0.009\,pb^2(1+2\alpha^2-\alpha^4) = 0.009(0.540)(1000)^2[1+2(0.5)^2-0.;\ ] = 6{,}990 \text{ N.mm}$$

$$M = M_{bc} + \nu M_{ac} = 20{,}770 + 0.29(6990) = 22{,}800 \text{ N.mm}$$

$$Y = \frac{6M}{h^2}\ ;\quad h = \sqrt{\frac{6(22{,}800)}{280}} = \underline{22.1 \text{ mm}}$$

$$w_{max} = \frac{0.032}{1+0.5^4}(1-0.29^2)\frac{0.270(1000)^4}{200{,}000(22.1)^3} = \underline{3.45 \text{ mm}}$$

13.9

$\alpha = 1$; $M_Y = M_{bc} + \nu M_{ac} = \frac{1}{24}p_Y b^2(1+\nu)$ Table 13.1 ; $Y = \frac{6M_Y}{h^2} = \frac{p_Y b^2}{4h^2}(1+\nu)$

$$p_Y = \frac{4Yh^2}{b^2(1+\nu)} = \frac{4(240)(15)^2}{1500^2(1+0.29)} = 0.0744 \text{ MPa} = \underline{74.4 \text{ kPa}}$$

$$w_{max} = \frac{0.16}{1+2.4\alpha^3}(1-\nu^2)\frac{p_Y b^4}{Eh^3} = \frac{0.16}{3.4}(1-0.29^2)\frac{0.0744(1500)^4}{200{,}000(15)^3} = \underline{24.1 \text{ mm}}$$

13.10 We want to show that

$$w = \frac{p_0 r^4}{64D} + A_1 + A_2 \log r + B_1 r^2 + B_2 r^2 \log r \qquad (1)$$

is a solution of the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{p_0}{D} \qquad (2)$$

The first derivative of Eq.(1) is

$$\frac{dw}{dr} = \frac{p_0 r^3}{16D} + A_2\frac{1}{r} + 2B_1 r + 2B_2 r \log r + B_2 r^2 \frac{1}{r} \qquad (3)$$

The second derivative of Eq.(1) is

$$\frac{d^2 w}{dr^2} = \frac{3p_0 r^2}{16D} - \frac{A_2}{r^2} + 2B_1 + 2B_2 \log r + 2B_2 r \frac{1}{r} + B_2 \qquad (4)$$

Substitute Eqs. (3) and (4) into Eq. (2).

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{p_0 r^2}{4D} + 4B_1 + 4B_2 + 4B_2 \log r\right) = \frac{p_0}{D}$$

$$\frac{p_0}{2D} - \frac{4B_2}{r^2} + \frac{p_0}{2D} + \frac{4B_2}{r^2} = \frac{p_0}{D} \qquad \text{(verified)}$$

13.11 For a solid circular plate $A_2 = B_2 = 0$ in Eq. (13.73)

$$w = \frac{p_0 r^4}{64D} + A_1 + B_1 r^2 \qquad (1)$$

From Eqs. (13.74),

$$-\frac{1}{D}M_{rr} = w_{rr} + \nu\left(\frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2}\right) = 2(1+\nu)B_1 + (3+\nu)\frac{p_0 r^2}{16D} \qquad (2)$$

But $w = M_{rr} = 0$ at $r = a$

$$\frac{p_0 a^4}{64D} + A_1 + B_1 a^2 = 2(1+\nu)B_1 + (3+\nu)\frac{p_0 a^2}{16D} = 0$$

Solve for A_1 and B_1.

$$B_1 = -\frac{(3+\nu)p_0 a^2}{32(1+\nu)D}$$

$$A_1 = \frac{(3+\nu)p_0 a^4}{32(1+\nu)D} - \frac{p_0 a^4}{64D} = \frac{(5+\nu)p_0 a^4}{64(1+\nu)D}$$

Substitute these into Eqs. (1) and (2) and into 2nd of Eqs. (13.74).

$$w = \frac{p_0 r^4}{64D} + \frac{(5+\nu)p_0 a^4}{64(1+\nu)D} - \frac{(3+\nu)p_0 a^2 r^2}{32(1+\nu)D} = \frac{p_0 a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]\left[\frac{5+\nu}{1+\nu} - \left(\frac{r}{a}\right)^2\right].$$

$$M_{rr} = -D\left[-2(1+\nu)\frac{(3+\nu)p_0 a^2}{32(1+\nu)D} + (3+\nu)\frac{p_0 r^2}{16D}\right] = \frac{p_0 a^2}{16}(3+\nu)\left[1 - \left(\frac{r}{a}\right)^2\right]$$

$$M_{\theta\theta} = -D\left[\frac{w_r}{r} + \nu w_{rr}\right] = -D\left[\frac{p_0 r^2}{16D} - \frac{2(3+\nu)p_0 a^2}{32(1+\nu)D} + \nu\frac{3p_0 r^2}{16D} - \frac{2\nu(3+\nu)p_0 a^2}{32(1+\nu)D}\right]$$

$$= \frac{p_0 a^2}{16}\left[3 + \nu - (1+3\nu)\left(\frac{r}{a}\right)^2\right]$$

13.12 Eqs. (1) and (2) of Problem 13.11 are valid.

$$w(a) = \frac{p_0 a^4}{64D} + A_1 + B_1 a^2 = 0$$

$$w_r(a) = \frac{p_0 a^3}{16D} + 2B_1 a = 0$$

Solve for A_1 and B_1.

$$B_1 = -\frac{p_0 a^2}{32D}; \quad A_1 = \frac{p_0 a^4}{32D} - \frac{p_0 a^4}{64D} = \frac{p_0 a^4}{64D}$$

Substitute these into Eqs. (1) and (2) above and into 2nd of Eqs. (13.74).

$$w = \frac{p_0 r^4}{64D} + \frac{p_0 a^4}{64D} - \frac{2p_0 a^2 r^2}{64D} = \frac{p_0 a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2$$

$$M_{rr} = -D\left[-2(1+\nu)\frac{p_0 a^2}{32D} + (3+\nu)\frac{p_0 r^2}{16D}\right] = \frac{p_0 a^2}{16}\left[1 + \nu - (3+\nu)\left(\frac{r}{a}\right)^2\right]$$

$$M_{\theta\theta} = -D\left[\frac{w_r}{r} + \nu w_{rr}\right] = -D\left[\frac{p_0 r^2}{16D} - \frac{2p_0 a^2}{32D} + \nu\frac{3p_0 r^2}{16D} - \frac{2\nu p_0 a^2}{32D}\right]$$

$$= \frac{p_0 a^2}{16}\left[1 + \nu - (1+3\nu)\left(\frac{r}{a}\right)^2\right]$$

$\boxed{13.13}$ Start with Eqs. (13.77) and (13.78).

$$V_r(b) = -D\left[\frac{4B_2}{b} + \frac{p_0 b}{2D}\right] = 0; \quad B_2 = -\frac{p_0 b^2}{8D}$$

$$M_{rr}(b) = -D\left\{-(1-\nu)\frac{A_2}{b^2} + 2B_1(1+\nu) - \frac{p_0 b^2}{8D}\left[3+\nu+2(1+\nu)\log b\right] + \frac{(3+\nu)p_0 b^2}{16D}\right\} = 0 \qquad (1)$$

$$w(a) = A_1 + A_2 \log a + B_1 a^2 - \frac{p_0 b^2}{8D} a^2 \log a + \frac{p_0 a^4}{64D} = 0 \qquad (2)$$

$$M_{rr}(a) = -D\left\{-(1-\nu)\frac{A_2}{a^2} + 2B_1(1+\nu) - \frac{p_0 b^2}{8D}\left[3+\nu+2(1+\nu)\log a\right] + \frac{(3+\nu)p_0 a^2}{16D}\right\} = 0 \qquad (3)$$

Eliminate $\left[2B_1(1+\nu) - \frac{p_0 b^2}{8D}(3+\nu)\right]$ from Eqs (1) and (3)

$$-(1-\nu)\frac{A_2}{b^2} - \frac{p_0 b^2}{8D}\left[2(1+\nu)\log b\right] + \frac{(3+\nu)p_0 b^2}{16D} = -(1-\nu)\frac{A_2}{a^2} - \frac{p_0 b^2}{8D}\left[2(1+\nu)\log a\right] + \frac{(3+\nu)p_0 a^2}{16D}$$

$$A_2(1-\nu)\frac{a^2-b^2}{a^2 b^2} = \frac{(3+\nu)p_0}{16D}(b^2-a^2) + \frac{p_0 b^2}{4D}(1+\nu)\log\frac{a}{b}$$

$$A_2 = -\frac{(3+\nu)a^2 b^2 p_0}{16(1-\nu)D} + \frac{(1+\nu)a^2 b^4 p_0}{(1-\nu)(a^2-b^2)}\log\frac{a}{b} = \frac{p_0 a^4}{4D}\left\{\frac{(1+\nu)\log\frac{a}{b}}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{3+\nu}{4(1-\nu)\left(\frac{a}{b}\right)^2}\right\}$$

Substitute this value of A_2 into Eq. (1).

$$2B_1(1+\nu) = \frac{(1-\nu)p_0 a^4}{4D b^2}\left\{\frac{(1+\nu)\log\frac{a}{b}}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{3+\nu}{4(1-\nu)\left(\frac{a}{b}\right)^2}\right\} + \frac{p_0 b^2}{8D}\left[3+\nu+2(1+\nu)\log b\right] - \frac{(3+\nu)p_0 b^2}{16D}$$

$$B_1 = \frac{p_0 a^2}{8D}\left\{\frac{\left(\frac{a}{b}\right)^2(\log a - \log b)}{\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{\left(\frac{a}{b}\right)^2(3+\nu)}{4(1+\nu)\left(\frac{a}{b}\right)^2}{}^2 + \frac{3+\nu}{2(1+\nu)\left(\frac{a}{b}\right)^2} + \frac{\log b}{\left(\frac{a}{b}\right)^2} - \frac{3+\nu}{4(1+\nu)\left(\frac{a}{b}\right)^2}\right\}$$

$$= \frac{p_0 a^2}{8D}\left\{\frac{\left(\frac{a}{b}\right)^2\log a - \log b}{\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)\left[\left(\frac{a}{b}\right)^2-1\right]}{4(1+\nu)\left(\frac{a}{b}\right)^2}\right\}$$

Substitute values of A_2 and B_1 into Eq. (2).

$$A_1 = -\frac{p_0 a^4}{4D}\log a\left\{\frac{(1+\nu)\log\frac{a}{b}}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)}{4(1-\nu)\left(\frac{a}{b}\right)^2}\right\} - \frac{p_0 a^4}{8D}\left\{\frac{\left(\frac{a}{b}\right)^2\log a - \log b}{\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)\left[\left(\frac{a}{b}\right)^2-1\right]}{4(1+\nu)\left(\frac{a}{b}\right)^2}\right\}$$

$$+ \frac{p_0 b^2}{8D}a^2\log a - \frac{p_0 a^4}{64D}$$

$$A_1 = -\frac{p_0 a^4}{4D}\left\{\frac{(1+\nu)\log\frac{a}{b}\log a}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(5-\nu)\log a}{4(1-\nu)\left(\frac{a}{b}\right)^2} + \frac{\left(\frac{a}{b}\right)^2\log a - \log b}{2\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)\left[\left(\frac{a}{b}\right)^2-1\right]}{8(1+\nu)\left(\frac{a}{b}\right)^2} + \frac{1}{16}\right\}$$

The deflection at $r=b$ is given by Eq. (13.73).

$$w(d) = A_1 + A_2\log b + B_1 b^2 + B_2 b^2\log b + \frac{p_0 b^4}{64D}; \quad \text{For } \frac{a}{b} = 2$$

$$A_1 = \frac{p_0 a^4}{4D}(0.14298\log a + 0.04167\log b + 0.17548)$$

$$A_2 = \frac{p_0 a^4}{4D}(-0.187372)$$

$$B_1 = \frac{p_0 a^2}{4D}(0.166667\log a - 0.041667\log b - 0.237983); \quad B_2 = -\frac{p_0 a^2}{4D}\left(\frac{1}{8}\right)$$

$$w(b) = \frac{p_0 a}{4D}(0.119894 + 0.187370\log a - 0.187375\log b) = 0.24977\frac{p_0 a^4}{4D} = \underline{0.6819\frac{p_0 a^4}{Eh^3}}$$

13.14

$$\sigma_{max} = \sigma_w = \frac{3pa^2}{4h^2} \quad \text{(From Table 13.2)}$$

$$h = \sqrt{\frac{3pa^2}{4\sigma_w}} = \sqrt{\frac{3(0.690)(200)^2}{4(82.0)}} = \underline{15.9 \text{ mm}}$$

$$w_{max} = \frac{3}{16}(1-\nu^2)\frac{pa^4}{Eh^3} = \frac{3(1-0.29^2)(0.690)(200)^4}{16(200{,}000)(15.9)^3} = \underline{0.236 \text{ mm}}$$

13.15 From Table 13.2,

$$\sigma_{max} = \sigma_w = \frac{3}{8}(3+\nu)\frac{pa^2}{h^2}; \quad p = 60(9.80) = 588 \text{ kPa} = 0.588 \text{ MPa}$$

$$h = \sqrt{\frac{3(3+\nu)pa^2}{8\sigma_w}} = \sqrt{\frac{3(3+0.20)(0.588)(150)^2}{8(14)}} = \underline{33.7 \text{ mm}}$$

$$w_{max} = \frac{3}{16}(1-\nu)(5+\nu)\frac{pa^4}{Eh^3} = \frac{3(1-0.20)(5+0.20)(0.588)(150)^4}{16(100{,}000)(33.7)^3} = \underline{0.061 \text{ mm}}$$

13.16 From Table 13.2,

$$\text{(a) } \sigma_{max} = \frac{3}{8}(3+\nu)\frac{pa^2}{h^2} = \frac{3(3+0.29)(1.38)(250)^2}{8(25)^2} = \underline{170 \text{ MPa}}$$

$$w_{max} = \frac{3}{16}(1-\nu)(5+\nu)\frac{pa^4}{Eh^3} = \frac{3(1-0.29)(5+0.29)(1.38)(250)^4}{16(200{,}000)(25)^3} = \underline{1.215 \text{ mm}}$$

$$\text{(b) } \sigma_{max} = Y = \frac{3}{8}(3+\nu)\frac{p_Y a^2}{h^2}$$

$$p_Y = \frac{8Yh^2}{3(3+\nu)a^2} = \frac{8(276)(25)^2}{3(3+0.29)(250)^2} = \underline{2.24 \text{ MPa}}$$

$$SF = \frac{p_Y}{p} = \frac{2.24}{1.38} = \underline{1.62}$$

13.17 Use Eqs. (13.81) and (13.82) along with Table 13.3

$$\frac{a}{r_0} = \frac{300}{100} = 3.00$$

$$\text{(a) } \sigma_{max} = k_1\frac{pa^2}{h^2} = \frac{2.15(0.10)(300)^2}{10^2} = \underline{193.5 \text{ MPa}}$$

$$w_{max} = k_2\frac{pa^4}{Eh^3} = \frac{0.293(0.10)(300)^4}{200{,}000(10)^3} = \underline{1.187 \text{ mm}}$$

$$\text{(b) } \sigma_{max} = Y = k_1\frac{p_Y a^2}{h^2}$$

$$p_Y = \frac{Yh^2}{k_1 a^2} = \frac{290(10)^2}{2.15(300)^2} = 0.150 \text{ MPa} = 150 \text{ kPa}$$

$$SF = \frac{p_Y}{p} = \frac{150}{100} = \underline{1.50}$$

13.18

$\frac{\sigma a^2}{Eh^2} = \frac{Ya^2}{Eh^2} = \frac{241(127)^2}{200,000(2.54)^2} = 3.01$; From Fig. 13.15a, we read $\frac{w_{max}}{h} = 1.1$ for $\frac{\sigma a^2}{Eh^2} = 3.01$. From Fig. 13.15b, we read $\frac{p_Y a^4}{Eh^4} = 10$ for $\frac{w_{max}}{h} = 1.1$.

$$p_Y = \frac{10Eh^4}{a^4} = \frac{10(200,000)(2.54)^4}{127^4} = 0.320 \text{ MPa} = 320 \text{ kPa}$$

$$p = \frac{1}{3} p_Y = \frac{320}{3} = \underline{107 \text{ kPa}}$$

13.19

By trial and error $h = \underline{2.0 \text{ mm}}$; $SF = \frac{3}{2} = 1.50$;

$p_Y = SF(p) = 1.50(73.8) = 110.7$ kPa. Calculate

$$\frac{p_Y a^4}{Eh^4} = \frac{0.1107(127)^4}{72,000(2.0)^4} = 25.0 \quad \text{and} \quad \frac{Ya^2}{Eh^2} = \frac{276(127)^2}{72,000(2.0)^2} = 15.5$$

Enter Figs. 13.15 a and b, respectively, with these two values and read same values for $\frac{w_{max}}{h}$ indicating that $h = \underline{2.0 \text{ mm}}$

13.20

$\frac{\sigma a^2}{Eh^2} = \frac{Ya^2}{Eh^2} = \frac{207(1270)^2}{200,000(12.7)^2} = 10.35$; From Fig. 13.16 a we read $\frac{w_{max}}{h} = 3.7$ for $\frac{\sigma a^2}{Eh^2} = 10.35$. From Fig. 13.16b, we read $\frac{p_Y a^4}{Eh^4} = 25$ for $\frac{w_{max}}{h} = 3.7$.

$$p_Y = \frac{25Eh^4}{a^4} = \frac{25(200,000)(12.7)^4}{(1270)^4} = 0.050 \text{ MPa} = \underline{50 \text{ kPa}}$$

$$w_{max} = 3.7h = 3.7(12.7) = \underline{47.0 \text{ mm}}$$

13.21

$\sigma = \frac{Y}{2} = \frac{207}{2} = 103.5$ MPa; $\frac{\sigma a^2}{Eh^2} = \frac{103.5(1270)^2}{200,000(12.7)^2} = 5.18$; From Fig. 13.16a we read $\frac{w_{max}}{h} = 2.25$ for $\frac{\sigma a^2}{Eh^2} = 5.18$. From Fig. 13-16b, we read $\frac{pa^4}{Eh^4} = 7$ for $\frac{w_{max}}{h} = 2.25$.

$$p = \frac{7Eh^4}{a^4} = \frac{7(200,000)(12.7)^4}{(1270)^4} = 0.014 \text{ MPa} = \underline{14 \text{ kPa}}$$

The ratio of Y to σ is 2.00 while the ratio of p_Y to p is 3.57. The two ratios would have been the same for a linear problem. Large deflections make the problem non linear.

13.22

$\frac{\sigma a^2}{Eh^2} = \frac{Ya^2}{Eh^2} = \frac{241(127)^2}{200{,}000(12.7)^2} = 3.01$; From Fig. 13.16a, we read $\frac{w_{max}}{h} = 1.5$ for $\frac{\sigma a^2}{Eh^2} = 3.01$. From Fig. 13.16b, we read $\frac{p_Y a^4}{Eh^4} = 3$.

$$p_Y = \frac{3Eh^4}{a^4} = \frac{3(200{,}000)(2.54)^4}{(127)^4} = 0.096\ \text{MPa} = 96\ \text{kPa}$$

$$p = \frac{p_Y}{SF} = \frac{96}{3.00} = \underline{32\ \text{kPa}}$$

13.23

$SF = \frac{3}{2} = 1.50$; $\frac{\sigma a^2}{Eh^2} = \frac{Ya^2}{Eh^2} = \frac{276(127)^2}{200{,}000(2)^2} = 15.5$; From Fig. 13.16a, we read $\frac{w_{max}}{h} = 5$ for $\frac{\sigma a^2}{Eh^2} = 15.5$. From Fig. 13.16b, we read

$$\frac{p_Y a^4}{Eh^4} = 52$$

$$p_Y = \frac{52Eh^4}{a^4} = \frac{52(72{,}000)(2)^4}{(127)^4} = 0.2303\ \text{MPa} = 230.3\ \text{kPa}$$

$$p = \frac{p_Y}{SF} = \frac{230.3}{1.50} = \underline{154\ \text{kPa}}$$

14.1

$\sqrt{\frac{t}{\rho}} = 1.00$; $\sqrt{\frac{b}{\rho}} = \sqrt{16} = 4.00$. From Fig. 14-10, $S_{cc} = \underline{2.95}$

14.2

(a) $\sqrt{\frac{t}{\rho}} = 1.00$; $\sqrt{\frac{b}{\rho}} = \sqrt{4} = 2.00$. From Fig. 14.10, $S_{cc} = \underline{2.7}$

(b) $\sqrt{\frac{t}{\rho}} = 1.00$; $\sqrt{\frac{b}{\rho}} = \sqrt{1} = 1.00$. From Fig. 14.10, $S_{cc} = \underline{2.3}$

14.3

$\sqrt{\frac{t}{\rho}} = \sqrt{4} = 2.00$; $\sqrt{\frac{b}{\rho}} = \sqrt{16} = 4.00$. From Fig. 14.10

(a) $S_{cc(T)} = \underline{3.9}$

(b) $S_{cc(M)} = \underline{3.2}$

14.4

$\sqrt{\frac{t}{\rho}} = \sqrt{\frac{6}{2.2}} = 1.65$; $\sqrt{\frac{b}{\rho}} = \sqrt{\frac{60}{2.2}} = 5.22$. From Fig. 14.10

(a) $S_{cc(P)} = \underline{3.7}$

(b) $S_{cc(M)} = \underline{3.3}$

(c) $S_{cc(T)} = \underline{2.1}$

14.5

By Fig. C, Table 14-3, and Fig. P14-5, $2t = 2\rho = 25$ mm, $2b = 100 - 2t = 75$ mm. (also $h = 20$ mm.) Therefore $\sqrt{\frac{t}{\rho}} = 1.00$ and $\sqrt{\frac{b}{\rho}} = \sqrt{\frac{75}{25}} = 1.732$

Using curve 5 and scale f in Fig. 14.10, we obtain $S_{cc} = 2.68$. By the first of Eqs. (14.2), $\sigma_{max} = \sigma_c = S_{cc}\sigma_n$,

or $\sigma_n = \frac{\sigma_{max}}{S_{cc}} = \frac{\sigma_u}{S_{cc}}$. also, $\sigma_n = (SF)\sigma_{design} = (SF)\frac{P_{design}}{(2b)(h)}$.

Therefore, $P_{design} = \frac{\sigma_u (2b)(h)}{(SF)(S_{cc})}$

or $P_{design} = \frac{(300)(75)(20)}{(3.5)(2.68)} = 47.97$ kN

14.6

Let $\epsilon_g = 0.00100$ be the strain at the bottom of the groove and let $\epsilon_a = 0.00032$ be the strain far removed from the groove.

Then
$$S_{cc} = \frac{\sigma_{max}}{\sigma_n} = \frac{E\epsilon_g}{E\epsilon_a} = \frac{\epsilon_g}{\epsilon_a} = \frac{0.00100}{0.00032} = 3.125$$

The load P is given by
$$P = \sigma_n A = E\epsilon_a A = (200\times 10^3)(0.00032)(\pi)(100)^2$$

or $\quad P = 2.01$ kN

14.7

By Fig. P14.7, $I_{y'} = \frac{1}{12}(50)(200)^3 - \frac{2}{12}(50)(10)^3 - 2(50)(10)(y'-5)^2$
$$= 33{,}325{,}000 - 1000(y')^2 + 10{,}000y' - 25000$$

For $y' = 50$, $I_{50} = 31.300\times 10^6$ mm^4

For $y' = 75$, $I_{75} = 28.425\times 10^6$ mm^4

Consider the stress at y': $\sigma_{y'} = \sigma_{Limit} = \dfrac{(SF)My'}{I_{y'}}$

For $y' = 50$, $\sigma_{Limit} = 120\text{ MPa} = \dfrac{(3.0)M(50)}{31.30\times 10^6}$; Therefore,

$M = 25.04$ kN·m.

For $y' = 75$, $\sigma_{Limit} = 120\text{MPa} = \dfrac{(3.0)M(75)}{28.425\times 10^6}$; $M = 15.16$ kN·m

Check the stress at the outer fibers of the beam.

For $y' = 50$, $\sigma_{Limit} = 120\text{MPa} = \dfrac{M(100)}{31.30\times 10^6}$; $M = 37.56$ kN·m

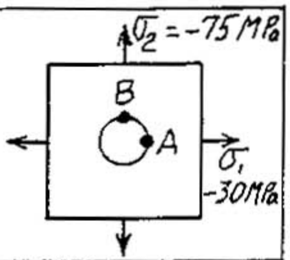
For $y' = 75$ $\sigma_{Limit} = 120\text{MPa} = \dfrac{M(100)}{28.425\times 10^6}$; $M = 34.11$ kN.m.

Thus, the limiting moment is controlled by the stress concentration at the drilled hole, assuming ~~that the concentration~~ has dissipated at the outer fibers.

14.8

$\sigma_A = 3\sigma_2 - \sigma_1 = 3(-75)-(-20) = \underline{-205 \text{ MPa}}$

$\sigma_B = 3\sigma_1 - \sigma_2 = 3(-20)-(-75) = \underline{15 \text{ MPa}}$



14.9

$\coth \alpha_0 = \frac{1}{\tanh \alpha_0} = \frac{a}{b} = 5; \quad \alpha_0 = 0.2027 \text{ rad}$

$\sinh 2\alpha_0 = 0.4167; \quad \cosh 2\alpha_0 = 1.0833; \quad \coth 2\alpha_0 = 2.6000$

$e^{2\alpha_0} = 1.500;$ By Eq. (14.34)

$$\cos 2\theta = -\frac{-1.0833-0.4167+2\left(\frac{-20-75}{-20+75}\right)^2(0.4167)}{2\left(\frac{-20-75}{-20+75}\right)} = 0.2855$$

$\theta = \underline{0.6406 \text{ rad}}$

By Eq. (14.38),

$$\sigma_{max} = \sigma_{\beta\beta(max)} = \sigma_{\beta\beta_1} = -\frac{(-20+75)^2}{2(-20-75)}(1+2.6000) = \underline{57.3 \text{ MPa}}$$

By Eq. (14.35),

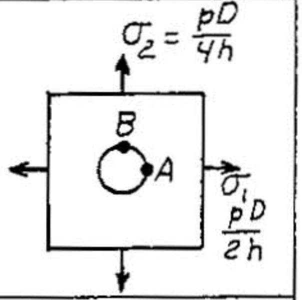
$$\cos 2\beta = 1.0833 - \frac{2}{1.500}\left(\frac{-20-75}{-20+75}\right)^2(0.4167)^2 = 0.3926$$

$\beta = \underline{0.5837 \text{ rad}}$

14.10

$\sigma_A = -\sigma_1 + \sigma_2\left(1+2\frac{a}{b}\right) = -\frac{pD}{2h} + \frac{pD}{4h}(3) = \underline{\frac{pD}{4h}}$

$\sigma_B = \sigma_1\left(1+2\frac{a}{b}\right) - \sigma_2 = \frac{pD}{2h}(3) - \frac{pD}{4h} = \underline{\frac{5pD}{4h}}$



14.11 Take z-axis along axis of vessel.

$\sigma_{\theta\theta} = \frac{pD}{2h}; \quad \sigma_{zz} = \frac{pD}{4h} \pm \sigma_{Bending}$

The bending moment at center of pressure vessel is $\frac{wL^2}{8}$ where $w = \frac{\rho\pi D^2}{4}$. Since $c = D/2$ and $I \cong \frac{\pi D^3 h}{8}$,

$$\sigma_{Bending} = \frac{Mc}{I} = \frac{\rho\pi D^2 L^2}{32}\left(\frac{D}{2}\right)\left(\frac{8}{\pi D^3 h}\right) = \frac{\rho L^2}{8h}$$

For small hole at center top of tank,

$$\sigma_{max} = 3\sigma_{\theta\theta} - \sigma_{zz} = 3\frac{pD}{2h} - \frac{pD}{4h} + \frac{\rho L^2}{8h} = \underline{\frac{5pD}{4h} + \frac{\rho L^2}{8h}}$$

(cont.)

14.11 cont. For a small hole at the center bottom of the tank,

$$\sigma_{max} = 3\sigma_{\theta\theta} - \sigma_{zz}$$

or

$$\sigma_{max} = 3\frac{pD}{2h} - \frac{pD}{4h} - \frac{\rho L^2}{8h} = \frac{5p}{4h} - \frac{\rho L^2}{8h}$$

14.12

$$\sigma = S_{cc}\frac{Mc}{I}$$

$$= \frac{3.3(15{,}000{,}000)(60)}{10{,}180{,}000}$$

or

$$\sigma = 291.7 \text{ MPa}$$

$$\tau = S_{cc}\frac{Tc}{J}$$

$$= \frac{2.1(30{,}000{,}000)(60)}{20{,}330{,}000}$$

or

$$\tau = 185.7 \text{ MPa}$$

(a) $\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = 145.9 + 236.1 = 382.0 \text{ MPa}$

$\sigma_2 = 145.9 - 236.1 = -90.2 \text{ MPa}$

$\sigma_3 = 0$

(b) $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{382.0 - (-90.2)}{2}$

or

$$\tau_{max} = 236.1 \text{ MPa}$$

(c) $\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_3-\sigma_2)^2}$

$$= \frac{1}{3}\sqrt{(382.0+90.2)^2 + (382.0)^2 + (90.2)^2}$$

or

$$\tau_{oct} = 204.7 \text{ MPa} < \tau_{max} = 236.1 \text{ MPa}$$

14.13 By Fig. a and Fig. 14.16, we have for the fillet $\rho = 11$ mm, $d = 50 - 22 = 28$ mm, and $t = \rho = 11$ mm. Hence, $\rho/d = 0.393$. Then, by Fig. 14.16,

$$S_{cc} \approx 1.5 \qquad (a)$$

For the circular hole, $\rho = 14$ mm, $d = 50 - 28 = 22$ mm, and $\rho/d = 0.636$. Then, by Fig. 14.16,

$$S_{cc} \approx 2.1 \qquad (b)$$

For the semicircular groove, $d = 25$ mm, $\rho = (50-25)/2 = 12.5$mm, and $\rho/d = 0.50$. Then, by Fig. 14.16,

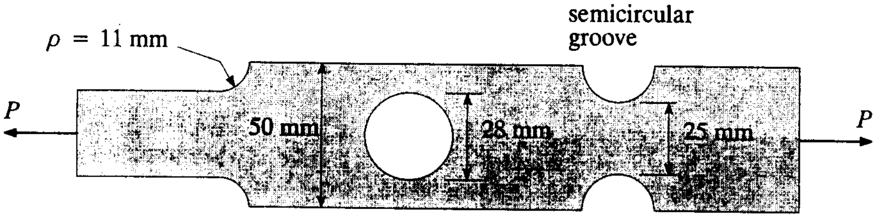


Figure a

$$S_{cc} \approx 1.53 \qquad (c)$$

(a) To determine when yield first occurs, we calculate the load require to cause yield at each section.

For the fillet, $S_{cc} P_{fillet} = Y A_{fillet}$, or with Eq. (a), and $Y = 276$ MPa and thickness 7 mm,

$$P_{fillet} = (276\times10^6)(0.028)(0.007)/1.5 = 36.06 \text{ kN} \quad (d)$$

For the circular hole, $S_{cc} P_{hole} = Y A_{hole}$, or with Eq. (b) and $Y = 276$ MPa and ~~thickness~~ 7 mm,

$$P_{hole} = (276\times10^6)(0.022)(0.007)/2.1 = 20.24 \text{ kN} \quad (e)$$

(Cont.)

14.13 Cont. For the semicircular groove, $S_{cc} P_{groove} = Y A_{groove}$ or with Eq. (c),

$$P_{groove} = (276 \times 10^6)(0.025)(0.007)/1.53 = 31.58 \text{ kN} \qquad (f)$$

Comparison of Eqs. (d), (e), and (f) shows that yield occurs first at the circular hole section for a load $P = 20.24$ kN.

(b) To determine the section that first becomes fully plastic, we calculate the force P_{FP} required to cause each section to become fully plastic. Thus, for the fillet

$$P_{FP} = Y A_{fillet} = (276 \times 10^6)(0.028)(0.007) = 54.10 \text{ kN} \qquad (g)$$

For the circular hole,

$$P_{FP} = Y A_{hole} = (276 \times 10^6)(0.022)(0.007) = 42.50 \text{ kN} \qquad (h)$$

For the semicircular groove,

$$P_{FP} = Y A_{groove} = (276 \times 10^6)(0.025)(0.007) = 48.30 \text{ kN} \qquad (i)$$

Comparison of Eqs. (g), (h), and (i) shows that the section at the circular hole is the first to become fully plastic for a load $P = 42.50$ kN.

14.14 (a) As in the solution of Problem 14.13, the yield loads for the fillet and the groove remain unchanged; that is

$$P_{fillet} = 36.06 \text{ kN and } P_{groove} = 31.58 \text{ kN} \quad (a)$$

Now for the hole with a diameter of 26 mm,
$\rho = 13$ mm, $d = 50 - 26 = 24$ mm, and $\rho/d = 0.542$. Then, by Fig. 14.16, $S_{cc} \approx 2.1$ and $S_{cc} P_{hole} = Y A_{hole}$, or

$$P_{hole} = (276\times10^6)(0.026)(0.007)/2.1 = 23.92 \text{ kN} \quad (b)$$

By Eqs. (a) and (b),

$$P_{fillet} = 36.06 \text{ kN} > P_{groove} = 31.57 \text{ kN} > P_{hole} = 23.92 \text{ kN}.$$

Hence, yielding occurs first at the hole for a load $P = 23.92$ kN.

(b) The fully plastic loads for the fillet and the groove remain unchanged (see Problem 14.13); that is, for fully plastic conditions,

$$P_{FP} = P_{fillet} = 54.10 \text{ kN}; \; P_{FP} = P_{groove} = 48.30 \text{ kN} \quad (c)$$

Now for the hole,

$$P_{FP} = P_{hole} = Y A_{hole} = (276\times10^6)(0.026)(0.007) = 50.23 \text{ kN} \quad (d)$$

Comparison of Eqs. (c) and (d) shows that the first section to become fully plastic is the groove section for a load $P = 48.30$ kN

14.15 By Figs. P14.13 and (4.16), $\rho = 10$ mm, $d = h = 40$ mm and therefore $\rho/d = 0.25$. Then by Fig. 14.16, $S_{cc} \approx 2.25$. At the reduced section, $\sigma_n = \frac{P}{(40)h} = \frac{P}{(40)(40)} = \frac{P}{1600}$. Thus, the maximum stress is $\sigma_{max} = S_{cc}\sigma_n = \frac{(2.25)P}{1600} = 350$ MPa. Hence, $P = \frac{350 \times 10^6 (N/m^2) \times 1600\ mm^2}{2.25} \cdot \frac{1\ m^2}{10^6\ mm^2} = 248.9$ kN.

14.16 By Figs. P14.14 and 14.16, for the circular hole, $\rho = 10$ mm, $D = H = 120$ mm, $d = \frac{120-20}{2} = 50$ mm. Therefore, $\rho/d = 0.20$. Hence, by Fig. 14.16, $S_{cc} \approx 2.3$. At the circular hole, $\sigma_n = \frac{P}{(2d)(20)} = \frac{80000\ N}{(100)(20)\ mm^2} = 40$ MPa Therefore, $\sigma_{max} = S_{cc}\sigma_n = (2.3)(40) = 92$ MPa at the hole.

Similarly for the fillets, $\rho = 10$ mm, $D = 120$ mm, $d = h = 100$ mm, $t = \rho = 10$ mm, and $\rho/d = 0.1$. By Fig. 14.16, $S_{cc} \approx 1.79$. at the base of the fillet $\sigma_n = \frac{P}{20h}$ or $\sigma_n = \frac{80000\ N}{(20)(100)\ mm^2} = 40$ MPa. Therefore, $\sigma_{max} = S_{cc}\sigma_n = (1.79)(40) = 71.6$ MPa. Thus, $\sigma_{max} = 92$ MPa at the hole, and $\sigma_{max} = 71.6$ MPa at the fillet.

14.17 Since $H = h + 2\rho$, $\rho = 0.20h$, we have $\rho/h = 0.20$ and $H/h = 1.4$. Therefore, by Fig. 14.25, $S_{cc} \approx 1.59$.

14.18 At the fillet, $\sigma_{max} = E\epsilon = 200\times10^9 \times 0.0008 = 160$ MPa. The nomimal stress is $\sigma_n = \frac{Mc}{I} = \frac{(600)(3000)(30)4}{\pi(30)^4}\frac{N}{mm^2}$

or $\sigma_n = 84.88$ MPa. Therefore, $S_{cc} = \frac{\sigma_{max}}{\sigma_n} = \frac{160}{84.88} = 1.88$

14.19 By the strain measurements, $S_{cc} = \frac{0.00250}{0.00100} = 2.50$. At the gage removed from the notch,

$\sigma_n = E\epsilon = 72\times10^9 \times 0.001 = 72 \text{ MPa} = \frac{Mc}{I}$.

$I = \frac{1}{12}(50)(150)^3 = 14{,}062{,}500 \text{ mm}^4$, $c = 75$ mm, $M = 500P$.

$\therefore \frac{(500)(P)(75)}{14.0625\times10^6} = 72$ MPa. Hence, $P = 27$ kN.

14.20 $\sigma_u = 450 \text{ MPa} = \sigma_{max}$. $h = 125$ mm, $\rho = 15$ mm, $SF = 3.50$

$I = \frac{1}{12}(60)(250)^3 = 78{,}125{,}000 \text{ mm}^4$. Since $\rho/h = 0.12$ and $H/h = 250/125 = 2$, Fig. 14.25 yields

$S_{cc} \approx 2.0$. At the notch, $\sigma_n = (SF)Mc/I$, $M = 1.6P$, $c = h/2 = 62.5$ mm. Therefore,

$$\sigma_{max} = \sigma_u = 450 \text{ MPa} = S_{cc}\sigma_n = (S_{cc})(SF)Mc/I \text{ or}$$

$$450 \text{ MPa} = (2.0)(3.5)(1.6P)(62.5)/(78.125\times10^6)$$

Therefore,

$$P = 50.22 \text{ kN}$$

14.21

$$t = \frac{D-d}{2} = \frac{68-50}{2} = 9\text{ mm} = 3\rho ; \quad \frac{\rho}{d} = \frac{3}{50} = 0.060$$

From Fig. 14.16, $S_{cc} = 2.35$

$$S_{ce} = q(S_{cc}-1)+1 = 0.8(2.35-1)+1 = 2.08$$

$$P_{max} = \frac{\sigma_W A}{S_{ce}} = \frac{14(24)(50)}{2.08} = \underline{8.08\text{ kN}}$$

14.22 From Problem 14.4, $S_{cc} = 3.3$ in bending and $S_{cc} = 2.1$ in torsion.

In Bending

$$S_{ce} = q(S_{cc}-1)+1 = 0.8(3.3-1)+1 = 2.84$$

In Torsion

$$S_{ce} = q(S_{cc}-1)+1 = 0.8(2.1-1)+1 = 1.88$$

$$\sigma = S_{ce}\frac{Mc}{I} = \frac{2.84(15{,}000{,}000)(60)}{10{,}180{,}000} = 251.1\text{ MPa}$$

$$\tau = S_{ce}\frac{Tc}{J} = \frac{1.88(30{,}000{,}000)(60)}{20{,}360{,}000} = 166.2\text{ MPa}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{251.1}{2}\right)^2 + 166.2^2} = \underline{208.3\text{ MPa}}$$

$$\sigma_{max} = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{251.1}{2} + 208.3 = \underline{333.8\text{ MPa}}$$

14.23

$$\sqrt{\frac{t}{\rho}} = \sqrt{\frac{5}{5}} = 1.00; \quad \sqrt{\frac{a}{\rho}} = \sqrt{\frac{30}{5}} = 2.449$$

From the nomograph (Fig. 14.10), $S_{cc} = 2.5$

$$\sigma_n = \frac{P}{A} = \frac{P}{2ad} = \frac{110{,}000}{2(30)(12.5)} = 147\text{ MPa}$$

In Fig. E14.4a $E = 206.8$ GPa; thus,

$$\epsilon_n = \frac{\sigma_n}{E} = \frac{147}{206{,}800} = 0.000711$$; Equation (14.49) gives

$$\sigma_{max}\epsilon_{max} = S_{cc}^2\sigma_n\epsilon_n = 2.5^2(147)(0.000711) = 0.653$$

The intersection of this equation with the stress-strain diagram in Fig. E14.4a gives $\sigma_{max} = 258$ MPa and $\epsilon_{max} = 0.00253$

$$S_{ce} = \frac{\sigma_{max}}{\sigma_n} = \frac{258}{147} = \underline{1.76}$$

14.24

$\sigma_n = \frac{\sigma_{max}}{S_{ce}} = \frac{258}{1.10} = 235$ MPa; $\epsilon_n = \frac{\sigma_n}{E} = \frac{235}{206{,}800} = 0.001136$

$\sigma_{max}\,\epsilon_{max} = S_{cc}^2\,\sigma_n\,\epsilon_n = 2.5^2(235)(0.001136) = 1.669$

$\epsilon_{max} = \frac{1.669}{258} = \underline{0.0065}$

15.1

$$\sigma = SF\frac{Mc}{I} = \frac{4.00(5{,}500{,}000)(d)(64)}{2\pi d^4} = \frac{224{,}100{,}000}{d^3}\ (MPa)$$

$$\tau = SF\frac{Tc}{J} = \frac{4.00(5{,}000{,}000)(d)(32)}{2\pi d^4} = \frac{101{,}900{,}000}{d^3}\ (MPa)$$

$$\sigma_1 = \sigma_{max} = \sigma_u = 145 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{224{,}100{,}000}{2d^3} + \frac{1}{d^3}\sqrt{\left(\frac{224{,}100{,}000}{2}\right)^2 + 101{,}900{,}000^2}$$

$d = \underline{122.0 \text{ mm}}$

15.2

$$\sigma = \frac{P}{A} = \frac{\sigma_u \pi d^2(4)}{12\pi d^2} = \frac{\sigma_u}{3}; \quad \tau = \frac{Tc}{J} = \frac{T(d)(32)}{2\pi d^4} = \frac{16T}{\pi d^3}$$

$$\sigma_1 = \sigma_{max} = \sigma_u = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{\sigma_u}{6} + \sqrt{\left(\frac{\sigma_u}{6}\right)^2 + \tau^2}$$

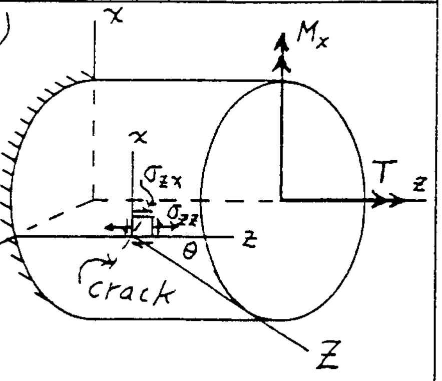
$$\tau = \frac{\sqrt{2}}{\sqrt{3}}\sigma_u$$

$$T = \frac{\pi d^3}{16}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\sigma_u = \underline{\frac{1}{8\sqrt{6}}\pi d^3 \sigma_u}$$

15.3

$$\sigma_{zz} = \frac{M_x c}{I_x} = \frac{M_x(25)(4)}{\pi(25)^4} = 0.00008149 M_x\ (MPa)$$

$$-\tau = \sigma_{zx} = -\frac{Tc}{J} = -\frac{1{,}200{,}000(25)(2)}{\pi(25)^4} = -48.89 \text{ MPa}$$



For T positive, $\sigma_1 = \sigma_{max}$ occurs for θ negative

$$\theta = -\frac{\pi}{2} + 1.0000 = -0.5708 \text{ rad}; \ 2\theta = -1.1416 \text{ rad}$$

$$\tan 2\theta = -2.1850 = \frac{2\sigma_{zx}}{\sigma_{zz}} = \frac{2(-48.89)}{\sigma_{zz}}$$

$$\sigma_{zz} = 44.75 \text{ MPa}$$

$$M_x = \frac{44.75}{0.00008149} = \underline{549.8 \text{ N.m}}$$

$$\sigma_u = \sigma_{max} = \frac{\sigma_{zz}}{2} + \sqrt{\left(\frac{\sigma_{zz}}{2}\right)^2 + \tau^2} = \frac{44.75}{2} + \sqrt{\left(\frac{44.75}{2}\right)^2 + 48.89^2}$$

$$= \underline{76.14 \text{ MPa}}$$

15.4

$$\sigma_{zz} = SF\frac{Mc}{I} = \frac{3.00(7{,}500{,}000(62.5)(64)}{\pi(125)^4} = 117.3 \text{ MPa}$$

$$\tau = SF\frac{Tc}{J} = \frac{3.00T(62.5)(32)}{\pi(125)^4} = 0.00000782T\ (MPa)$$

$$\sigma_1 = \sigma_{max} = \sigma_u = 150 = \frac{\sigma_{zz}}{2} + \sqrt{\left(\frac{\sigma_{zz}}{2}\right)^2 + \tau^2} = \frac{117.3}{2} + \sqrt{\left(\frac{117.3}{2}\right)^2 + \tau^2}$$

$$\tau = 70.0 \text{ MPa}$$

$$T = \frac{70.0}{0.00000782} = \underline{8.98 \text{ kN.m}}$$

15.5

$$\tau = \frac{5.00}{3.00}\,0.00000782 = 0.00001303T$$

$$T = \frac{70.0}{0.00001303} = \underline{5.37 \text{ kN.m}}$$

15.6 $\lambda = \frac{a}{c} = \frac{9}{150} = 0.060$; $f(\lambda) = 1.195$; $K_{IC} = 23 (MPa\sqrt{m}) = 23\sqrt{1000} (MPa\sqrt{mm})$

$$\sigma = \frac{K_{IC}}{\sqrt{a\pi}\, f(\lambda)} = \frac{23\sqrt{1000}}{\sqrt{9\pi}(1.195)} = 114.5 \text{ MPa}$$

$$P = \sigma A = 114.5(8)(150) = \underline{137.4 \text{ kN}}$$

15.7 $\lambda = \frac{a}{c} = \frac{9}{75} = 0.120$; $f(\lambda) = 1.12$

$$\sigma = \frac{K_{IC}}{\sqrt{a\pi}\, f(\lambda)} = \frac{23\sqrt{1000}}{\sqrt{9\pi}\,(1.12)} = 122.1 \text{ MPa}$$

$$P = \sigma A = 122.1(8)(150) = \underline{146.5 \text{ kN}}$$

15.8 $\text{area} = \frac{\pi}{4}(100^2 - 84^2) = 2310 \text{ mm}^2$

$a = 7.50$ mm

$c = 144.5$ mm

$\lambda = \frac{a}{c} = \frac{7.50}{144.5} = 0.0519$

$f(\lambda) = 1.01$

Failure is brittle fracture since both a and thickness is greater than 6.7 mm.

$$\sigma = \frac{K_{IC}}{\sqrt{a\pi}\, f(\lambda)} = \frac{23\sqrt{1000}}{\sqrt{7.5\pi}(1.01)} = 148.4 \text{ MPa}$$

$$P = \sigma A = 148.4(2310) = \underline{342.8 \text{ kN}}$$

15.9 Since both a and the thickness is less than 31.1, the failure is not expected to be brittle fracture since K_I is greater than K_{IC}. The lower limit for the failure load is obtained by assuming that $K_I = K_{IC}$

$$\sigma_{\text{lower limit}} = \frac{K_{IC}}{\sqrt{a\pi}\, f(\lambda)} = 148.4\,\frac{77}{23} = 496.8 \text{ MPa}$$

$$P_{\text{lower limit}} = \sigma A = 496.8(2310) = \underline{1.148 \text{ MN}}$$

15.10 $\lambda = \frac{a}{2c} = \frac{8}{60} = 0.133$

$f(\lambda) = 1.033$

$$K_{IC} = 59 (MPa\sqrt{m}) = 59\sqrt{1000} (MPa\sqrt{mm}) = \frac{3M}{2tc^2}\sqrt{a\pi}\, f(\lambda)$$

$$M = \frac{2tc^2 K_{IC}}{3\sqrt{a\pi}\, f(\lambda)} = \frac{2(60)(30^2)(59\sqrt{1000})}{3\sqrt{8\pi}\,(1.033)} = \underline{12.97 \text{ kN.m}}$$

15.11

$$I = \frac{1}{12} b h^3 = \frac{100(250)^3}{12} = 130{,}200{,}000 \text{ mm}^4$$

$$M_y = \frac{y I}{c} = \frac{299(130{,}200{,}000)}{125} = 311.4 \text{ kN}\cdot\text{m}$$

$$M = \frac{M_y}{SF} = \frac{311.4}{3.00} = 103.8 \frac{\text{kN}}{4}$$

$$\frac{PL}{4} = M, \text{ or } P = \frac{4M}{L} = \frac{4(103.8)}{4.0} = 103.8 \text{ kN}$$

$$\lambda = \frac{a}{c} = \frac{24}{250} = 0.096$$

$$f(\lambda) = 1.02$$

$$\therefore \quad M_F(1500 \text{mm from end}) = \frac{2 t c^2 K_{IC}}{3\sqrt{a\pi}\; f(\lambda)} = \frac{2(100)(125)^2(29\sqrt{1000})}{3\sqrt{24\pi}\,(1.02}$$

$$= 107{,}900{,}000 = 1500\left(\frac{P_F}{2}\right)$$

or

$$P_F = 143.8 \text{ kN}$$

$$SF = \frac{P_F}{P} = \frac{143.8}{103.8} = 1.39$$

15.12 (a) Let $r = x - a$. Then, $x = r + a$. Substitution of $x = r + a$ into Eq. (15.22) yields for $r << a$,

$$\sigma_{yy} = \frac{\sigma(r+a)}{\sqrt{r^2 + 2ra + a^2 - a^2}} \approx \frac{\sigma a}{\sqrt{2ra}} = \frac{\sigma\sqrt{a}}{\sqrt{2r}} \qquad (a)$$

Multiplying the numerator and denominator of Eq. (a) by $\sqrt{\pi}$, we have

$$\sigma_{yy} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} = \frac{K_I}{\sqrt{2\pi r}} \qquad (b)$$

since $K_I = \sigma\sqrt{\pi a}$. Equation (a) is the first of Eqs. (15.23)

(b) For $r = 0.02a$, Eq. (b) yields

$$\sigma_{yy} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi(0.02a)}} = 5\sigma \qquad (c)$$

By Eq. (15.22), with $x = r + a = 1.02a$,

$$\sigma_{yy} = \frac{\sigma(1.02a)}{\sqrt{(1.02a)^2 - a^2}} = 5.0747\sigma \qquad (d)$$

By Eqs. (c) and (d), the percent error is $\frac{5.0747 - 5}{5.0745} \times 100 = 1.47\%$

For $r = 0.06a$, Eq. (b) yields

$$\sigma_{yy} = \frac{\sigma}{\sqrt{2(0.06)}} = 2.887\sigma \qquad (e)$$

For $x = r + a = 1.06a$, Eq. (15.22) yields

$$\sigma_{yy} = \frac{\sigma(1.06a)}{\sqrt{(1.06a)^2 - a^2}} = 3.015\sigma \qquad (f)$$

By Eqs. (e) and (f), the percent error is $\frac{3.015 - 2.887}{3.015} \times 100 = 4.245\%$

For $r = 0.10a$, Eq. (b) yields

$$\sigma_{yy} = \frac{\sigma}{\sqrt{2(0.10)}} = 2.236\sigma \qquad (g)$$

(Cont.)

15.12 cont. For $x = r + a = 1.1a$, Eq. (15.22) yields

$$\sigma_{yy} = \frac{\sigma(1.10a)}{\sqrt{(1.10a)^2 - a^2}} = 2.400\sigma \qquad (h)$$

By Eqs. (g) and (h), the percent error is $\frac{2.400 - 2.236}{2.400} \times 100 = 6.83\%$
This is about the largest permissible error; any larger error (larger value of r) would be unacceptable and Eq. (15.22) should be used

---

15.13 (a) By Table 15.2, Case 3, $a = 45\text{ mm}$, $c = 75\text{ mm}$, $\lambda = a/c = 0.6$. Hence, with $\sigma = 220\text{ MPa}$, $K_{IC} = \sigma\sqrt{\pi a}\, f(\lambda) \approx 220\sqrt{\pi(0.045)}\,(130)$, or $K_{IC} = 107.5\text{ MPa}\sqrt{m}$.

(b) For an infinitely wide plate (Table 15.2, Case 1), $K_I = \sigma\sqrt{\pi a}$. Hence, at fracture $K_I = K_{IC}$ and $\sigma = \frac{K_{IC}}{\pi a} = \frac{107.5}{\sqrt{\pi(0.045)}} = 286\text{ MPa}$.
For a plate 360 mm wide with a 90 mm central crack, Case 3, Table 15.2, $a = 45\text{ mm}$, $c = 90\text{ mm}$, $\lambda = a/c = 0.50$. Then, with $K_I = K_{IC}$ and $f(\lambda) = 1.19$,

$$\sigma = \frac{K_{IC}}{\sqrt{\pi a}\, f(\lambda)} = \frac{107.5}{\sqrt{\pi(0.045)}(1.19)} = 240.3\text{ MPa}.$$

15.14 By Fig. (15.2a) and Table 15.2, Case 6, $M = \frac{P}{2}(2W) = PW = 0.12P$ [N·m], $a = 0.060\text{ m}$, $2c = W = 120\text{ mm}$, $\lambda = \frac{a}{2c} = 0.50$. By Case 6, $f(\lambda) = 1.62$. Therefore,

$$\sigma = \frac{K_{IC}}{\sqrt{\pi a}\, f(\lambda)} = \frac{107.5}{\sqrt{\pi(0.060)}(1.62} = 152.8\text{ MPa} \qquad (a)$$

also, by Case 6, with $t = B = 60\text{ mm}$, $M = PW$, we have

$$\sigma = \frac{3M}{2BC^2} = \frac{3PW}{2BC^2} = \frac{3(0.120)P}{2(0.060)(0.060)^2}. \qquad (b)$$

Then, by Eqs. (a) and (b),

$$P = 183.4\text{ kN}$$

15.15 (a) From Problem 15.14, $K_{Ic} = 107.5\ \text{MPa}\sqrt{m}$. By Eq. (15.1), with $Y = 850$ MPa,

$$2.5\left(\frac{K_{Ic}}{Y}\right)^2 = 2.5\left(\frac{107.5}{850}\right) = 0.04\ m$$

Hence, $B = 0.06 > 0.04$. Therefore, a specimen thickness of $B = 60$ mm is sufficient to guarantee plain strain fracture.

(b) By Eq. (15.1), we must have

$$B = 0.06 \geq 2.5\left(\frac{K_{Ic}}{Y}\right)^2 = 2.5\left(\frac{107.5}{Y}\right)^2$$

or

$$Y \geq 693.9\ MPa.$$

The minimum required yield stress is $Y = 693.9$ MPa

15.16 (a) For $a = 5$ mm, $\lambda = \frac{a}{2c} = \frac{5}{50} = 0.100$. For tension loading (Case 4 of Table 15.2, $f(\lambda) = 1.25$. For beam loading (Case 6 of Table 15.2), $f_2(\lambda) = 1.02$. Hence,

$$\sigma f(\lambda) = \frac{P}{A} f_1(\lambda) + \frac{3M}{2tc^2} f_2(\lambda) = \frac{P}{25(50)}(1.25) + \frac{3(225P)}{2(25)(25)^2}(1.02) = 0.02303\ P$$

$K_{Ic} = \sigma f(\lambda)\sqrt{\pi a} = 59\sqrt{1000}$ ($K_{Ic} = 59\sqrt{1000}$ from Table 15.1.

$$P = \frac{59\sqrt{1000}}{0.02303\sqrt{5\pi}} = 20.4\ kN$$

(b) For $a = 10$ mm, $\lambda = \frac{a}{2c} = \frac{10}{50} = 0.20$. For tension loading (Case 4 of Table 15.2, $f_1(\lambda) = 1.37$. For beam loading (case 6 of Table 15.2, $f_2(\lambda) = 1.06$.

$$\sigma f(\lambda) = \frac{P}{25(50)}(1.37) + \frac{3(225P)}{2(25)(25)^2}(1.06) = 0.02399\ P$$

$$P = \frac{59\sqrt{1000}}{0.02399\sqrt{10\pi}} = 13.9\ kN.$$

15.17 (a) Note that the crack length a and width of beam t both satisfy Eq. (15.38); therefore, plane strain conditions are satisfied.

(b) $K_{Ic} = \frac{3M}{2tc^2}\sqrt{\pi a}\, f(\lambda) = 77\sqrt{1000}$; $M = \frac{PL}{4}$; Since $\lambda = \frac{a}{2c} = 0.10$, $f(\lambda) = 1.02$

$$P = \frac{8tc^2 K_{Ic}}{3L\sqrt{\pi a}\, f(\lambda)} = \frac{8(25)(75^2)(77\sqrt{1000})}{3(2000)(\sqrt{15\pi})(1.02)} = \underline{65.2 \text{ kN}}$$

15.18 (a) $a, t > 6.7$ mm Plane strain conditions are satisfied.

(b) $$P = \frac{8tc^2 K_{Ic}}{3L\sqrt{\pi a}\, f(\lambda)} = \frac{8(25)(75^2)(23\sqrt{1000})}{3(2000)(\sqrt{15\pi})(1.02)} = \underline{19.5 \text{ kN}}$$

15.19

$\sigma = \frac{pD}{2t} = \frac{6(1000)}{2(20)} = 150$ MPa; $K_{Ic} = \sigma\sqrt{\pi a}$; $a = \frac{K_{Ic}^2}{\pi\sigma^2} = \frac{26^2(1000)}{\pi(150^2)} = 9.56$ mm

$2a = 2(9.56) = \underline{19.1 \text{ mm}}$; $a, t > 6.7$; $\underline{\text{Eq. (15.38) is satisfied.}}$

15.20

$$\sigma = \frac{K_{Ic}}{\sqrt{\pi a}} = \frac{26\sqrt{1000}}{\sqrt{9.56\pi}} = 150 \text{ MPa} = \frac{pD}{4t}$$

$$p = \frac{150(4)(20)}{1000} = \underline{12.0 \text{ MPa}}$$

15.21

For $a = 28$ mm, $\lambda = \frac{a}{2c} = \frac{28}{300} = 0.0933$. For tension loading (Case 4 of Table 15.2), $f_1(\lambda) = 1.24$. For beam loading (Case 6 of Table 15.2), $f_2(\lambda) = 1.02$.

$$\sigma f(\lambda) = \frac{P}{A} f_1(\lambda) + \frac{3M}{2tc^2} f_2(\lambda) = \frac{P}{30(300)}(1.24) + \frac{3(150P)}{2(30)(150^2)}(1.02) = 0.000478P$$

$$K_{Ic} = \sigma f(\lambda)\sqrt{\pi a} = 93\sqrt{1000} = 0.000478P\sqrt{28\pi}$$

$$P = \frac{93\sqrt{1000}}{0.000478(\sqrt{28\pi})} = \underline{656 \text{ kN}}$$

15.22

$$\sigma f(\lambda) = \frac{P}{30(300)}(1.24) + \frac{3(122P)}{2(30)(150^2)}(1.02) = 0.000414P$$

$$P = \frac{93\sqrt{1000}}{0.000414(\sqrt{28\pi})} = \underline{757 \text{ kN}}$$

15.23

(a) To check whether or not $K_c = 55\ \text{MPa}\sqrt{m}$ is invalid for K_{IC}, we need to check the dimension B of the test specimen. Thus,

$B \overset{?}{\geq} 2.5\left[\frac{K_{IC}}{Y}\right]^2 = 2.5\left[\frac{55}{689}\right]^2 = 15.93$ mm. Thus, since $B = 12.7$ mm, K_c is not valid for K_{IC}.

(b) The maximum value of K_{IC} that can be measured is given by the condition

$$0.0127 \geq 2.5\left[\frac{K_{IC}}{689}\right]^2$$

or

$$K_{IC} \leq 49.1\ \text{MPa}\sqrt{m}\text{; or } (K_{IC})_{max} = 49.1\ \text{MPa}\sqrt{m}.$$

(c) The actual K_{IC} for the material is therefore greater than $49.1\ \text{MPa}\sqrt{m}$ and less than $55\ \text{MPa}\sqrt{m}$.

Hence,

$$49.1\ \text{MPa}\sqrt{m} < K_{IC\,(actual)} < 55\ \text{MPa}\sqrt{m}.$$

If we use an average value $\frac{1}{2}(49.1 + 55) \cong 52\ \text{MPa}\sqrt{m}$, we will be accurate to within 6%, since $\frac{52-49.1}{52} \times 100 = 5.58\%$ and $\frac{55-52}{52} \times 100 = 5.77\%$.

(d) If the range of K_{IC} is from 49.1 to 55, we need to assure that B is sufficiently large. Therefore,

$B \geq 2.5\left[\frac{55}{689}\right]^2 = 15.93$ mm.

16.1

$$\sigma_{max} = \frac{SF(P_{max})}{A} = \frac{2.20(16{,}000)(4)}{\pi D^2} = \frac{44{,}820}{D^2} \ (MPa)$$

$$\sigma_{min} = \frac{-10}{16}\sigma_{max} = -0.625\,\sigma_{max}$$

$$\sigma_{max} - \sigma_{min} = 2\sigma_a \ ; \ 1.625\,\sigma_{max} = 2\sigma_a \ ; \ \sigma_{max} = 1.231\,\sigma_a$$

$$\sigma_{max} = \sigma_m + \sigma_a \ ; \ 1.231\,\sigma_a = \sigma_m + \sigma_a \ ; \ \sigma_m = 0.231\,\sigma_a$$

$$\frac{\sigma_a}{\sigma_{am}} + \left(\frac{\sigma_m}{\sigma_u}\right)^2 = 1 \ ; \ \frac{\sigma_a}{350} + \left(\frac{0.231\,\sigma_a}{700}\right)^2 = 0 \ ; \ \sigma_a^2 + 26{,}236\,\sigma_a - 9{,}182{,}736 = 0$$

$\sigma_a = 345.4\,MPa$

$\sigma_{max} = 1.231\,\sigma_a = 1.231(345.4) = 425.2\,MPa$

Since σ_{max} is less than Y, the mode of failure is fatigue.

$$\sigma_{max} = 425.2 = \frac{44{,}820}{D^2}$$

$D = \underline{10.27\ mm}$

16.2

$$\sigma_a = \sigma_m \ ; \ \sigma_{max} = 2\sigma_a \ ; \ \frac{\sigma_a}{\sigma_{am}} + \left(\frac{\sigma_m}{\sigma_u}\right)^2 = 1 \ ; \ \sigma_a^2 + 1400\,\sigma_a - 490{,}000 = 0$$

$\sigma_a = 290\ MPa$; $\sigma_{max} = 2\sigma_a = 2(290) = 580\ MPa$

Since Y is less than σ_{max}, the mode of failure is general yielding.

$$\sigma_{max} = Y = 450 = \frac{44{,}820}{D^2}$$

$D = \underline{9.98\ mm}$

16.3

$$I = \frac{100(150^3)}{12} - \frac{80(110^3)}{12} = 19,250,000 \text{ mm}^4$$

$$\sigma_{min} = \frac{SF(M_{min})c}{I} = \frac{2.50(5,000,000)(75)}{19,250,000} = 48.7 \text{ MPa}$$

$$\sigma_{max} - \sigma_{min} = 2\sigma_a ; \quad \sigma_{max} = \sigma_m + \sigma_a = \sigma_{min} + 2\sigma_a$$

$$\sigma_m = \sigma_a + 48.7$$

$$\frac{\sigma_a}{\sigma_{am}} + \frac{\sigma_m}{\sigma_u} = 1 ; \quad \frac{\sigma_a}{90} + \frac{\sigma_a + 48.7}{200} = 1 ; \quad \sigma_a = 47.0 \text{ MPa}$$

$$\sigma_{max} = 2\sigma_a + \sigma_{min} = 2(47.0) + 48.7 = 142.7 \text{ MPa}$$

Since σ_{max} is less than σ_u, the mode of failure is fatigue.

$$\sigma_{max} = 142.7 = \frac{SF(M_{max})c}{I}$$

$$M_{max} = \frac{\sigma_{max} I}{SF(c)} = \frac{142.7(19,250,000)}{2.50(75)} = \underline{14.7 \text{ kN.m}}$$

16.4 Instability theory for thin-wall pressure vessels indicate that buckling is not a problem for an external pressure of 2.00 MPa.

$$\sigma_{min} = -\frac{2}{7}\sigma_{max} ; \quad 2\sigma_a = \sigma_{max} - \sigma_{min} = \frac{9}{7}\sigma_{max} ; \quad \sigma_{max} = \frac{14}{9}\sigma_a = \sigma_m + \sigma_a$$

$$\sigma_m = \frac{5}{9}\sigma_a$$

$$\frac{\sigma_a}{\sigma_{am}} + \left(\frac{\sigma_m}{\sigma_u}\right)^2 = 1 ; \quad \frac{\sigma_a}{190} + \left[\frac{5\sigma_a}{9(430)}\right]^2 = 1 ; \quad \sigma_a^2 + 3153\sigma_a - 599,100 = 0$$

$$\sigma_a = 179.8 \text{ MPa} ; \quad \sigma_{max} = \frac{14}{9}\sigma_a = \frac{14(179.8)}{9} = 279.7 \text{ MPa}$$

Since σ_{max} is less than Y, the mode of failure is fatigue.

$$\sigma_{max} = 279.7 = SF\frac{pD}{2h}$$

$$SF = \frac{2h\sigma_{max}}{pD} = \frac{2(8)(279.7)}{7(300)} = \underline{2.13}$$

16.5

$$\sigma = SF\frac{Mc}{I} = \frac{1.75(500)(80)(d)(64)}{2\pi d^4} = \frac{713,000}{d^3} \text{ (MPa)}$$

$$\tau = SF\frac{Tc}{J} = \frac{1.75(500)(55)(d)(32)}{2\pi d^4} = \frac{245,100}{d^3} \text{ (MPa)}$$

$$\tau_{oct(max)} = \frac{1}{3}\sqrt{2\sigma_{am}^2} = \frac{1}{3}\sqrt{2\sigma^2 + 6\tau^2} ; \quad \sigma_{am} = \sqrt{\sigma^2 + 3\tau^2}$$

$$380 = \frac{1}{d^3}\sqrt{713,000^2 + 3(245,100)^2}$$

$$d = \underline{13.0 \text{ mm}}$$

16.6

$$\sigma = SF\frac{Mc}{I} = SF\frac{500(80)(7.5)(64)}{\pi(15)^4} = 120.7\ SF\ (MPa)$$

$$\tau = SF\frac{Tc}{J} = SF\frac{500(55)(7.5)(32)}{\pi(15)^4} = 41.5\ SF\ (MPa)$$

$$\sigma_{max} - \sigma_{min} = \sigma_{max} = 2\sigma_a;\ \sigma_m = \sigma_a$$

$$\frac{\sigma_a}{\sigma_{am}} + \left(\frac{\sigma_m}{\sigma_u}\right)^2 = 1;\ \frac{\sigma_a}{160} + \left(\frac{\sigma_a}{430}\right)^2 = 1;\ \sigma_a^2 + 1155.6\sigma_a - 184{,}900 = 0$$

$$\sigma_a = 142.5\ MPa$$

$$\sigma_{max} = 2\sigma_a = 2(142.5) = 285.0\ MPa$$

Since σ_{max} is less than Y, the mode of failure is fatigue.

$$\tau_{oct(max)} = \frac{1}{3}\sqrt{2\sigma_{max}^2} = \frac{1}{3}\sqrt{2\sigma^2 + 6\tau^2};\ \sigma_{max} = \sqrt{\sigma^2 + 3\tau^2}$$

$$285.0 = SF\sqrt{120.7^2 + 3(41.5)^2}$$

$$SF = \underline{2.03}$$

16.7

$$\sigma = SF\frac{Mc}{I} = \frac{2.00(150)(2400)(d)(64)}{2\pi d^4} = \frac{7{,}330{,}000}{d^3}\ (MPa)$$

$$\tau = SF\frac{Tc}{J} = \frac{2.00(150)(3000)(d)(32)}{2\pi d^4} = \frac{4{,}580{,}000}{d^3}\ (MPa)$$

$$\tau_{oct(max)} = \frac{1}{3}\sqrt{2\sigma_{am}^2} = \frac{1}{3}\sqrt{2\sigma^2 + 6\tau^2};\ \sigma_{am} = \sqrt{\sigma^2 + 3\tau^2}$$

$$380 = \frac{1{,}000{,}000}{d^3}\sqrt{7.33^2 + 3(4.58)^2}$$

$$d = \underline{30.5\ mm}$$

16.8

$$\sigma = SF\frac{Mc}{I} = \frac{1.80(200P)(15)(64)}{\pi(30)^4} = 0.1358P\ (MPa)$$

$$\tau = SF\frac{Tc}{J} = \frac{1.80(150P)(15)(32)}{\pi(30)^4} = 0.0509P\ (MPa)$$

$$\sigma_{max} - \sigma_{min} = 1.6\sigma_{max} = 2\sigma_a;\ \sigma_{max} = 1.25\sigma_a = \sigma_m + \sigma_a;\ \sigma_m = 0.250\sigma_a$$

$$\frac{\sigma_a}{\sigma_{am}} + \left(\frac{\sigma_m}{\sigma_u}\right)^2 = 1;\ \frac{\sigma_a}{410} + \left(\frac{0.250\sigma_a}{810}\right)^2 = 1;\ \sigma_a^2 + 25{,}604\sigma_a - 10{,}498{,}000 = 0$$

$$\sigma_a = 403.6\ MPa;\ \sigma_{max} = 1.25\sigma_a = 1.25(403.6) = 504.5\ MPa$$

Since σ_{max} is less than Y, the mode of failure is fatigue.

$$\tau_{oct(max)} = \frac{1}{3}\sqrt{2\sigma_{max}^2} = \frac{1}{3}\sqrt{2\sigma^2 + 6\tau^2};\ \sigma_{max} = \sqrt{\sigma^2 + 3\tau^2}$$

$$504.5 = P\sqrt{0.1358^2 + 3(0.0509)^2}$$

$$P = \underline{3.12\ kN}$$

16.9

$$\sigma = SF\,\frac{Mc}{I} = \frac{1.75(50P)(13.0)(64)}{2\pi(13.0)^4} = 0.4057\,P\ (\text{MPa})$$

$$\tau = SF\,\frac{Tc}{J} = \frac{1.75(55P)(13.0)(32)}{2\pi(13.0)^4} = 0.2231\,P\ (\text{MPa})$$

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{0.4057\,P}{2} + P\sqrt{\left(\frac{0.4057}{2}\right)^2 + 0.2231^2} = 0.5044\,P$$

$$\sigma_2 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = -0.0987\,P$$

$$\sigma_{max} = 3\sigma_1 - \sigma_2 = 3(0.5044\,P) - (-0.0987\,P) = 1.6119\,P = 380$$

$$P = \underline{235.7\ \text{N}}$$

16.10 When the stress concentration is included, $\sigma_{max} = 1.6119P$ where P has a $SF = 1.75$. When the stress concentration is not included $\sigma_{max} = \sigma_1 = 0.5044P$.

$$S_{cc} = \frac{\sigma_{max}}{\sigma_1} = \frac{1.6119P}{0.5044P} = 3.196;\quad S_{ce} = q(S_{cc}-1)+1) = 0.9(3.196-1)+1 = 2.98$$

$$\sigma_{n(min)} = 0;\quad \sigma_{nm} = \sigma_{na};\quad \sigma_{n(max)} = \sigma_1 = 0.5044\,P$$

$$\sigma_{na} = \frac{\sigma_{n(max)}}{2} = 0.2522\,P;\quad \sigma_{nam} = \frac{\sigma_{am}}{S_{ce}} = \frac{380}{2.98} = 127.5\ \text{MPa}$$

$$\frac{\sigma_{na}}{\sigma_{nam}} + \left(\frac{\sigma_{na}}{\sigma_u}\right)^2 = 1;\quad \frac{\sigma_{na}}{127.5} + \left(\frac{\sigma_{na}}{830}\right)^2 = 1;\quad \sigma_{na}^2 + 5403.1\,\sigma_{na} - 688{,}900 = 0$$

$$\sigma_{na} = 124.6\ \text{MPa} = 0.2522\ P$$

$$P = \underline{494.1\ \text{N}}$$

16.11

$$S_{ce(M)} = 0.85(2.50-1)+1 = 2.28;\quad S_{ce(T)} = 0.85(2.00-1)+1 = 1.85$$

$$\sigma = S_{ce(M)}\,SF\,\frac{Mc}{I} = \frac{2.28(2.20)(200\,P)(15)(64)}{\pi(30)^4} = 0.3785\,P\ (\text{MPa})$$

$$\tau = S_{ce(T)}\,SF\,\frac{Tc}{J} = \frac{1.85(2.20)(180\,P(15)(32)}{\pi(30)^4} = 0.1382\,P\ (\text{MPa})$$

$$\tau_{max} = \frac{\sigma_{am}}{2} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

$$\frac{410}{2} = P\sqrt{\left(\frac{0.3785}{2}\right)^2 + 0.1382^2}$$

$$P = \underline{874.8\ \text{N}}$$

16.12 Nominal values of σ and τ are

$$\sigma_n = SF\frac{Mc}{I} = \frac{0.3785P}{2.28} = 0.1660P; \quad \tau_n = SF\frac{Tc}{J} = \frac{0.1382P}{1.85} = 0.0747P$$

$$\tau_{n(max)} = \sqrt{\left(\frac{\sigma_n}{2}\right)^2 + \tau_n^2} = P\sqrt{\left(\frac{0.1660}{2}\right)^2 + 0.0747^2} = 0.1117P$$

$$S_{ce} = \frac{0.2343P}{0.1117P} = 2.098; \quad \sigma_{am} = \frac{\sigma_{am}}{S_{ce}} = \frac{410}{2.098} = 195.4\text{ MPa.}$$

$$\sigma_{nm} = \sigma_{na}. \quad \frac{\sigma_{na}}{\sigma_{nam}} + \left(\frac{\sigma_{na}}{\sigma_u}\right)^2 = 1; \quad \frac{\sigma_{na}}{195.4} + \left(\frac{\sigma_{na}}{810}\right)^2 = 1 \text{ or}$$

$\sigma_{na}^2 + 3357.7\sigma_{na} - 656{,}100 = 0$. Hence, $\sigma_{na} = 185.2$ MPa.

$$\tau_{n(max)} = 0.1117P = \frac{\sigma_{na} + \sigma_{nm}}{2} = \frac{185.2 + 185.2}{2} = 185.2\text{ MPa}$$

$\therefore P = 1.66$ kN

16.13 With the given data and Fig. 14.16 of the text, $d = D - 2\rho = 60 - 10 = 50$ mm $\therefore \frac{\rho}{d} = \frac{5}{50} = 0.10$. By Fig. 14.16, $S_{cc} \approx 2.48$. Since $q = 0.85$, Eq. (14.43) yields $S_{ce} = 1 + q(S_{cc} - 1) = 1 + 0.95(2.48 - 1) = 2.406$. Since 2024-T4 Aluminum Alloy is a ductile material, the Gerber relation applies; that is

$$\frac{\sigma_{na}}{\sigma_{nam}} + \left(\frac{\sigma_{nm}}{\sigma_u}\right)^2 = 1 \qquad \text{(a)}$$

where by Fig. 16.4

$$\sigma_{n(min)} = \frac{P_{min}}{A} = \frac{30\text{ kN}}{(0.05)(0.02)[\text{m}^2]} = 30\text{ MPa} \qquad \text{(b)}$$

$$\sigma_{nm} = \sigma_{n(min)} + \sigma_{na} = 30 + \sigma_{na} \qquad \text{(c)}$$

$$\sigma_{n(max)} = \frac{P_{max}}{A} = \sigma_{n(min)} + 2\sigma_{na} \qquad \text{(d)}$$

$$\sigma_{nam} = \frac{\sigma_{am}}{S_{ce}} = \frac{190}{2.406} = 78.97\text{ MPa} \qquad \text{(e)}$$

Substitution of Eqs. (c) and (e) into Eq. (a) gives $\frac{\sigma_{na}}{78.97} + \left(\frac{30 + \sigma_{na}}{470}\right)^2 = 1$

or $\sigma_{na}^2 + 2857.3 - 220{,}000 = 0$. Therefore, $\sigma_{na} = 75.0$ MPa.

By Eq. (d), $\sigma_{n(max)} = 30 + 2(75.0) = 180$ MPa, and thus,

$$P_{max} = \sigma_{n(max)}A = 180(0.05)(0.02) = 180\text{ kN}$$

(Continued)

16.13 (Continued)

(b) Assuming that the stress concentration factor need be applied only to the alternating component of stress, we have (see the discussion in Example 16.4, following Eqs. (e) and (f))

$$\sigma_{max} = \sigma_{nm} + S_{ce}\sigma_{na} = 105 + (2.406)(75) = 285.4 \text{ MPa}$$

$$\sigma_{min} = \sigma_{nm} - S_{ce}\sigma_{na} = 105 - (2.406)(75) = -75.4 \text{ MPa}$$

16.14 Fig. P16-14 and Fig. 14.16, $D = d + 2\rho$, or $\rho = \frac{D-d}{2} = \frac{60-40}{2} = 10 \text{ mm}$

$\therefore \frac{\rho}{d} = \frac{10}{40} = 0.25$, and by Fig. 14.16, $S_{cc} \approx 1.9$.

(a) For static loading, the failure load occurs at yield. Therefore,

$$Y = S_{cc}\frac{P}{A} = \frac{1.9P}{(40)(40)}; \quad P = \frac{(350)(1600)}{1.9} = 294.7 \text{ kN}$$

(b) For completely reversible load,

$$\sigma_{am} = 280 \text{ MPa} = S_{cc}\frac{P}{A}; \quad P = \frac{(280)(1600)}{1.9} = 235.8 \text{ kN}$$

16.15 By Prob. 16.14, $S_{cc} = 1.9$. Since q is not given, assume $q = 1.0$. Then $S_{ce} = S_{cc} = 1.9$. Consider nominal stresses (see Fig. 16.4)

$$\sigma_{n(min)} = \frac{P_{min}}{A} = 0 \qquad (a)$$

$$\sigma_{nm} = \sigma_{n(min)} + \sigma_{na} = \sigma_{na} \qquad (b)$$

$$\sigma_{n(max)} = \sigma_{n(min)} + 2\sigma_{na} = 2\sigma_{na} \qquad (c)$$

$$\sigma_{nam} = \frac{\sigma_{am}}{S_{ce}} = \frac{280}{1.9} = 147.4 \text{ MPa} \qquad (d)$$

Since σ_u is not given, we use Soderberg's relation [Eq. (16.1)]

$$\frac{\sigma_{na}}{\sigma_{nam}} + \frac{\sigma_{nm}}{Y} = 1 \qquad (e)$$

Substitution of Eqs. (b) and (d) into Eq. (e), we obtain

$$\frac{\sigma_{na}}{147.4} + \frac{\sigma_{na}}{350} = 1 \quad \text{or} \quad \sigma_{na} = 103.7 \text{ MPa}.$$

(Continued)

16.15 (continued)

$$\therefore P_{max} = \sigma_{n(max)} A = 2(103.7)(1600) = 331.8 \text{ kN}$$

As noted in the discussion in Section 16.1, the Soderberg relation is conservative for ductile metals for predictions of σ_{na}. Thus, the actual failure load for the member is probably greater than 331.8 kN. See also the discussion in Example 16.4, following Eqs. (e) and (f).

16.16 As in Prob. 16.15, we define nominal stresses

$$\sigma_{n(min)} = \frac{P_{min}}{A} = \frac{-100}{1600} = -62.5 \text{ MPa} \quad (a)$$

$$\sigma_{nm} = \sigma_{n(min)} + \sigma_{na} = -62.5 + \sigma_{na} \quad (b)$$

$$\sigma_{n(max)} = \sigma_{n(min)} + 2\sigma_{na} = -62.5 + 2\sigma_{ma} \quad (c)$$

$$\sigma_{nam} = \frac{\sigma_{am}}{S_{ce}} = \frac{280}{1.9} = 147.4 \text{ MPa} \quad (d)$$

and by Soderberg's relation [Eq. (16.1)]

$$\frac{\sigma_{na}}{147.4} + \frac{(-62.5+\sigma_{na})}{350} = 1 \quad (e)$$

Hence,

$$\sigma_{na} = 122.2 \text{ MPa}$$

and

$$P_{max} = \sigma_{n(max)} A = (-62.5 + 2\sigma_{na})A$$

$$= (-62.5 + 2 \times 122.2)(1600)$$

or

$$P_{max} = 291 \text{ kN}$$

See also the discussion in Example 16.4, following Eqs. (e) and (f).

16.17 By Fig. P16.7, $I = \frac{1}{12}(40)(100)^3 = 3,333,333 \text{ mm}^4$

The moment at the fillet is $M = 500P$. Therefore, the nominal stress at the fillet is

$$\sigma_n = \frac{Mc}{I} = \frac{(500P)(50)}{3,333,333} = 0.0075P$$

By Fig. P16.7 and Fig. 14.24, $\frac{H}{h} = \frac{125}{100} = 1.25$ and $\frac{\rho}{h} = \frac{12.5}{100} = 0.125$. Therefore, by Fig. 14.24,

$S_{cc} = 1.58$, and $\sigma_{max} = \sigma_{am} = S_{cc}\sigma_n = (1.58)(0.0075P) = 0.01185P$.

With a safety factor $SF = 2.20$, the working stress is

$\sigma_w = \frac{\sigma_{max}}{SF} = \frac{220}{2.20} = 0.01185\,P_w$. Therefore, $P_w = 8.44$ kN.

The load $P_w = 8.44$ kN is the failure load based upon a safety factor of 2.20.

16.18 By Prob. 16.17, $S_{cc} = 1.58$. Since q is not specified, see Eq. (14.43), let $q = 1.0$. Then $S_{ce} = S_{cc}$. For a ductile material Gerber's relation is valid. Written in terms of nominal stresses, it is

$$\frac{\sigma_{na}}{\sigma_{nam}} + \left(\frac{\sigma_{nm}}{\sigma_u}\right)^2 = 1 \qquad (a)$$

where (Fig. 16.4)

$$\sigma_{n(min)} = \frac{P_{min}}{A} = 0 \qquad (b)$$

$$\sigma_{nm} = \sigma_{n(min)} + \sigma_{na} = \sigma_{na} \qquad (c)$$

$$\sigma_{n(max)} = \sigma_{n(min)} + 2\sigma_{na} = 2\sigma_{na} \qquad (d)$$

$$\sigma_{nam} = \frac{\sigma_{am}}{S_{ce}} = \frac{220}{1.58} = 139.2\text{ MPa} \qquad (e)$$

$$\sigma_u = 590\text{ MPa} \qquad (f)$$

(continued)

16.18 (Continued)

Substitution of Eqs. (c), (e), and (f) into Eq. (a) yields

$$\frac{\sigma_{na}}{139.2} + \left(\frac{\sigma_{na}}{590}\right)^2 = 1 \quad \text{or} \quad \sigma_{na}^2 + 2500.7\sigma_{na} - 348{,}100 = 0$$

Therefore, $\sigma_{na} = 132.2$ MPa, and $\sigma_{n(max)} = 2\sigma_{na} = 264.4$ MPa

On the basis of a safety factor $SF = 2.20$,

$$\sigma_{n(max)} = 264.4 = 0.0075(2.20P) \quad \text{or} \quad P = 16.0 \text{ kN}.$$

See also the discussion in Example 16.4, following Eqs. (e) and (f).

16.19 By Fig. P16.19, $I = \frac{1}{12}(60)(200)^3 = 40{,}000{,}000 \text{ mm}^4$ at the notch. The moment at the motch is $M = 1600P$. Therefore, the nominal stress at the notch is $\sigma_n = \frac{Mc}{I} = \frac{(1600P)(100)}{40{,}000{,}000} = 0.004P$.

By Figures P16.19 and 14.25, with the given data,

$$\frac{H}{h} = \frac{250}{200} = 1.25 \quad \text{and} \quad \frac{\rho}{h} = \frac{25}{200} = 0.125.$$

Therefore, by Fig. 14.25, $S_{cc} = 1.8$, and $\sigma_{max} = 1.8\sigma_n = (1.8)(0.004P)$.

With a safety factor $SF = 1.8$, the load P is given by

$$\sigma_{max} = \sigma_{am} = 170 \text{ MPa} = (SF)(0.0072P) = (1.8)(0.0072P)$$

or

$$P = 13.12 \text{ kN}$$

16.20 By Prob. 16.19 $S_{cc} = 1.8$ or with $q = 1.0$, $S_{ce} = 1.8$. Also, for ductile material, Gerber's relation is valid. Hence, in terms of nominal stresses

$$\frac{\sigma_{na}}{\sigma_{nam}} + \left(\frac{\sigma_{nm}}{\sigma_u}\right)^2 = 1 \qquad (a)$$

where (Fig. 16.4), the nominal stresses are

$$\sigma_{n(min)} = \frac{P_{min}}{A} = 0 \qquad (b)$$

$$\sigma_{nm} = \sigma_{n(min)} + \sigma_{na} = \sigma_{na} \qquad (c)$$

$$\sigma_{n(max)} = \sigma_{n(min)} + 2\sigma_{na} = 2\sigma_{na} \qquad (d)$$

$$\sigma_{nam} = \frac{\sigma_{am}}{S_{ce}} = \frac{170}{1.8} = 94.44 \text{ MPa} \qquad (e)$$

$$\sigma_u = 470 \text{ MPa} \qquad (f)$$

where S_{ce} is applied to σ_{am} only (See Example 16.4). Substitution of Eqs. (c), (e) and (f) into Eq. (a) yields

$$\left(\frac{\sigma_{na}}{94.44}\right) + \left(\frac{\sigma_{na}}{470}\right)^2 = 1 \quad \text{or} \quad \sigma_{na}^2 + 2339.0\sigma_m - 220900 = 0$$

Therefore, $\sigma_{na} = 91.9$ MPa, and with a safety factor $SF = 1.8$, the load P is given by

$$\sigma_{max} = 2\sigma_{na} = 2(91.9) = (0.004)(SF)P = 0.004(1.8)P$$

$$= 0.0072P$$

or $P = 25.53$ kN

See also the discussion in Example 16.4, following Eqs. (e) and (f).

16.21 By Fig. P16.21, the cross-sectional area at the fillet is $A = (60)(60) = 3600\ mm^2$. By the data given in Prob. 16.19, $E = 72$ GPa, $\nu = 0.33$, $\sigma_u = 470$ MPa, and $Y = 330$ MPa. also, $\sigma_{am} = 220$ MPa and the safety factor is $SF = 2.20$ By Figs. P16.21 and 14.16, $t = 3\rho = 30$, $\rho = 10$ mm, and $\rho/d = 10/60 = 0.167$. Therefore, by Fig. 14.16, $S_{cc} \approx 1.91$. The nominal stress at the fillet is

$$\sigma_n = \frac{P}{A} = \frac{P}{3600} = 2.778 \times 10^{-4} P$$

and $\sigma_{am} = \sigma_{max} = S_{cc}\sigma_n = 1.91 \times 2.7778 \times 10^{-4} P = 5.306 \times 10^{-4} P$

therefore $P = \dfrac{\sigma_{am}}{5.306 \times 10^{-4}} = \dfrac{220}{5.306 \times 10^{-4}} = 414.6\ kN$

The design load is

$$P_{design} = \frac{P}{SF} = \frac{414.6}{2.20} = 188.5\ kN$$

16.22 By Prob. 16.21, $S_{cc} = 1.91$, $A = 3600\ mm^2$, $\sigma_{am} = 220$ MPa, $SF = 2.20$, and $\sigma_u = 470$ MPa. Let $q = 1$ [Eq. (14.43)]. Then $S_{cc} = S_{ce}$. For a ductile metal, Gerber's relation is applicable. In terms of nominal stresses, it is

$$\left(\frac{\sigma_{na}}{\sigma_{nam}}\right) + \left(\frac{\sigma_{nm}}{\sigma_u}\right)^2 = 1 \qquad (a)$$

where (Fig. 16.4) the nominal stresses are

(continued)

16.22 (continued)

$$\sigma_{n(min)} = \frac{P_{min}}{A} = 0 \qquad \text{(b)}$$

$$\sigma_{nm} = \sigma_{n(min)} + \sigma_{na} = \sigma_{na} \qquad \text{(c)}$$

$$\sigma_{n(max)} = \sigma_{n(min)} + 2\sigma_{na} = 2\sigma_{na} \qquad \text{(d)}$$

$$\sigma_{nam} = \frac{\sigma_{am}}{S_{ce}} = \frac{220}{1.91} = 115.2 \text{ MPa} \qquad \text{(e)}$$

where S_{ce} is applied to σ_{am} only (see Example 16.4) and

$$\sigma_u = 470 \text{ MPa} \qquad \text{(f)}$$

Substitution of Eqs. (c), (e), and (f) into Eq. (a) yields

$$\left(\frac{\sigma_{na}}{115.2}\right) + \left(\frac{\sigma_{na}}{470}\right)^2 = 1 \text{ or } \sigma_{na}^2 + 1917.5\sigma_{na} - 220,900 = 0$$

$\therefore$ $\sigma_{na} = 109$ MPa, and $\sigma_{n(max)} = 2\sigma_{na} = 218$ MPa

The load P is given by $P = 2\sigma_{na} A = (218)(3600) = 784.8$ kN, and the design load is

$$P_{design} = \frac{P}{SF} = \frac{784.8}{2.20} = 356.7 \text{ kN}$$

16.23 Given $S = P/A_0$, $e = \Delta l_0/l_0$, $\sigma = P/A$ and

$$\epsilon = \int_{l_0}^{l} \frac{dl}{l} \; ; \quad l = l_0 + \Delta l_0 .$$

(a) With $Al = A_0 l_0$, we find

$$\sigma = \frac{P}{A} = \frac{Pl}{A_0 l_0} = S\left(\frac{l_0 + \Delta l_0}{l_0}\right) = S\left(1 + \frac{\Delta l_0}{l_0}\right) = S(1+e) \quad \text{(a)}$$

and

$$\epsilon = \int_{l_0}^{l = l_0 + \Delta l_0} \frac{dl}{l} = \ln l \Big|_{l_0}^{l_0 + \Delta l_0} = \ln\left(\frac{l_0 + \Delta l_0}{l_0}\right) = \ln(1+e) \quad \text{(b)}$$

(b) By Eq(a) the differences between σ and S for engineering strains $e = 1\%$, 2%, 4%, and 10% are tabulated in Table a. Similarly, by Eq.(b), the differences between true strain ϵ and engineering strain e are tabulated in Table b.

Table a

e	σ	$\sigma - S$
0.01	1.01 S	0.01 S
0.02	1.02 S	0.02 S
0.04	1.04 S	0.04 S
0.10	1.10 S	0.10 S

Table b

e	ϵ	$e - \epsilon$
0.01	0.00995	5×10^{-5}
0.02	0.01980	2×10^{-4}
0.04	0.03922	7.8×10^{-4}
0.10	0.09531	4.69×10^{-3}

(c) For 4340 annealed steel, we have by Eq. 16.7

$$\epsilon_t = \frac{\sigma_f'}{E}(2N)^b + \epsilon_f'(2N)^c \qquad \text{(c)}$$

where σ_f' and ϵ_f' are true stress and true strain, respectively. To estimate the effect of using engineering stress and strain, consider the case where e (engineering strain) is 0.10 (Tables a and b). Then,

$$\frac{S_f'}{1.0} = \frac{\sigma_f'}{1.1} \quad \text{and} \quad \frac{e_f'}{0.10} = \frac{\epsilon_f'}{0.09531} \; ; \; e_f' = \frac{\epsilon_f'}{0.9531}$$

(cont.)

16.23 Hence, in terms of engineering stress and strain, the coefficients in Eq. (a) of Example 16.5 must be divided by 1.1 and 0.9531, respectively. Thus,

$$\epsilon_t = \frac{0.58}{1.1}(2N)^{-0.57} + \frac{0.0062}{0.9531}(2N)^{-0.09}$$

or

$$\epsilon_t = 0.527(2N)^{-0.57} + 0.0065(2N)^{-0.09} \qquad \text{(d)}$$

Hence, the cross over frequency is obtained from Eq. (d), namely

$$0.527(2N_c)^{-0.57} = 0.0065(2N_c)^{-0.09}$$

or

$$(2N_c)^{0.48} = 81.077\,;\quad N_c = 4741 \text{ cycles}$$

This result compares to $N_c = 6390$ cycles obtained using true stress and true strain in Example 16.5.

For $2N = 10^6$ cycles, Eq. (d) yields $\epsilon_t = 0.00207$, compared to $\epsilon_t = 0.00201$ in Example 16.5. Also, for $\epsilon_t = 0.01$, Eq. (d) yields $N = 1049$ cycles compared to $N = 1184$ cycles in Example 16.5. Thus, for $e = 0.10$, Eq. (d) gives a good estimate of ϵ_t, a reasonable estimate for N at 10^6 cycles, but greatly underestimates N_C. Hence, the use of engineering stress and strain in Eq. (16.7) is questionable for engineering strains of magnitude 0.10.

Similarly, for $e = 0.04$, we find $N_c = 5650$ cycles, $\epsilon_t = 0.00204$ (for $2N = 10^6$) and $N = 1126$ cycles (for $\epsilon_t = 0.01$), compared to $N_c = 6390$ cycles, $\epsilon_t = 0.00201$, and $N = 1184$ cycles, still somewhat inaccurate.

(cont.)

16.23 cont. For $e = 0.02$, we find by Eq. (d),
$N_c = 6010$ cycles, $\epsilon_t = 0.00202$ (for $2N = 10^6$), and
$N = 1156$ cycles (for $\epsilon_t = 0.01$), which, given the uncertainty in fatigue data, is probably acceptable. Hence, the use of engineering stress and strain in Eq. (16.7) is reasonable for engineering strain $e \leq 0.02$ (2%).

16.24 For 1040 steel, as forged, $\sigma_f' = 1500$ MPa, $\epsilon_f' = 0.61$, $b = -0.14$, $C = -0.57$, and $E = 200$ GPa. Then, by Eq. (16.7),

$$\epsilon_t = \epsilon_p + \epsilon_e = 0.61(2N)^{-0.57} + 0.0075(2N)^{-0.14} \qquad (a)$$

(a) For $N = 10^3$, Eq. (a) yields $\epsilon_t = 0.01060$; $\epsilon_p = 0.00801$; $\epsilon_e = 0.002588$

For $N = 10^5$, Eq. (a) yields $\epsilon_t = 0.001938$; $\epsilon_p = 0.0005804$; $\epsilon_e = 0.001358$

For $N = 10^7$, Eq. (a) yields $\epsilon_t = 0.0007548$, $\epsilon_p = 0.00004205$, $\epsilon_e = 0.0007127$

Note that as N increases the elastic strain ϵ_e becomes a larger percentage of the total strain ϵ_t.

(b) At the cross-over cycles, $N = N_c$, $\epsilon_p = \epsilon_e$. Hence, by Eq. (a),

$$0.61(2N_c)^{-0.57} = 0.0075(2N_c)^{-0.14}$$

or

$$(2N_c)^{0.43} = 81.33\cdots; \quad N_c = 13{,}850 \text{ cycles}$$

Therefore, the total strain amplitude at $N = N_c$ is

$$\epsilon_{tc} = 0.61(27{,}700)^{-0.57} + 0.0075(27{,}700)^{-0.14} = 0.003582$$

This value of ϵ_{tc} lies between the values of ϵ_t for $N = 10^3$ and $N = 10^5$, as it should.

(c) For $\epsilon_t = 0.01$, Eq. (a) yields

$$0.01 = 0.61(2N)^{-0.57} + 0.0075(2N)^{-0.14} \qquad (b)$$

(cont.)

16.24 cont. The solution of Eq. (b) is $N = 1134$ cycles. Note that since $\epsilon_t = 0.01$ is smaller than $\epsilon_t = 0.0106$ for $N = 1000$, $N = 1134$ is larger than 1000 for $\epsilon_t = 0.01$ (see Fig. 16.10).

16.25 In fatigue tests, $N = 10^3, 10^5$, and 10^7 cycles. The time of testing is required for frequencies of

1 Hz, 35 Hz, and 160 Hz for $N = 10^3, 10^5$, and 10^7 cycles.

(a) Since 1 Hz = 1 cycle per second, for $N = 1000$ cycles, the time required for a frequency of 1 Hz is

$$t_{10^3} = \frac{1000}{1} = 1000\ \text{s} = 16.6\bar{6}\ \text{hours} = 0.694\ \text{days}$$

$$t_{10^5} = \frac{100{,}000}{1} = 100{,}000\ \text{s} = 1666.\bar{6}\ \text{hours} = 69.44\ \text{days}$$

$$t_{10^7} = \frac{10^7}{1} = 10^7\ \text{s} = 1.6\bar{6} \times 10^5\ \text{hours} = 6944.\bar{4}\ \text{days} \approx 19\ \text{years}$$

(b) For 35 Hz frequency,

$$t_{10^3} = \frac{1000}{35} = 28.57\ \text{s} = 0.476\ \text{hours} \approx 0.02\ \text{day}$$

$$t_{10^5} = \frac{100{,}000}{35} = 2857\ \text{s} = 47.62\ \text{hours} \approx 1.98\ \text{days}$$

$$t_{10^7} = \frac{10^7}{35} = 285714\ \text{s} = 4762\ \text{hours} \approx 198\ \text{days}$$

(c) For 160 Hz frequency,

$$t_{10^3} = \frac{1000}{65} = 6.25\ \text{s} = 0.104\ \text{hours} \approx 0.0043\ \text{day}$$

$$t_{10^5} = \frac{100{,}000}{65} = 625\ \text{s} = 10.4\ \text{hours} \approx 0.43\ \text{day}$$

$$t_{10^7} = \frac{10^7}{65} = 62500\ \text{s} = 1042\ \text{hours} \approx 43.4\ \text{days}$$

16.26 The Manson universal slope equation is

$$\epsilon_t = 1.75\left(\frac{S_u}{E}\right)N^{-0.12} + 0.5\left(\epsilon_f^{0.6}\right)N^{-0.6} \qquad (a)$$

where for 4340 annealed steel ($S_u = 827$ MPa, $E = 200$ GPa, and $\epsilon_f = 0.57$).

(a) Reworking Example 16.5 using Eq. (a) for 4340 annealed steel we have

$$\epsilon_p = 0.5\epsilon_f^{0.6}N^{-0.6} \text{ and } \epsilon_e = 1.75\left(\frac{S_u}{E}\right)N^{-0.12}$$

and by $\epsilon_p = \epsilon_e$, we obtain $N_C = 3365$ cycles. Then, by Eq. (a), with $N = N_C = 3365$,

$$\epsilon_{tc} = 0.007236(3365)^{-0.12} + 0.356858(3365)^{-0.6} = 0.00546$$

(b) For $2N = 10^6$ ($N = 5\times10^5$), Eq. (a) yields

$$\epsilon_t = 0.007236(5\times10^5)^{-0.12} + 0.356858(5\times10^5)^{-0.6} = 0.00163$$

(c) For $\epsilon_t = 0.01$, Eq. (a) yields

$$0.01 = 0.007236\,N^{-0.12} + 0.356858\,N^{-0.6}; \quad N = 748 \text{ cycles}$$

Comparing the above results with those of Example 16.5, we see that the values of ϵ_{tc} agree fairly well; however, the values of N_C differ by a factor of approximately 2. The value of ϵ_t predicted by Eq. (a) for $2N = 10^6$ is approximately 19% less than that predicted by Example 16.5. Likewise, the life cycles predicted by Eq. (a), for $\epsilon_t = 0.01$, is less, 748 cycles compared to 1184 cycles. Therefore, it appears that Eq. (a) is more conservative in predicting fatigue life; that is, it predicts fewer cycles to fracture.

16.27 The Manson universal slope equation is

$$\epsilon_t = 1.75\left(\frac{S_u}{E}\right)N^{-0.12} + 0.5\,\epsilon_f^{0.6}N^{-0.6} \qquad \text{(a)}$$

Rework Problem 16.24 using Eq.(a). For 1040 steel as forged $S_u = 621$ MPa, $E = 200$ GPa, and $\epsilon_f = 0.93$.

(a) The plastic, elastic, and total strain amplitudes for life to $N = 10^3$, 10^5, and 10^7 cycles are calculated using Eq.(a) as follows:

For $N = 10^3$,

$$\epsilon_p = 0.5\,\epsilon_f^{0.6}N^{-0.6} = 0.007587$$

$$\epsilon_e = 1.75\left(\frac{S_u}{E}\right)N^{-0.12} = 0.002372$$

$$\epsilon_t = \epsilon_p + \epsilon_e = 0.009959$$

For $N = 10^5$,

$$\epsilon_p = 0.0004787, \quad \epsilon_e = 0.007854, \quad \epsilon_t = 0.0008156$$

Compare the above results to those obtained in Problem 16.24

(b) At the cross-over cycles $N = N_c$, $\epsilon_p = \epsilon_e$. Therefore, by Eq.(a),

$$N_c = 11,272 \text{ Cycles}. \qquad \text{(b)}$$

Then, by Eqs. (a) and (b), the total strain amplitude at $N = N_c$ is $\epsilon_{tc} = 0.003547$. This value lies between the values of ϵ_t for $N = 10^3$ and $N = 10^5$, as it should.

(c) For $\epsilon_t = 0.01$, Eq.(a) yields $N = 992$ cycles ≈ 1000 cycles.

The above results are somewhat comparable to those obtained in Problem 16.4.

17.1 $R_1 = \infty;\ R_1' = 440\ \text{mm};\ R_2 = 330\ \text{mm};\ R_2' = \infty;\ \alpha = 0$

$$B = \frac{1}{4}\left(\frac{1}{R_2} + \frac{1}{R_1'}\right) + \frac{1}{4}\sqrt{\left(-\frac{1}{R_1'} + \frac{1}{R_2}\right)^2} = \frac{1}{2R_2} = \frac{1}{2(330)} = 0.001515\ \text{mm}^{-1}$$

$$A = \frac{1}{4}\left(\frac{1}{R_2} + \frac{1}{R_1'}\right) - \frac{1}{4}\sqrt{\left(-\frac{1}{R_1'} + \frac{1}{R_2}\right)^2} = \frac{1}{2R_1'} = \frac{1}{2(440)} = 0.001136\ \text{mm}^{-1}$$

$$\frac{B}{A} = \frac{0.001515}{0.001136} = 1.333$$

$$\Delta = \frac{1}{A+B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{(0.001136+0.001515)(200{,}000)} = 0.003454\ \text{mm}^3/\text{N}$$

From Fig. 17.10, $C_b = k = 0.84$, $C_\sigma = 0.69$, $C_\tau = 0.23$, and $C_G = 0.21$.

$$b = C_b\sqrt[3]{P\Delta} = 0.84\sqrt[3]{110{,}000(0.003454)} = 6.08\ \text{mm};\quad a = \frac{b}{k} = \frac{6.08}{0.84} = 7.24\ \text{mm}$$

$$e = \frac{a}{b} = \frac{7.24}{6.08} = 1.19;\qquad \frac{b}{\Delta} = \frac{6.08}{0.003454} = 1760\ \text{MPa}$$

$$\sigma_{max} = -C_\sigma\frac{b}{\Delta} = -0.69(1760) = \underline{-1215\ \text{MPa}}$$

$$\tau_{max} = C_\tau\frac{b}{\Delta} = 0.23(1760) = \underline{405\ \text{MPa}}$$

$$\tau_{oct} = C_G\frac{b}{\Delta} = 0.21(1760) = \underline{370\ \text{MPa}}$$

For $e = 1.19$, Fig. 17.8 gives $\frac{2\tau_0\Delta}{a} = 0.23$. $2\tau_0 = \frac{0.23(7.24)}{0.003454} = \underline{482\ \text{MPa}}$

At yield, $\tau_{max} = \frac{Y}{2} = 440\ \text{MPa}$ and $\frac{b}{\Delta} = \frac{\tau_{max}}{C_\tau} = \frac{440}{0.23} = 1913\ \text{MPa}$

$$b = 1913(0.003454) = 6.61\ \text{mm} = C_b\sqrt[3]{SF\,P\Delta} = 0.84\sqrt[3]{SF(110{,}000)(0.003454)}$$

$$SF = \left(\frac{6.61}{0.84}\right)^3\frac{1}{110{,}000(0.003454)} = \underline{1.28}$$

17.2 For $B/A = 1.333$, Fig. 17.10 gives $C_\delta = 2.2$

$$\delta = C_\delta\,\frac{P}{\pi}\,\frac{A+B}{b/\Delta} = \frac{2.2(110{,}000)(0.001136+0.001515)}{\pi(1760)} = \underline{0.116\ \text{mm}}$$

17.3 $B = A = 1/R = 1/100 = 0.010\ \text{mm}^{-1};\ B/A = 1.00;\ C_b = 0.90$

$$\Delta = \frac{1}{A+B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{2(0.010)(200{,}000)} = 0.000458\ \text{mm}^3/\text{N};\quad b = C_b\sqrt[3]{P\Delta}$$

$$\frac{b}{\Delta} = \frac{0.90\sqrt[3]{P(0.000458)}}{0.000458} = 151\sqrt[3]{P};\quad C_\sigma = 0.64;\ C_\tau = 0.21;\ C_G = 0.19$$

$$\sigma_{max} = -C_\sigma\frac{b}{\Delta} = -0.64(151)\sqrt[3]{P} = \underline{-97\sqrt[3]{P}\ \text{MPa}}$$

$$\tau_{max} = C_\tau\frac{b}{\Delta} = 0.21(151)\sqrt[3]{P} = \underline{32\sqrt[3]{P}\ \text{MPa}}$$

$$\tau_{oct(max)} = C_G\frac{b}{\Delta} = 0.19(151)\sqrt[3]{P} = \underline{29\sqrt[3]{P}\ \text{MPa}}$$

17.4 $B = A = 1/2R = 1/2(100) = 0.005 \text{ mm}^{-1}; \; B/A = 1.00; \; C_b = 0.90$

$$\Delta = \frac{1}{A+B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{2(0.005)(200{,}000)} = 0.000916 \text{ mm}^3/\text{N}; \quad b = C_b\sqrt[3]{P\Delta}$$

$$\frac{b}{\Delta} = \frac{0.90\sqrt[3]{P(0.000916)}}{0.000916} = 95.4\sqrt[3]{P}\text{ MPa}; \quad C_\sigma = 0.64; \quad C_\tau = 0.21; \quad C_G = 0.19$$

$$\sigma_{max} = -C_\sigma\frac{b}{\Delta} = -0.64(95.4)\sqrt[3]{P} = \underline{-61\sqrt[3]{P}\text{ MPa}}$$

$$\tau_{max} = C_\tau\frac{b}{\Delta} = 0.21(95.4)\sqrt[3]{P} = \underline{20\sqrt[3]{P}\text{ MPa}}$$

$$\tau_{oct(max)} = C_G\frac{b}{\Delta} = 0.19(95.4)\sqrt[3]{P} = \underline{18\sqrt[3]{P}\text{ MPa}}$$

17.5 $B = A = \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{2}\left(\frac{1}{100} - \frac{1}{200}\right) = 0.0025 \text{ mm}^{-1}; \; \frac{B}{A} = 1.00; \; C_b = 0.90$

$$\Delta = \frac{1}{A+B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{2(0.0025)(200{,}000)} = 0.001832 \text{ mm}^3/\text{N}; \quad b = C_b\sqrt[3]{P\Delta}$$

$$\frac{b}{\Delta} = \frac{0.90\sqrt[3]{P(0.001832)}}{0.001832} = 60.1\sqrt[3]{P}\text{ MPa}; \quad C_\sigma = 0.64; \quad C_\tau = 0.21; \quad C_G = 0.19$$

$$\sigma_{max} = -C_\sigma\frac{b}{\Delta} = -0.64(60.1)\sqrt[3]{P} = \underline{-38.5\sqrt[3]{P}\text{ MPa}}$$

$$\tau_{max} = C_\tau\frac{b}{\Delta} = 0.21(60.1)\sqrt[3]{P} = \underline{12.6\sqrt[3]{P}\text{ MPa}}$$

$$\tau_{oct(max)} = C_G\frac{b}{\Delta} = 0.19(60.1)\sqrt[3]{P} = \underline{11.4\sqrt[3]{P}\text{ MPa}}$$

17.6 Let $R_1' = R_2' = \infty; \; R_2 = 100 \text{ mm}; \; R_1 = 30 \text{ mm}; \; \alpha = \frac{\pi}{2}$

$$B = \frac{1}{2R_1} = \frac{1}{2(30)} = 0.01667 \text{ mm}^{-1}; \quad A = \frac{1}{2R_2} = \frac{1}{2(100)} = 0.0050 \text{ mm}^{-1}; \quad \frac{B}{A} = \frac{0.01667}{0.0050} = 3.33$$

$$\Delta = \frac{1}{A+B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{(0.00500+0.01667)(200{,}000)} = 0.0004227 \text{ mm}^3/\text{N}; \quad C_b = 0.63$$

$$b = C_b\sqrt[3]{P\Delta} = 0.63\sqrt[3]{4500(0.0004227)} = 0.781 \text{ mm}$$

$$\frac{b}{\Delta} = \frac{0.781}{0.0004227} = 1848 \text{ MPa}; \quad C_\sigma = 0.86; \quad C_\tau = 0.28; \quad C_{zs} = 0.66$$

$$\sigma_{max} = -C_\sigma\frac{b}{\Delta} = -0.86(1848) = \underline{-1589 \text{ MPa}}$$

$$\tau_{max} = C_\tau\frac{b}{\Delta} = 0.28(1848) = \underline{517 \text{ MPa}}$$

$$z_s = C_{zs}\,b = 0.66(0.781) = \underline{0.515 \text{ mm}}$$

17.7 Let $R_1' = R_2' = \infty$, $R_1 = 30$ mm, $R_2 = 100$, and $\alpha = \frac{\pi}{12}$

$$B = \frac{1}{4}\left(\frac{1}{30} + \frac{1}{100}\right) + \frac{1}{4}\sqrt{\left(\frac{1}{30} + \frac{1}{100}\right)^2 - 4\left(\frac{1}{30}\right)\left(\frac{1}{100}\right)\sin^2\frac{\pi}{12}} = 0.021406 \text{ mm}^{-1}$$

$$A = \frac{1}{4}\left(\frac{1}{30} + \frac{1}{100}\right) - \frac{1}{4}\sqrt{\left(\frac{1}{30} + \frac{1}{100}\right)^2 - 4\left(\frac{1}{30}\right)\left(\frac{1}{100}\right)\sin^2\frac{\pi}{12}} = 0.000261 \text{ mm}^{-1}$$

$$\frac{B}{A} = \frac{0.021406}{0.000261} = 82.0; \quad C_b = 0.305; \quad C_\sigma = 1.00; \quad C_\tau = 0.30; \quad C_{zs} = 0.80$$

$$\Delta = \frac{1}{A+B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{(0.021406+0.000261)(200{,}000)} = 0.000423 \text{ mm}^3/\text{N}$$

$$b = C_b\sqrt[3]{P\Delta} = 0.305\sqrt[3]{4500(0.000423)} = 0.3780 \text{ mm}; \quad \frac{b}{\Delta} = \frac{0.}{0.000423} = 894 \text{ MPa}$$

$$\sigma_{max} = -C_\sigma\frac{b}{\Delta} = -1.00(894) = \underline{-894 \text{ MPa}}$$

$$\tau_{max} = C_\tau\frac{b}{\Delta} = 0.30(894) = \underline{268 \text{ MPa}}$$

$$z_s = C_{zs}\, b = 0.80(0.3780) = \underline{0.302 \text{ mm}}$$

17.8 $R_1 = 5$ mm; $R_1' = \infty$; $R_2 = R_2' = 4000$ mm; $\alpha = 0$

$$B = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\left(\frac{1}{5} + \frac{1}{4000}\right) = 0.100125 \text{ mm}^{-1}$$

$$A = \frac{1}{2R_2} = \frac{1}{2(4000)} = 0.000125 \text{ mm}^{-1}; \quad \frac{B}{A} = \frac{0.100125}{0.000125} = 801; \quad C_b = 0.20; \quad C_\sigma = 1.00$$

$$\Delta = \frac{1}{A+B}\left[\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right] = \frac{1}{0.000125+0.100125}\left[\frac{1-0.20^2}{117{,}000} + \frac{1-0.29^2}{200{,}000}\right] = 0.0001275 \text{ mm}^3/\text{N}$$

$$\sigma_{max} = -1400 = -C_\sigma\frac{b}{\Delta} = -1.00\frac{b}{\Delta}$$

$$b = \frac{1400\Delta}{1.00} = 1400(0.000125) = 0.1785 \text{ mm} = C_b\sqrt[3]{P\Delta} \approx 0.20\sqrt[3]{P(0.0001275)}$$

$$P = \underline{5.58 \text{ kN}}$$

17.9

$$B = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{R_1} = \frac{1}{32} = 0.03125 \text{ mm}^{-1}; \quad A = \frac{1}{2R_2'} = \frac{1}{2(20)} = 0.02500 \text{ mm}^{-1}$$

$$\frac{B}{A} = \frac{0.03125}{0.02500} = 1.25; \quad C_b = 0.83; \quad C_\sigma = 0.69$$

$$\Delta = \frac{1}{A+B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{(0.03125+0.02500)(200{,}000)} = 0.0001628 \text{ mm}^3/\text{N}$$

(a) $$\sigma_{max} = -C_\sigma\frac{b}{\Delta} = -C_\sigma C_b\frac{\sqrt[3]{P\Delta}}{\Delta} = -0.69(0.83)\frac{\sqrt[3]{P(0.0001628)}}{0.0001628} = \underline{-192.1\sqrt[3]{P} \text{ MPa}}$$

(b) $$\sigma_{max} = -2758 = -192.1\sqrt[3]{P}$$

$$P = \underline{2.96 \text{ kN}}$$

17.10

$B = \frac{1}{2R_2} = \frac{1}{2(12.8)} = 0.03906\ mm^{-1}$; $A = \frac{1}{2}\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) = \frac{1}{R_1'} = \frac{1}{32} = 0.03125\ mm^{-1}$

$\frac{B}{A} = \frac{0.03906}{0.03125} = 1.25$; $k = 0.83 = C_b$; $C_\sigma = 0.69$

$\Delta = \frac{1}{A+B}\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{(0.03906+0.03125)(200{,}000)} = 0.0001303\ mm^3/N$

In Problem 17-9, $\sigma_{max1} = -2758$ MPa and ellipse of contact has the same shape as that for this problem except that the major axis of the ellipse is perpendicular to that for this problem. For this problem $e = \frac{1}{k} = \frac{1}{0.83} = 1.20$. For Problem 17.9 $e = k = 0.83$. Since $2\tau_{01} = 2\tau_{02}$, Fig. 14-6.6 gives

(a) $2\tau_{01} = 2\tau_{02} = 0.450\,\sigma_{max1} = 0.405\,\sigma_{max2}$

$\sigma_{max2} = \frac{0.450(-2758)}{0.405} = \underline{-3064\ MPa} = -C_\sigma \frac{b}{\Delta} = -C_\sigma C_b \frac{\sqrt[3]{P\Delta}}{\Delta}$

(b) $P = \left[\frac{3064(0.0001303)}{0.69(0.83)}\right]^3 \frac{1}{0.0001303} = \underline{2.60\ kN}$

17.11

$B = A = \frac{1}{2R} = \frac{1}{2(25)} = 0.020\ mm^{-1}$

$\frac{B}{A} = 1.00$; $C_b = 0.90$; $C_G = 0.19$

$\Delta = \frac{1}{A+B}\left[\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right] = \frac{1}{2(0.020)}\left[\frac{1-0.29^2}{200{,}000} + \frac{1-0.33^2}{72{,}000}\right] = 0.0004239\ mm^3/N$

$\tau_{oct(max)} = \frac{\sqrt{2}Y}{3} = \frac{\sqrt{2}(450)}{3} = 212.1\ MPa = C_G \frac{b}{\Delta} = C_G C_b \frac{\sqrt[3]{P_Y \Delta}}{\Delta}$

$P_Y = \left[\frac{212.1(0.0004239)}{0.19(0.90)}\right]^3 \frac{1}{0.0004239} = \underline{343\ N}$

17.12

(a) $P = \frac{P_Y}{SF} = \frac{343}{1.75} = \underline{196\ N}$; For $\frac{B}{A} = 1.00$, $C_\delta = 2.30$

(b) $b = C_b\sqrt[3]{P\Delta} = 0.90\sqrt[3]{196(0.0004239)} = 0.393\ N$

$\frac{b}{\Delta} = \frac{0.393}{0.0004239} = 927\ MPa$

$\delta = C_\delta \frac{P}{\pi}\left(\frac{A+B}{\frac{b}{\Delta}}\right) = 2.30\frac{196}{\pi}\left(\frac{2(0.020)}{927}\right) = \underline{0.0062\ mm}$

17.13

$B = \frac{1}{R} = \frac{1}{40} = 0.0250\ mm^{-1};\quad A = 0;\quad \frac{B}{A} = \infty;\quad e = 0;\quad \frac{2\tau_0}{\sigma_{max}} = 0.50$

$\Delta = R\,\frac{2(1-\nu^2)}{E} = \frac{40(2)(1-0.29^2)}{200,000} = 0.0003664\ mm^3/N;\quad w = \frac{P}{h}$

$b = \sqrt{\frac{2P\Delta}{h\pi}} = \sqrt{\frac{2P(0.0003664)}{20\pi}} = 0.00342\sqrt{P};\quad \frac{b}{\Delta} = \frac{0.00342\sqrt{P}}{0.0003664} = 9.33\sqrt{P}\ MPa$

$\sigma_{max} = -\frac{b}{\Delta} = \underline{-9.33\sqrt{P}\ MPa}$

$\tau_{max} = 0.30\frac{b}{\Delta} = 0.30(9.33\sqrt{P}) = \underline{2.80\sqrt{P}\ MPa}$

$\tau_{oct(max)} = 0.27\frac{b}{\Delta} = 0.27(9.33\sqrt{P}) = \underline{2.52\sqrt{P}\ MPa}$

$\tau_0 = \frac{0.50\,\sigma_{max}}{2} = 0.25(9.33\sqrt{P}) = \underline{2.33\sqrt{P}\ MPa}$

17.14

(a) $\sigma_{max} = -1380 = -9.33\sqrt{P};\quad P = \left(\frac{1380}{9.33}\right)^2 = \underline{21.88\ kN}$

(b) $P_w = \frac{P}{SF} = \frac{21.88}{2.50} = 8.75\ kN$

$b = \sqrt{\frac{2P_w\Delta}{h\pi}} = \sqrt{\frac{2(8750)(0.0003664)}{20\pi}} = 0.3195\ mm$

$\frac{b}{\Delta} = \frac{0.3195}{0.0003664} = 872\ MPa;\quad \sigma_{max} = -\frac{b}{\Delta} = \underline{-872\ MPa}$

17.15

(a) $\Delta = 2R\,\frac{2(1-\nu^2)}{E} = \frac{4(440)(1-0.29^2)}{200,000} = 0.00806\ mm^3/N$

$b = \sqrt{\frac{2P\Delta}{h\pi}} = \sqrt{\frac{2(110,000)(0.00806)}{100\pi}} = 2.38\ mm;\quad \frac{b}{\Delta} = \frac{2.38}{0.00806} = 295.3\ MPa$

$\sigma_{max} = -\frac{b}{\Delta} = \underline{-295.3\ MPa};\quad \tau_{max} = 0.30\frac{b}{\Delta} = 0.30(295.3) = \underline{88.6\ MPa}$

(b) At yield $\tau_{max} = \frac{Y}{2} = 440\ MPa = 0.30\frac{b}{\Delta};\quad \frac{b}{\Delta} = 1467\ MPa$

$b = 1467(0.00806) = 11.82\ mm = \sqrt{\frac{2P_Y\Delta}{h\pi}}$

$P_Y = \frac{11.82^2(100\pi)}{2(0.00806)} = 2.72\ MN$

$SF = \frac{P_Y}{P} = \frac{2,720,000}{110,000} = \underline{24.8}$

17.16

$B = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\left(\frac{1}{15} + \frac{1}{6}\right) = 0.1167$

$\Delta = \frac{1}{B}\,\frac{2(1-\nu^2)}{E} = \frac{2(1-0.29^2)}{0.1167(200,000)} = 0.0000785\ mm^3/N$

$\sigma_{max} = -1000 = -\frac{b}{\Delta};\quad b = 1000(0.0000785) = 0.0785\ mm = \sqrt{\frac{2P\Delta}{h\pi}}$

$P = \frac{0.0785^2(15\pi)}{2(0.0000785)} = \underline{1.85\ kN}$

17.17 From Table 17.1, $\sigma_{max} = -1.40\frac{b}{\Delta}$, $\tau_{max} = 0.435\frac{b}{\Delta}$, and $\tau_{oct(max)} = 0.368\frac{b}{\Delta}$.

$$\Delta = R\,\frac{2(1-\nu^2)}{E} = \frac{40(2)(1-0.29^2)}{200{,}000} = 0.0003664 \text{ mm}^3/\text{N}$$

$$b = \sqrt{\frac{2P\Delta}{h\pi}} = \sqrt{\frac{2(80{,}000)(0.0003664)}{150\,\pi}} = 0.3527 \text{ mm};\quad \frac{b}{\Delta} = \frac{0.3527}{0.0003664} = 962.6 \text{ MPa}$$

$$\sigma_{max} = -1.40(962.6) = \underline{-1348 \text{ MPa}}$$

$$\tau_{max} = 0.435(962.6) = \underline{419 \text{ MPa}}$$

$$\tau_{oct(max)} = 0.368(962.6) = \underline{354 \text{ MPa}}$$

17.18

(a) $$b = \sqrt{\frac{2P\Delta}{h\pi}} = \sqrt{\frac{2P(0.0003664)}{150\,\pi}} = 0.001247\sqrt{P} \text{ mm}$$

$$\frac{b}{\Delta} = \frac{0.001247\sqrt{P}}{0.0003664} = 3.403\sqrt{P} \text{ MPa};\quad \sigma_{max} = -1500 = -\frac{b}{\Delta} = -3.403\sqrt{P}$$

$$P = \left(\frac{1500}{3.403}\right)^2 = \underline{194 \text{ kN}}$$

(b) $$\frac{\sigma_{max(\beta=1/3)}}{\sigma_{max(\beta=0)}} = \frac{\text{Range } \tau_{max(\beta=0)}}{\text{Range } \tau_{max(\beta=1/3)}} = \frac{0.5\frac{b}{\Delta}}{0.67\frac{b}{\Delta}} = 0.746$$

$$\sigma_{max(\beta=1/3)} = 0.746(-1500) = -1119 \text{ MPa} = -1.40\frac{b}{\Delta} = 1.40(3.403\sqrt{P})$$

$$P = \left(\frac{1119}{3.403}\right)^2 = \underline{55.2 \text{ kN}}$$

19.1

a) i) $@\ x=0,\ v=0 \Rightarrow a_0=0$

$@\ x=0,\ v'=0 \Rightarrow a_1=0$

$$\underline{\underline{v(x) = a_2 x^2}}$$

ii) NO SIMPLIFICATION POSSIBLE

iii) $@\ x=0,\ v=0 \Rightarrow c_0=0$

$@\ x=0,\ v'=0 \Rightarrow c_1=0$

$$\underline{\underline{v(x) = c_2x^2 + c_3x^3}}$$

b) $U = \frac{EI}{2}\int_0^L (v'')^2\,dx \ ; \ \Omega = -P\,v(L)$

i) $\underline{\underline{U = \frac{EI}{2}\int_0^L 4a_2^2\,dx = 2EI\,a_2^2 L}}$

$$\underline{\underline{\Omega = -Pa_2L^2}}$$

ii) $U = \frac{EI}{2}\int_0^L b^2\frac{\pi^4}{16L^4}\left(\cos^2\frac{\pi x}{2L}\right)dx$

$$\underline{\underline{U = \frac{EIb^2\pi^4}{64L^3}}}$$

$$\underline{\underline{\Omega = -Pb}}$$

iii) $U = \frac{EI}{2}\int_0^L 4\left(c_2^2 + 6c_2c_3x + 9c_3^2x^2\right)dx$

$$\underline{\underline{U = 2EI\left(c_2^2L + 3c_2c_3L^2 + 3c_3^2L^3\right)}}$$

$$\underline{\underline{\Omega = -P\left(c_2L^2 + c_3L^3\right)}}$$

c) i) $\Pi = 2EI\,a_2^2L - Pa_2L^2$

$$\frac{\partial\Pi}{\partial a_2} = 0 \Rightarrow \underline{\underline{a_2 = \frac{PL}{4EI}}}$$

ii) $\Pi = \frac{EIb^2\pi^4}{64L^3} - Pb$

$$\frac{\partial\Pi}{\partial b} = 0 \Rightarrow \underline{\underline{b = \frac{32PL^3}{EI\pi^4}}}$$

iii) $\Pi = 2EI\left(c_2^2L + 3c_2c_3L^2 + 3c_3^2L^3\right) - P\left(c_2L^2 + c_3L^3\right)$

$$\frac{\partial\Pi}{\partial c_2} = 0 \Rightarrow 4EIc_2 + 6EILc_3 = PL \quad (a)$$

$$\frac{\partial\Pi}{\partial c_3} = 0 \Rightarrow 6EIc_2 + 12EILc_3 = PL \quad (b)$$

SOLVING (a) & (b):

$$\underline{\underline{c_2 = \frac{PL}{2EI}}}\ ,\quad \underline{\underline{c_3 = \frac{-P}{6EI}}}$$

d.) i) $v(x) = \frac{PL}{4EI}x^2$

$$\underline{\underline{v(L) = \frac{PL^3}{4EI}}}$$

$$\Pi = 2EIL\left(\frac{PL}{4EI}\right)^2 - PL^2\left(\frac{PL}{4EI}\right)$$

$$\underline{\underline{\Pi = -\frac{P^2L^3}{8EI}}}$$

ii) $v(x) = \frac{32PL^3}{EI\pi^4}\left(1 - \cos\frac{\pi x}{2L}\right)$

$$\underline{\underline{v(L) = \frac{32PL^3}{EI\pi^4}}}$$

$$\Pi = \frac{EI\pi^4}{64L^3}\left(\frac{32PL^3}{EI\pi^4}\right)^2 - P\left(\frac{32PL^3}{EI\pi^4}\right)$$

$$\underline{\underline{\Pi = -\frac{16P^2L^3}{EI\pi^4}}}$$

iii) $v(x) = \frac{PL}{2EI}x^2 - \frac{P}{6EI}x^3$

$$\underline{\underline{v(L) = \frac{PL^3}{3EI}}}$$

$$\underline{\underline{\Pi = -\frac{P^2L^3}{6EI}}}$$

e) FUNCTION iii IS EXACT AND HAS MINIMUM POTENTIAL ENERGY. FUNCTION ii IS A BETTER APPROXIMATION THAN i AS SEEN BY COMPARING TIP DISPLACEMENTS AND TOTAL POTENTIALS.

19.2

a) $A = 2\,,\quad \alpha_1 = 4\,,\quad \beta_1 = -2\,,\quad \gamma_1 = -2\,;\quad N_1 = 1 - x/2 - y/2$

$\alpha_2 = 0\,,\quad \beta_2 = 2\,,\quad \gamma_2 = 0\,;\quad N_2 = x/2$

$\alpha_3 = 0\,,\quad \beta_3 = 0\,,\quad \gamma_3 = 2\,;\quad N_3 = y/2$

b) $N_1(0.5, 1) = 0.25\,;\quad N_2(0.5, 1) = 0.25\,;\quad N_3(0.5, 1) = 0.5$

c) $A_{P12} = 1/2(2)(1) = 1.0\,,\quad A_{P13} = 1/2(2)(0.5) = 0.5\,,\quad A_{P23} = A - A_{P12} - A_{P13} = 0.5$

$N_1(0.5, 1) = \dfrac{A_{P23}}{A} = 0.25\,,\quad N_2(0.5, 1) = \dfrac{A_{P13}}{A} = 0.25\,,\quad N_3 = \dfrac{A_{P12}}{A} = 0.5$

19.3

$$[M_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[M_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

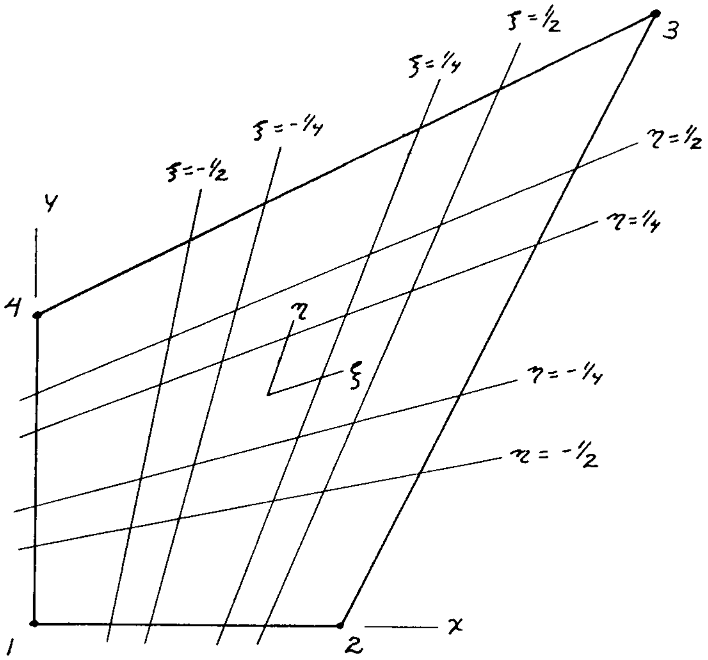
$$[M_4] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
k^1_{11} + k^2_{11} & k^1_{12} + k^2_{12} & k^1_{15} & k^1_{16} & k^2_{13} & k^2_{14} & k^1_{13} + k^2_{15} & k^1_{14} + k^2_{16} & & & & \\
k^1_{21} + k^2_{21} & k^1_{22} + k^2_{22} & k^1_{25} & k^1_{26} & k^2_{23} & k^2_{24} & k^1_{23} + k^2_{25} & k^1_{24} + k^2_{26} & & & & \\
\hline
k^1_{51} & k^1_{52} & k^1_{55} & k^1_{56} & & & k^1_{53} & k^1_{54} & & & & \\
k^1_{61} & k^1_{62} & k^1_{65} & k^1_{66} & & & k^1_{63} & k^1_{64} & & & & \\
\hline
k^2_{31} & k^2_{32} & & & k^2_{33} + k^3_{11} + k^4_{11} & k^2_{34} + k^3_{12} + k^4_{12} & k^2_{35} + k^3_{15} & k^2_{36} + k^3_{16} & k^4_{13} & k^4_{14} & k^3_{13} + k^4_{15} & k^3_{14} + k^4_{16} \\
k^2_{41} & k^2_{42} & & & k^2_{43} + k^3_{21} + k^4_{21} & k^2_{44} + k^3_{22} + k^4_{22} & k^2_{45} + k^3_{25} & k^2_{46} + k^3_{26} & k^4_{23} & k^4_{24} & k^3_{23} + k^4_{25} & k^3_{24} + k^4_{26} \\
\hline
k^1_{31} + k^2_{51} & k^1_{32} + k^2_{52} & k^1_{35} & k^1_{36} & k^2_{53} + k^3_{51} & k^2_{54} + k^3_{52} & k^1_{33} + k^2_{55} + k^3_{55} & k^1_{34} + k^2_{56} + k^3_{56} & & & k^3_{53} & k^3_{54} \\
k^1_{41} + k^2_{61} & k^1_{42} + k^2_{62} & k^1_{45} & k^1_{46} & k^2_{63} + k^3_{61} & k^2_{64} + k^3_{62} & k^1_{43} + k^2_{65} + k^3_{65} & k^1_{44} + k^2_{66} + k^3_{66} & & & k^3_{63} & k^3_{64} \\
\hline
 & & & & k^4_{31} & k^4_{32} & & & k^4_{33} & k^4_{34} & k^4_{35} & k^4_{36} \\
 & & & & k^4_{41} & k^4_{42} & & & k^4_{43} & k^4_{44} & k^4_{45} & k^4_{46} \\
\hline
 & & & & k^3_{31} + k^4_{51} & k^3_{32} + k^4_{52} & k^3_{35} & k^3_{36} & k^4_{53} & k^4_{54} & k^3_{33} + k^4_{55} & k^3_{34} + k^4_{56} \\
 & & & & k^3_{41} + k^4_{61} & k^3_{42} + k^4_{62} & k^3_{45} & k^3_{46} & k^4_{63} & k^4_{64} & k^3_{43} + k^4_{65} & k^3_{44} + k^4_{66}
\end{array}\right]
$$

Problem 19.4 – Assembled Stiffness Matrix (partitioned into 2x2 nodal submatrices)

19.5

a)



b) USE SHAPE FUNCTIONS GIVEN BY EQ. (19.51). SINCE $x_1 = 0$ & $x_4 = 0$,

$x = N_2 x_2 + N_3 x_3 = 1/4(3 + 3\xi + \eta + \xi\eta)$. SINCE $y_1 = 0$ & $y_2 = 0$,

$y = N_3 y_3 + N_4 y_4 = 1/4(3 + \xi + 3\eta + \xi\eta)$.

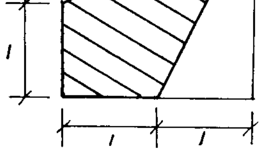
c) $\partial x/\partial\xi = 1/4(3+\eta)$, $\partial y/\partial\xi = 1/4(1+\eta)$

$\partial x/\partial\eta = 1/4(1+\xi)$, $\partial y/\partial\eta = 1/4(3+\xi)$

d) $$|J| \text{ @ } \xi=0, \eta=0 = \begin{vmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{vmatrix} = 1/2$$

IN NATURAL COORDINATES, $A_{\xi\eta} = 2(2) = 4$

IN PHYSICAL COORDINATES, $A_{xy} = 2(2) - 2(1/2)2(1) = 2$



$$\frac{A_{xy}}{A_{\xi\eta}} = 2/4 = 1/2 = |J| \text{ @ } \xi=0, \eta=0$$

19.6

USE EQ. (19.57) TO FIND $[B_1]$ WITH SHAPE FUNCTIONS FROM EQ. (19.51).

$$\frac{\partial N_1}{\partial \xi} = \tfrac{1}{4}(-1)(1-\eta), \quad \frac{\partial N_2}{\partial \xi} = \tfrac{1}{4}(1)(1-\eta), \quad \frac{\partial N_3}{\partial \xi} = \tfrac{1}{4}(1)(1+\eta), \quad \frac{\partial N_4}{\partial \xi} = \tfrac{1}{4}(-1)(1+\eta)$$

$$\frac{\partial N_1}{\partial \eta} = \tfrac{1}{4}(1-\xi)(-1), \quad \frac{\partial N_2}{\partial \eta} = \tfrac{1}{4}(1+\xi)(-1), \quad \frac{\partial N_3}{\partial \eta} = \tfrac{1}{4}(1+\xi)(1), \quad \frac{\partial N_4}{\partial \eta} = \tfrac{1}{4}(1-\xi)(1)$$

$$@\ \xi=0, \eta=0: \frac{\partial N_1}{\partial \xi} = -\tfrac{1}{4}, \quad \frac{\partial N_2}{\partial \xi} = \tfrac{1}{4}, \quad \frac{\partial N_3}{\partial \xi} = \tfrac{1}{4}, \quad \frac{\partial N_4}{\partial \xi} = -\tfrac{1}{4},$$

$$\frac{\partial N_1}{\partial \eta} = -\tfrac{1}{4}, \quad \frac{\partial N_2}{\partial \eta} = -\tfrac{1}{4}, \quad \frac{\partial N_3}{\partial \eta} = \tfrac{1}{4}, \quad \frac{\partial N_4}{\partial \eta} = \tfrac{1}{4}$$

$$\frac{\partial x}{\partial \xi} = \sum_1^4 \frac{\partial N_i}{\partial \xi} x_i = \tfrac{1}{2}, \quad \frac{\partial y}{\partial \xi} = \sum_1^4 \frac{\partial N_i}{\partial \xi} y_i = \tfrac{1}{4}, \quad \frac{\partial x}{\partial \eta} = \sum_1^4 \frac{\partial N_i}{\partial \eta} x_i = 0$$

$$\frac{\partial y}{\partial \eta} = \sum_1^4 \frac{\partial N_i}{\partial \eta} y_i = \tfrac{3}{4}; \qquad [J] = \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{4} \\ 0 & \tfrac{3}{4} \end{bmatrix}; \ |J| = \tfrac{3}{8}; \ [J]^{-1} = \tfrac{8}{3}\begin{bmatrix} \tfrac{3}{4} & -\tfrac{1}{4} \\ 0 & \tfrac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -\tfrac{2}{3} \\ 0 & \tfrac{4}{3} \end{bmatrix}$$

$$[B_1] = \begin{bmatrix} 2(-\tfrac{1}{4}) + (-\tfrac{2}{3})(-\tfrac{1}{4}) & 0 \\ 0 & 0(-\tfrac{1}{4}) + \tfrac{4}{3}(-\tfrac{1}{4}) \\ 0(-\tfrac{1}{4}) + \tfrac{4}{3}(-\tfrac{1}{4}) & 2(-\tfrac{1}{4}) + (-\tfrac{2}{3})(-\tfrac{1}{4}) \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{3} & 0 \\ 0 & -\tfrac{1}{3} \\ -\tfrac{1}{3} & -\tfrac{1}{3} \end{bmatrix}$$

19.7 SAMPLING POINT LOCATIONS & WEIGHTS TAKEN FROM FIG 19.10, ξ DIRECTION ONLY.

a) EXACT: $I = (\tfrac{6}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2 - 2)\Big|_{-1}^{1} = \underline{\underline{-\tfrac{20}{3}}}$; 1-POINT: $I = 2(-2) = \underline{\underline{-4\ (40\%\ \text{ERROR})}}$

2-POINT: $I = 1(6(\tfrac{-1}{\sqrt{3}})^3) - 4(\tfrac{-1}{\sqrt{3}})^2 + 3(\tfrac{-1}{\sqrt{3}}) - 2)$
$\quad + 1(6(\tfrac{1}{\sqrt{3}})^3) - 4(\tfrac{1}{\sqrt{3}})^2 + 3(\tfrac{1}{\sqrt{3}}) - 2) = \underline{\underline{-\tfrac{20}{3}\ (\text{EXACT})}}$

3-POINT: $I = \tfrac{5}{9}(6(-\sqrt{.6})^3 - 4(-\sqrt{.6})^2 + 3(-\sqrt{.6}) - 2) + \tfrac{8}{9}(-2)$
$\quad + \tfrac{5}{9}(6(\sqrt{.6})^3 - 4(\sqrt{.6})^2 + 3(\sqrt{.6}) - 2) = \underline{\underline{-\tfrac{20}{3}\ (\text{EXACT})}}$

b) EXACT: $I = \sinh \xi \Big|_{-1}^{1} = \underline{\underline{2.350402}}$; 1-POINT: $I = 2(1) = \underline{\underline{2.0\ (15\%\ \text{ERROR})}}$

2-POINT: $I = 1(1.171378) + 1(1.171378) = \underline{\underline{2.342696\ \ (0.33\%\ \text{ERROR})}}$

3-POINT: $I = \tfrac{5}{9}(1.315303) + \tfrac{8}{9}(1) + \tfrac{5}{9}(1.315303) = \underline{\underline{2.350337\ \ (0.003\%\ \text{ERROR})}}$

c) EXACT: $I = e^{\xi}\Big|_{-1}^{1} = 2.350402$; 1-POINT: $I = 2(1) = \underline{\underline{2.0\ \ (15\%\ \text{ERROR})}}$

2-POINT: $I = 1(0.561384) + 1(1.781312) = \underline{\underline{2.342696\ \ (0.33\%\ \text{ERROR})}}$

3-POINT: $I = \tfrac{5}{9}(0.460890) + \tfrac{8}{9}(1) + \tfrac{5}{9}(2.169717) = \underline{\underline{2.350337\ \ (0.003\%\ \text{ERROR})}}$

19.8 SAMPLING POINT LOCATIONS & WEIGHTS TAKEN FROM FIG. 19.10.

a) EXACT: $I = \left(\sin\xi\Big|_{-1}^{1}\right)\left(\sin\eta\Big|_{-1}^{1}\right) = 2.832294$

1-POINT: $I = (2)(2)(1)(1) = 4.0$ (41% ERROR)

2-POINT: $I = 4\left[(1)(1)(0.837912)^2\right] = 2.808386$ (0.84% ERROR)

3-POINT: $I = 4\left[(5/9)^2(0.714703)^2\right] + 4\left[(5/9)(8/9)(0.714703)(1)\right] + (8/9)^2(1)$

$I = 0.630618 + 1.411759 + 0.790123 = 2.832500$ (0.007% ERR)

b) EXACT: $I = \left[\left(\tfrac{1}{2}\xi - \tfrac{1}{4}\sin 2\xi\right)\Big|_{-1}^{1}\right]\left(\sin\eta\Big|_{-1}^{1}\right) = (0.545351)(1.682942) = 0.917794$

1-POINT: $I = (2)(2)(0)(1) = 0$

2-POINT: $I = 4\left[(1)(1)(0.297904)(0.837912)\right] = 0.998469$ (8.8% ERROR)

3-POINT: $I = 4\left[(5/9)^2(0.489199)(0.714703)\right] + 2\left[(5/9)(8/9)(0)(0.714703)\right]$

$+ 2\left[(5/9)(8/9)(0.489199)(1)\right] + (8/9)^2(0)(1)$

$I = 0.431644 + 0 + 0.483160 + 0 = 0.914804$ (0.33% ERROR)

19.9 BY ANALOGY TO EQ. (19.74): $\delta W_c = \delta\bar{u}\Big|_{\bar{x}=L_c} \bar{P}_c$.

INTERPOLATE DISPLACEMENT: $\delta W_c = \{\delta u_i\}^T [N]^T\Big|_{\bar{x}=L_c} \bar{P}_c$

$\therefore \{\bar{P}_{ci}\} = [N]^T\Big|_{\bar{x}=L_c} \bar{P}_c$; WHERE $[N] = [1-\bar{x}/L \quad \bar{x}/L]$

AT $\bar{x} = L_c$; $\{\bar{P}_{ci}\} = \begin{Bmatrix} \bar{P}_c(1 - L_c/L) \\ P_c(L_c/L) \end{Bmatrix}$.

19.10 BY ANALOGY TO EQ. (19.72): $\delta W_D = \int_{L_a}^{L_b} \delta\bar{u}\, \bar{q}_o\, d\bar{x}$

INTERPOLATE DISPLACEMENT: $\delta W_D = \{\delta\bar{u}_i\}^T \int_{L_a}^{L_b} [N]^T \bar{q}_o\, d\bar{x}$.

$\therefore \{\bar{P}_{oi}\} = \int_{L_a}^{L_b} [N]^T \bar{q}_o\, d\bar{x}$; WHERE $[N] = [1 - \bar{x}/L \quad \bar{x}/L]$

$$\{\bar{P}_{oi}\} = \bar{q}_o \begin{Bmatrix} L_b - L_a - \dfrac{L_b^2 - L_a^2}{2L} \\ \dfrac{L_b^2 - L_a^2}{2L} \end{Bmatrix}.$$

19.11

BY ANALOGY TO EQ. (19.74): $\delta \mathcal{W}_c = \delta \bar{v}'\Big|_{\bar{x}=L_c} \bar{M}_c$

INTERPOLATE ROTATION AT $\bar{x}=L_c$: $\delta \mathcal{W}_c = \{\delta \bar{v}_i\}^T \frac{d}{d\bar{x}}[N]^T\Big|_{\bar{x}=L_c} \bar{M}_c$.

$\therefore$ $\{\bar{P}_{ci}\} = \frac{d}{d\bar{x}}[N]^T\Big|_{\bar{x}=L_c} \bar{M}_c$; WHERE $[N]$ IS GIVEN BY EQ. (19.63).

$$\{P_{ci}\} = \begin{Bmatrix} \frac{6\bar{M}_c}{L^3}\left(L_c^2 - LL_c\right) \\ \frac{\bar{M}_c}{L^2}(L-L_c)(L-3L_c) \\ \frac{-6\bar{M}_c}{L^3}\left(L_c^2 - LL_c\right) \\ \frac{\bar{M}_c L_c}{L^2}\left(3L_c - 2L\right) \end{Bmatrix}$$

19.12

GIVEN: $\delta \mathcal{U} = \int_0^L (\delta \bar{v})'' EI\, \bar{v}''\, d\bar{x} + \int_0^L (\delta \bar{v})\, k\, \bar{v}\, d\bar{x}$.

SUBSTITUTE FOR $\bar{v}$, $\delta\bar{v}$, $(\delta\bar{v})''$ FROM EQS. (19.62) AND (19.69a).

$$\delta \mathcal{U} = \{\delta \bar{v}_i\}^T \int_0^L [B_B]^T EI [B_B]\{\bar{v}_i\}\, d\bar{x} + \{\delta \bar{v}_i\}^T \int_0^L [N]^T k [N]\{\bar{v}_i\}\, d\bar{x} .$$

$$\delta \mathcal{U} = \{\delta \bar{v}_i\}^T \left[\int_0^L [B_B]^T EI [B_B]\, d\bar{x} + \int_0^L [N]^T k [N]\, d\bar{x} \right] \{\bar{v}_i\}$$

$\therefore$ $[\bar{K}] = [\bar{K}_B] + [\bar{K}_{EF}]$ WHERE $[\bar{K}_B]$ IS THE BENDING STIFFNESS OF THE BEAM [EQ (19.67)] AND $[\bar{K}_{EF}]$ IS THE STIFFNESS MATRIX FOR THE ELASTIC FOUNDATION:

$$[\bar{K}_{EF}] = \int_0^L [N]^T k [N]\, d\bar{x}$$

AFTER INTEGRATION:

$$[\bar{K}_{EF}] = k \begin{bmatrix} \frac{13}{35}L & & \text{SYMMETRIC} & \\ \frac{11}{210}L^2 & \frac{1}{105}L^3 & & \\ \frac{9}{70}L & \frac{13}{420}L^2 & \frac{13}{35}L & \\ \frac{-13}{420}L^2 & \frac{-1}{140}L^3 & \frac{-11}{210}L^2 & \frac{1}{105}L^3 \end{bmatrix}$$